Fast Deterministic Pricing of Options on Lévy Driven Assets *

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Abstract

A partial integro-differential equation (PIDE) $\partial_t u + \mathcal{A}[u] = 0$ for European contracts on assets with general jump-diffusion price process of Lévy type is derived. The PIDE is localized to bounded domains and the error due to this localization is estimated. The localized PIDE is discretized by the θ-scheme in time and a wavelet Galerkin method with N degrees of freedom in space. The full Galerkin matrix for $\mathcal{A}$ can be replaced with a sparse matrix in the wavelet basis, and the linear systems for each time step are solved approximatively with GMRES in linear complexity. The total work of the algorithm for M time steps is bounded by $O(MN(\ln N)^2)$ operations and $O(N \ln(N))$ memory. The deterministic algorithm gives optimal convergence rates (up to logarithmic terms) for the computed solution in the same complexity as finite difference approximations of the standard Black-Scholes equation. Computational examples for various Lévy price processes (VG, CGMY) are presented.

1 Introduction

Since the seminal paper [6], the pricing of options by means of partial differential equations has become standard practice in quantitative finance, either by means of explicit solution formulas for the heat equation (e.g. [21, 22, 19]) in the case of European vanillas or by numerical methods in the case of American or Barrier options.

In recent years, awareness of the shortcomings of the Black-Scholes model has increased and more general models for the stochastic dynamics of the risky asset have been proposed: we mention only stochastic volatility models and 'stochastic clocks'. The latter lead to so-called jump-diffusion price processes: the Wiener process from the Black-Scholes model is replaced by a jump-diffusion Lévy process (see e.g. [24, 3, 15, 23, 9, 7, 8, 28, 30, 29] and the references there and [5, 31] for background information on Lévy processes).

Abandoning the Wiener process as price process renders the market in the model incomplete and the martingale measure in the pricing problem non-unique. After selection of an equivalent martingale measure $\mathbb{Q}$ the asset pricing problem becomes once again the problem of solving a deterministic equation. Contrary to the Black-Scholes case, this equation is now a parabolic integro-differential equation (PIDE) with non-integrable kernel if the jump activity of the Lévy process is infinite.

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In case of European vanillas and in logarithmic price, this equation is posed on the whole real line. The justification, numerical analysis and rigorous derivation of efficient solution algorithms for this PIDE is the purpose of the present paper. Its outline is as follows: after brief recapitulation of the Black-Scholes model of asset pricing, and in particular of the functional setting which accomodates exponentially growing pay-off functions we turn in Section 3 to the derivation of the PIDE for pricing options on Lévy driven assets. We prove its well-posedness in spaces of possibly exponentially growing solutions and give a suitable variational formulation. Section 4 is devoted to the truncation of the PIDE to a bounded domain – an essential step for numerical simulation as well as for modeling certain types of contracts. Due to the jump part of the Lévy process, this localization cannot be effected by simple restriction to the bounded domain plus suitable local boundary conditions, but must take into account information about the pay-off from beyond the computational domain, respectively from behind the barrier. We show that the localization error decays exponentially with the size of the truncation domain; contrary to earlier work in the Black-Scholes case [20] we do not use the maximum principle, but rather *a-priori* estimates in exponentially weighted spaces.
Section 5 is devoted to our solution algorithm – the θ-scheme for time-stepping and a wavelet-Galerkin discretization of the integro-differential operator. We show that the solution algorithm has the same complexity as the Finite Difference Method (FDM) for the Black-Scholes equation. Finally, we present numerical examples of Lévy pricing – European vanillas under Variance Gamma (VG), CGMY-processes with finite and infinite intensity can all be handled by our approach in a unified fashion.

Let us briefly comment on how our approach compares with Fourier techniques [9, 10] for the Lévy pricing. These methods require the characteristic function of the process and allow, via Fast Fourier Transform (FFT), the efficient pricing of European vanillas. Due to the poor localization of the Fourier transformed solution in the frequency space, however, this approach has severe difficulties in dealing with barrier options or, more importantly, with American contracts.
The present approach accomodates this rather naturally, but, on the other hand, requires the distributional kernel of the generator of the Lévy process, i.e., the inverse Fourier transform of the characteristic function. It allows to handle barrier, touch-and-out or no-touch type contracts with guaranteed error bounds, and without Monte-Carlo techniques. It also allows to price American puts and Asian contracts on Lévy driven underlyings – this, however, will be reported elsewhere.

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2 Pricing European vanillas in the Black-Scholes setting

The classical option pricing theory of Black and Scholes [22, 21] relies on the fact that the pay-off of every contingent claim can be duplicated by a portfolio consisting of investments in the underlying stock and in a bond paying a riskless rate of interest. The model of Black and Scholes consists of one risky asset, a share with price S_t at time t and a riskless asset with price S_t^0 at time t satisfying the following ordinary differential equation

$$dS_t^0 = rS_t^0 dt,$$

with $r > 0$ being the riskless interest rate. The price of the risky asset is modelled by the following stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dB_t),$$

with μ, σ being constants and B_t the standard Brownian motion built on a probability space $(\Omega, \mathcal{F}, P)$. We denote by $(\mathcal{F}_t)_t$ its natural filtration. It is also well-known, see e.g. [22], that there exists a unique probability measure $\mathbb{Q}$ under which the discounted stock price $\tilde{S}_t := e^{-rt}S_t$ is a martingale and any option defined by a non-negative, $\mathcal{F}_T$-measurable random variable g is replicable and the value at time t of any replicating portfolio is given by

$$f(t, S_t) = E_{\mathbb{Q}}[e^{-r(T-t)}g(S_T)|\mathcal{F}_t].$$

We will focus exemplarily on European call options with pay-off $g(x) = (x - K)_+$, where K denotes the so-called strike price, but emphasize that our framework accomodates general pay-off functions. By assumptions of no-arbitrage, see e.g. [22], the price $f(t, S)$ of the option has to satisfy the following partial differential equation

$$\frac{\partial f}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf = 0 \quad \text{in} \quad (0, T) \times (0, \infty) \tag{2.1}$$

together with the terminal condition at maturity

$$f(T, S) = (S - K)_+. \tag{2.2}$$

2.1 Black-Scholes equation

We introduce the following change of variables $x = \ln(S)$ and we write the Black-Scholes equation (2.1)–(2.2) in logarithmic price for $u(t, x) := f(t, e^x)$

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x} - ru &= 0 \quad \text{in} \quad (0, T) \times \mathbb{R} \\ u(T, x) &= h(x) := (e^x - K)_+. \end{aligned} \tag{2.3}$$

In the time to maturity $\tau = T - t$, (2.3) for $w(\tau, x) = u(T - \tau, x)$ reads:

$$\begin{aligned} \frac{\partial w}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial x^2} - \left(r - \frac{\sigma^2}{2}\right)\frac{\partial w}{\partial x} + rw &= 0 \quad \text{in} \quad (0, T) \times \mathbb{R} \\ w(0, x) &= h(x). \end{aligned} \tag{2.4}$$

2.2 Variational formulation

In this section we derive the variational formulation to (2.4). We observe that the pay-off function h in (2.3), (2.4) does not belong to $L^2(\mathbb{R})$. Moreover, since we switched to logarithmic price, this function has an exponential growth at infinity, therefore we cannot use standard Sobolev spaces as function spaces for this problem. We introduce weighted Sobolev spaces to account for the exponential growth of solutions at infinity, following [19].
For $\nu \in \mathbb{R}$ we define the *weighted Sobolev space* $H^1_\nu(\mathbb{R})$ by

$$H^1_\nu(\mathbb{R}) := \{v \in L^1_{\text{loc}}(\mathbb{R}) \mid ve^{\nu|x|}, v'e^{\nu|x|} \in L^2(\mathbb{R})\}.$$

Similarly, $L^2_\nu(\mathbb{R}) := \{v \in L^1_{\rm loc}(\mathbb{R}) \mid ve^{\nu|x|} \in L^2(\mathbb{R})\}$. With this notation, the pay-off function h in (2.4) belongs to $H^1_{-\mu}(\mathbb{R})$ for any $\mu > 1$.
In order to cast (2.4) in a variational form [19] we consider a test function $v \in C_0^\infty(\mathbb{R})$ and we multiply (2.4) by $ve^{-2\nu|x|}$, with $\nu \in \mathbb{R}$ arbitrary, fixed. By integration by parts over $\mathbb{R}$ we obtain

$$\begin{aligned}
&\frac{d}{d\tau}\int_{\mathbb{R}} w(\tau,x)v(x)e^{-2\nu|x|}\,dx - \left(r-\frac{\sigma^2}{2}\right)\int_{\mathbb{R}} \frac{\partial w}{\partial x}(\tau,x)v(x)e^{-2\nu|x|}\,dx\\
&+\frac{\sigma^2}{2}\int_{\mathbb{R}}\left\{\frac{\partial w}{\partial x}(\tau,x)\frac{\partial v}{\partial x}(x)e^{-2\nu|x|} - 2\nu\,\mathrm{sign}(x)\frac{\partial w}{\partial x}(\tau,x)v(x)e^{-2\nu|x|}\right\}dx\\
&+\int_{\mathbb{R}} rw(\tau,x)v(x)e^{-2\nu|x|}\,dx = 0.
\end{aligned}$$

We define the bilinear form $a^{-\nu}(\cdot,\cdot) : H^1_{-\nu}(\mathbb{R}) \times H^1_{-\nu}(\mathbb{R}) \to \mathbb{R}$ as being given by

$$\begin{aligned}
a^{-\nu}(v_1,v_2) \;:=\; &\frac{\sigma^2}{2}\int_{\mathbb{R}} v_1'(x)v_2'(x)e^{-2\nu|x|}\,dx + \int_{\mathbb{R}} rv_1(x)v_2(x)e^{-2\nu|x|}\,dx\\
&-\int_{\mathbb{R}}\left(\nu\sigma^2\,\mathrm{sign}(x)+\left(r-\frac{\sigma^2}{2}\right)\right)v_1'(x)v_2(x)e^{-2\nu|x|}\,dx.
\end{aligned} \tag{2.5}$$

With $\mu > 1$ the variational formulation to (2.4) reads:
Given $h \in H^1_{-\mu}(\mathbb{R})$, find $w \in L^2((0,T);H^1_{-\mu}(\mathbb{R})) \cap H^1((0,T);(H^1_{-\mu}(\mathbb{R}))^*)$ such that

$$\begin{aligned}
\frac{d}{d\tau}(w(\tau,\cdot),v)_{L^2_{-\mu}(\mathbb{R})} + a^{-\mu}(w(\tau,\cdot),v(\cdot)) &= 0 \quad \forall\, v \in H^1_{-\mu}(\mathbb{R})\\
w(0,x) &= h(x).
\end{aligned} \tag{2.6}$$

To prove existence and uniqueness for the solution of (2.6), we analyze the properties of the bilinear form $a^{-\nu}(\cdot,\cdot)$ with respect to the weighted Sobolev spaces $H^1_{-\nu}(\mathbb{R})$ for arbitrary $\nu \in \mathbb{R}$.

Proposition 2.1 *Let $\nu \in \mathbb{R}$ be arbitrary, fixed.*

1. *The bilinear form $a^{-\nu}(\cdot,\cdot) : H^1_{-\nu}(\mathbb{R}) \times H^1_{-\nu}(\mathbb{R}) \to \mathbb{R}$ is continuous, i.e., there exists a constant $M > 0$ such that*

$$|a^{-\nu}(u,v)| \le M\|u\|_{H^1_{-\nu}(\mathbb{R})}\|v\|_{H^1_{-\nu}(\mathbb{R})} \quad \forall\, u,v \in H^1_{-\nu}(\mathbb{R}).$$

2. *There exists $\lambda_0 > 0$ depending on ν such that for all $\lambda > \lambda_0$ the new bilinear form $a^{-\nu}(\cdot,\cdot) + \lambda(\cdot,\cdot)_{L^2_{-\nu}(\mathbb{R})}$ is* **coercive**, *i.e., there exists $\alpha > 0$ such that for all $\lambda > \lambda_0$ it holds: $a^{-\nu}(u,u) + \lambda\|u\|^2_{L^2_{-\nu}(\mathbb{R})} \ge \alpha\|u\|^2_{H^1_{-\nu}(\mathbb{R})}$ $\forall\, u \in H^1_{-\nu}(\mathbb{R})$.*

Proof. Take $v_1 = v_2 = u$ in the definition (2.5) of the bilinear form $a^{-\nu}(\cdot,\cdot)$. Then, there exist some constants $\gamma > 0$, $\beta \ge 0$ such that for all $u \in H^1_{-\nu}(\mathbb{R})$ it holds

$$\begin{aligned}
a^{-\nu}(u,u) &= \frac{\sigma^2}{2}\|u'e^{-\nu|x|}\|^2_{L^2(\mathbb{R})} - \int_{\mathbb{R}}\left(\nu\,\mathrm{sign}(x)\sigma^2 + r - \frac{\sigma^2}{2}\right)u'(x)u(x)e^{-2\nu|x|}\,dx + r\|ue^{-\nu|x|}\|^2_{L^2(\mathbb{R})}\\
&\ge \gamma\|u'e^{-\nu|x|}\|^2_{L^2(\mathbb{R})} - \beta\|ue^{-\nu|x|}\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Choosing now $\lambda_0 > \beta$ we obtain *2.* The assertion *1.* follows from the Cauchy-Schwarz inequality.

□

Remark 2.2 Without loss of generality we assume from now on that $a^{-\nu}(\cdot,\cdot)$ is coercive with coercivity constant $\alpha > 0$. Indeed, by the transformation $v(\tau,x) = e^{-\lambda\tau}w(\tau,x)$ the problem for the new function v reads

$$\begin{aligned} \frac{\partial v}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2} - \left(r - \frac{\sigma^2}{2}\right)\frac{\partial v}{\partial x} + (r+\lambda)v &= 0 \quad \in (0,T)\times\mathbb{R} \\ v(0,x) &= h(x) \end{aligned} \tag{2.7}$$

and the corresponding bilinear form $a^{-\nu}(\cdot,\cdot) + \lambda(\cdot,\cdot)_{L^2_{-\nu}(\mathbb{R})}$ is by Proposition 2.1 *2.* for all $\lambda > \lambda_0$ coercive.

2.3 Functional setting

We give a functional setting for the existence and continuous dependence of $H^1_{-\nu}(\mathbb{R})$-solutions to (2.4) which will also be used later for Lévy processes. It is based on the following Gelfand triple:

$$X := H^1_{-\nu}(\mathbb{R}) \hookrightarrow L^2_{-\nu}(\mathbb{R}) := L^2(\mathbb{R}, e^{-2\nu|x|}dx) \cong \left(L^2_{-\nu}(\mathbb{R})\right)^* \hookrightarrow X^* = \left(H^1_{-\nu}(\mathbb{R})\right)^* \tag{2.8}$$

with dense, but non-compact embeddings. We denote by L the operator

$$Lu := -\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - \left(r - \frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x} + ru.$$

In our treatment of Lévy processes we need a general parabolic existence result in the triple (2.8).

Theorem 2.3 *1. $L \in \mathcal{L}(X, X^*)$ is an isomorphism.*

2. $-L$ is the infinitesimal generator of a uniformly bounded analytic C^0-semi-group $T^{-\nu}(t)$ in $(H^1_{-\nu}(\mathbb{R}))^*$.

Then the evolution problem (2.6) has a unique solution $v \in L^2(0,T;X) \cap H^1(0,T;X^*)$ which can be represented as

$$v(\tau,\cdot) = T^{-\mu}(\tau)h.$$

Proof.

Step 1. L is a closed operator, since the graph norm $\|u\|_L := \|Lu\|_{X^*} + \|iu\|_{X^*}$, with $X \overset{i}{\hookrightarrow} X^*$, is an equivalent norm for X.

Step 2. For all $\lambda \in \mathbb{C}$, with $\mathrm{Re}\lambda > 0$,

$$(u,v) \mapsto ((L+\lambda I)u, v)_{X^*\times X}$$

is also elliptic and it holds

$$\begin{aligned} \|(\lambda I + L)^{-1} f\|_X &\le \frac{1}{\alpha}\|f\|_{X^*} \\ \|(\lambda I + L)^{-1} f\|_{X^*} &\le \frac{1}{|\lambda|}\left(\frac{M}{\alpha} + 1\right)\|f\|_{X^*}. \end{aligned} \tag{2.9}$$

Step 3. By Step 1. and Step 2. and since $0 \in \rho(-L)$ it follows that there exists $0 < \delta < \pi/2$ and there exists $C > 0$ such that

$$\rho(-L) \supset \Sigma_\delta := \{\lambda \in \mathbb{C} \ : \ |\arg\lambda| < \pi/2 + \delta\} \cup \{0\}$$

and

$$\|(\lambda I + L)^{-1}\|_{\mathcal{L}(X^*,X^*)} \leq \frac{C}{|\lambda|} \quad \forall\, \lambda \in \Sigma_\delta,\ \lambda \neq 0.$$

Indeed, by (2.9), $\|(\lambda I + L)^{-1}\|_{\mathcal{L}(X^*,X^*)} \leq C/|\mathrm{Im}\,\lambda|$ for all $\mathrm{Re}\,\lambda > 0$. For $\bar\lambda = \xi + i\zeta$ with $\xi > 0$, the Taylor expansion for $(\lambda I + L)^{-1}$ around $\bar\lambda$

$$(\lambda I + L)^{-1} = \sum_{k=0}^{\infty} \left((\bar\lambda I + L)^{-1}\right)^{k+1} (\bar\lambda - \lambda)^k$$

converges in $\mathcal{L}(X^*, X^*)$ for $\|(\bar\lambda I+L)^{-1}\|_{\mathcal{L}(X^*,X^*)}|\bar\lambda-\lambda| \leq q < 1$. Choosing $\mathrm{Im}\,\lambda = \zeta$ we see that the series converges uniformly in $\mathcal{L}(X^*, X^*)$ for $|\xi - \mathrm{Re}\,\lambda| \leq q|\zeta|/C$. Since $\xi > 0$ and $q \in (0, 1)$ are arbitrary, $\rho(-L)$ contains all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \leq 0$ and $|\mathrm{Re}\,\lambda|/|\mathrm{Im}\,\lambda| < 1/C$ and in particular $\rho(-L) \supset \{\lambda \in \mathbb{C} \ : \ |\arg\lambda| < \pi/2+\delta\}$ with $\delta = q\arctan(1/C)$, $0 < q < 1$, and in this region we also have $\|(\lambda I + L)^{-1}\|_{\mathcal{L}(X^*,X^*)} \leq C/|\lambda|$.

By Theorem 1.7.7 and Theorem 2.5.2 in [25] it follows that $-L$ is the infinitesimal generator of a uniformly bounded C^0-semigroup in X^*. Moreover, $T^{-\nu}(t)$ can be extended to an analytic semigroup in the sector $\Delta_\delta = \{z \in \mathbb{C} \ : \ |\arg z| < \delta\}$ and $\|T^{-\nu}(t)\|_{\mathcal{L}(X^*,X^*)}$ is uniformly bounded in every closed subsector $\Delta_{\delta'}$, $\delta' < \delta$, of Δ_δ.

□

Remark 2.4 The elements of $(H^1_{-\nu}(\mathbb{R}))^*$ can be characterized as follows. Consider $\phi \in H^1_{-\nu}(\mathbb{R})$ arbitrary, fixed and let $\phi_n \in H^1_{-\nu}(\mathbb{R})$, $n \in \mathbb{N}$ be given by $\phi_n(y) := \phi(y - n)$. Then for each $f \in (H^1_{-\nu}(\mathbb{R}))^*$ it holds that

$$|\langle f, \phi_n\rangle_{(H^1_{-\nu}(\mathbb{R}))^* \times H^1_{-\nu}(\mathbb{R})}| \leq Ce^{-\nu n}\|\phi\|_{H^1_{-\nu}(\mathbb{R})}.$$

We apply the abstract results to (2.4) and the solution w of (2.4) can be represented as

$$w(\tau, \cdot) = T^{-\nu}(\tau)h.$$

2.3.1 The case $r = 0$

We consider here the case when $r = 0$, i.e., w solves

$$\begin{aligned} \frac{\partial w}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial x^2} + \frac{\sigma^2}{2}\frac{\partial w}{\partial x} &= 0, \quad (\tau, x) \in (0, T) \times \mathbb{R} \\ w(0, x) &= h(x) := (e^x - K)_+. \end{aligned} \tag{2.10}$$

We have seen that given $h \in H^1_{-\zeta}(\mathbb{R})$, (2.4) admits a unique solution $w \in L^2(0, T; H^1_{-\zeta}(\mathbb{R})) \cap H^1(0, T; \left(H^1_{-\zeta}(\mathbb{R})\right)^*)$. Here we analyse the solution behaviour at infinity in more detail, and we prove that $\tilde w := w - h$ decays exponentially at infinity. To this end, we show that $\tilde w$ solves the following parabolic equation

$$\begin{aligned} \frac{\partial \tilde w}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 \tilde w}{\partial x^2} + \frac{\sigma^2}{2}\frac{\partial \tilde w}{\partial x} &= f \\ \tilde w(0, x) &= 0, \end{aligned} \tag{2.11}$$

with inhomogeneous right hand side given by $f := \frac{\sigma^2}{2} K \delta_{\ln(K)}$. Indeed, for $\mu > 1$, $Lh \in (H^1_{-\mu}(\mathbb{R}))^*$ is given by

$$\langle Lh, \varphi \rangle_{(H^1_{-\mu}(\mathbb{R}))^* \times H^1_{-\mu}(\mathbb{R})} = a^{-\mu}(h, \varphi) \quad \forall\, \varphi \in H^1_{-\mu}(\mathbb{R})$$

and there holds

$$\frac{d}{d\tau}(\tilde{w}(\tau, \cdot), \varphi)_{L^2_{-\mu}(\mathbb{R})} + a^{-\mu}(\tilde{w}(\tau, \cdot), \varphi) = -a^{-\mu}(h, \varphi) \quad \forall\, \varphi \in H^1_{-\mu}(\mathbb{R}). \tag{2.12}$$

By the definition of $a^{-\mu}(\cdot, \cdot)$ we obtain that the right hand side in (2.12) is given by

$$\begin{aligned} &-\frac{\sigma^2}{2} \int_{\ln(K)}^{\infty} e^x \varphi'(x) e^{-2\mu|x|}\, dx + \int_{\ln(K)}^{\infty} \left(\mu\sigma^2 \mathrm{sign}(x) - \frac{\sigma^2}{2}\right) e^x \varphi(x) e^{-2\mu|x|}\, dx \\ &= \frac{\sigma^2}{2} K e^{-2\mu|\ln(K)|} \varphi(\ln(K)). \end{aligned}$$

It follows therefore that $\tilde{w}$ solves (2.11). We observe that the right hand side in (2.11) $f \in \left(H^1_\nu(\mathbb{R})\right)^*$ for all $\nu > 0$, since for arbitrary $v \in H^1_\nu(\mathbb{R})$

$$|\langle f, v \rangle_{(H^1_\nu(\mathbb{R}))^* \times H^1_\nu(\mathbb{R})}| = \left| \frac{\sigma^2}{2} K v(\ln(K)) e^{2\nu|\ln(K)|} \right| \leq C(\nu, \sigma, K) |v(\ln(K))| \leq C(\nu, \sigma, K) \|v\|_{H^1_\nu(\mathbb{R})}.$$

Multiplying (2.11) by the test function $v(x) e^{2\nu|x|}$, with $v \in C^\infty_0(\mathbb{R})$ we obtain

$$\begin{aligned} \frac{d}{d\tau}(\tilde{w}(\tau, \cdot), v)_{L^2_\nu(\mathbb{R}) \times L^2_\nu(\mathbb{R})} + a^\nu(\tilde{w}, v) &= \langle f, v \rangle_{(H^1_\nu(\mathbb{R}))^* \times H^1_\nu(\mathbb{R})} \quad \forall\, v \in C^\infty_0(\mathbb{R}) \\ \tilde{w}(0, x) &= 0. \end{aligned} \tag{2.13}$$

By Proposition 2.1, there exists a unique $\tilde{w} \in L^2(0, T; H^1_\nu(\mathbb{R})) \cap H^1(0, T; (H^1_\nu(\mathbb{R}))^*)$ solution to (2.13).

2.3.2 The case $r \neq 0$

We now come back to the case when $r \neq 0$. The transformation

$$w(\tau, x) = e^{-r\tau} \tilde{w}(\tau, x + r\tau)$$

reduces the original problem for w to a Black-Scholes equation for $\tilde{w}$ with $r = 0$:

$$\frac{\partial \tilde{w}}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 \tilde{w}}{\partial x^2} + \frac{\sigma^2}{2} \frac{\partial \tilde{w}}{\partial x} = 0, \quad \tilde{w}(0, x) = h(x)$$

and we can apply the results of Section 2.3.1 to $\tilde{w}$.

3 Pricing European Vanilla options on Lévy driven assets

3.1 Lévy processes

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t < \infty}, \mathbf{P})$ be a filtered probability space satisfying the usual hypothesis. An adapted process $X = (X_t)_{0 \leq t < \infty}$ with $X_0 = 0$ a.s. is a Lévy process iff

1. X has increments independent of the past, i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$

2. X has stationary increments, i.e. $X_t - X_s$ has the same distribution as X_{t-s}, $0 \leq s < t < \infty$

3. X_t is continuous in probability.

The Lévy-Khintchine formula describes explicitly a Lévy process in terms of its Fourier transform $E_{\mathbb{Q}}[e^{-iuX_t}]$ under a chosen equivalent martingale measure $\mathbb{Q}$:

$$E_{\mathbb{Q}}[e^{-iuX_t}] = e^{-t\psi(u)} \tag{3.1}$$

for some function ψ called the Lévy exponent of X. The Lévy-Khintchine formula says that

$$\begin{aligned}\psi(u) = \frac{\sigma^2}{2}u^2 + i\alpha u \\ + \int_{|x|<1}(1 - e^{-iux} - iux)\nu_{\mathbb{Q}}(dx) + \int_{|x|\geq 1}(1 - e^{-iux})\nu_{\mathbb{Q}}(dx)\end{aligned} \tag{3.2}$$

for $\sigma, \alpha \in \mathbb{R}$ and for a measure $\nu_{\mathbb{Q}}$ on $\mathbb{R}\backslash\{0\}$ satisfying

$$\int \min(1, x^2)\nu_{\mathbb{Q}}(dx) < \infty. \tag{3.3}$$

The characteristic exponent ψ turns out to be the symbol of the pseudo-differential operator $L_X^{\mathbb{Q}}$ which is the infinitesimal generator of the transition semi-group of X_t under the equivalent martingale measure $\mathbb{Q}$. We assume here that the equivalent martingale measure $\mathbb{Q}$ has been chosen by some procedure, we refer to [13, 14, 17, 11] and the references therein for various results in this direction.

3.2 Examples of Lévy processes

In this section we give some examples of Lévy processes.

3.2.1 Variance Gamma Process

The variance gamma process [23, 24] is a Brownian motion with drift in which the calendaristic time has been changed to a 'business' time modeled by a gamma process $\gamma(t;\nu)$ with mean rate unity and variance rate ν

$$X_{VG}(t;\sigma,\nu,\theta) = \theta\gamma(t;\nu) + \sigma W_{\gamma(t;\nu)}. \tag{3.4}$$

From the density of the gamma process

$$f_{\gamma(t;\nu)}(x) = \frac{x^{t/\nu-1}e^{-x/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)}$$

one obtains that the characteristic function of the gamma process is given by

$$\phi_{\gamma(t;\nu)}(u) = E[e^{iu\gamma(t;\nu)}] = \left(\frac{1}{1 - i\nu u}\right)^{t/\nu}$$

and the characteristic function of the variance gamma process has the form

$$\phi_{X_{VG}(t;\sigma,\nu,\theta)}(u) = E[e^{iuX_{VG}(t;\sigma,\nu,\theta)}] = \left(\frac{1}{1 - i\theta\nu u + \sigma^2\nu u^2/2}\right)^{t/\nu}. \tag{3.5}$$

This expression of the characteristic function $\phi_{X_{VG}(t;\sigma,\nu,\theta)}$ together with the fact that

$$\frac{1}{1 - i\theta\nu u + \sigma^2\nu u^2/2} = \frac{1}{1 - i\eta_p u} \cdot \frac{1}{1 + \eta_n u}$$

with

$$\eta_p = \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} + \frac{\theta\nu}{2}, \qquad \eta_n = \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} - \frac{\theta\nu}{2} \tag{3.6}$$

lead to another interpretation of the variance gamma process, namely as the difference of two independent gamma processes

$$X_{VG}(t;\sigma,\nu,\theta) \overset{\text{law}}{=} \gamma_p(t;\eta_p/\nu,\eta_p^2/\nu) - \gamma_n(t;\eta_n/\nu,\eta_n^2/\nu). \tag{3.7}$$

The representation (3.7) of the variance gamma process and the representation of the Lévy density for the gamma process lead to the following form of the Lévy density for the VG process

$$k_{VG}(x) = \begin{cases} \dfrac{1}{\nu} \cdot \dfrac{e^{-\frac{1}{\eta_n}|x|}}{|x|} & \text{if} \quad x < 0 \\ \dfrac{1}{\nu} \cdot \dfrac{e^{-\frac{1}{\eta_p}|x|}}{|x|} & \text{if} \quad x > 0. \end{cases} \tag{3.8}$$

3.2.2 CGMY process

The CGMY process [9] generalizes the VG process by adding a new parameter in the Lévy density that allows the resulting Lévy process to have both finite or infinite activity and finite or infinite variation. Specifically, the Lévy density of the CGMY process is given by

$$k_{CGMY}(x) = \begin{cases} C\dfrac{e^{-G|x|}}{|x|^{1+Y}} & \text{if} \quad x < 0 \\ C\dfrac{e^{-M|x|}}{|x|^{1+Y}} & \text{if} \quad x > 0, \end{cases} \tag{3.9}$$

where $C > 0$, $G, M \geq 0$ and $Y < 2$. The case $Y = 0$ is the special case of the variance gamma process. The CGMY Lévy process with the Lévy density given by (3.9) has the characteristic function

$$\phi_{X_{CGMY}(t;C,G,M,Y)}(u) = e^{tC\Gamma(-Y)\{(M-iu)^Y - M^Y + (G+iu)^Y - G^Y\}}. \tag{3.10}$$

3.2.3 Regular Lévy processes of exponential type

The characteristic exponents of the regular Lévy processes of exponential type [7, 8] satisfy (both under historical and equivalent martingale measures from a wide class) the following conditions: For some constants $c > 0$, $\nu \in (0, 2]$, $\nu' < \nu$, $\alpha \in \mathbb{R}$, $\lambda_- < 0 \leq \lambda_+$ and $C > 0$

$$\psi(u) = i\alpha u + \phi(u), \tag{3.11}$$

where ϕ admits an analytic continuation from $\mathbb{R}$ into the strip $\operatorname{Im} u \in (\lambda_-, \lambda_+)$. The analytic continuation of ϕ into the strip is continuous up to the boundary and satisfies the following estimates

$$|\phi(u) - c|u|^\nu| \leq C\langle u\rangle^{\nu'} \quad \forall \operatorname{Im} u \in [\lambda_-, \lambda_+]$$

where $\langle u\rangle = (1+|u|^2)^{1/2}$. Moreover, it is assumed that for any $[\lambda'_-, \lambda'_+] \subset (\lambda^-, \lambda^+)$

$$|\phi'(u)| \leq C\langle u\rangle^{\nu-1} \quad \forall \operatorname{Im} u \in [\lambda'_-, \lambda'_+].$$

3.3 Partial integro-differential equation (PIDE)

We assume that the asset price process is given by the following geometric law:

$$S_t = S_0 e^{(r+c-\sigma^2/2)t+X_t} \tag{3.12}$$

where X_t is a Lévy process of the form $X_t = \sigma B_t + Y_t$, with B_t denoting the Brownian motion and Y_t being a quadratic pure jump Lévy process independent of B_t. The correction parameter c in (3.12) ensures that the mean rate of return on the asset is risk-neutrally r, i.e. that $e^{-ct} = E_{\mathbb{Q}}[e^{Y_t}]$.

Let $\mu(dx, dt)$ denote the integer valued random measure (the *jump measure*) that counts the number of jumps of Y_t in space-time.
By Ito's formula, see e.g. Theorem 4.57 in [18], S_t solves the following stochastic differential equation

$$dS_t = S_{t_-} dX_t + S_{t_-} \int_{\mathbb{R}} (e^y - 1 - y)\mu(dy, dt) + (r+c)dt. \tag{3.13}$$

By stationarity of Lévy processes, the compensator of the measure $\mu(dx, dt)$ has the form $\nu_{\mathbb{Q}}(dx) \times dt$, with dt being the Lebesgue measure. In the following we will assume that the Lévy measure $\nu_{\mathbb{Q}}(dx)$ has a density $k_{\mathbb{Q}}$, i.e., $\nu_{\mathbb{Q}}(dx) = k_{\mathbb{Q}}(x)dx$ and we will drop the subscript $\mathbb{Q}$.

Remark 3.1 By (3.12), (3.1)–(3.2) and by $E_{\mathbb{Q}}[S_t] < \infty$ we obtain that $E_{\mathbb{Q}}[e^{X_t}] = e^{-t\psi(i)} < \infty$, with ψ being the Lévy exponent in (3.2). As a consequence, the Lévy density k has to satisfy both the integrability condition (3.3) and $\int_{|x|\geq 1} e^x k(x)dx < \infty$. For the case of the CGMY-model (3.9) these integrability conditions for the Lévy density imply that $Y < 2$ and $M > 1$.

Let $f(t, S_t)$ denote the price at time t of a contingent claim on the asset S_t in (3.12). We consider here an European call option, i.e. $f(T, S_T) = g(S_T) := (S_T - K)_+$, with strike price K and maturity T. The price $f(t, S_t)$ can be calculated for all dates $t < T$ by taking conditional expectations. Assuming that the savings account process is given by $S_t^0 = e^{rt}$, the process $e^{-rt}S_t$ is a martingale under $\mathbb{Q}$, since $\mathbb{Q}$ is assumed to be the risk-neutral measure. The same holds true for the value process $f(t, S_t)$ of the option, therefore

$$f(t, S_t) = E_{\mathbb{Q}}[e^{r(t-T)} g(S_T)|\mathcal{F}_t].$$

The key to fast deterministic valuation of $f(t, S_t)$ is the following result which characterizes $f(t, S_t)$ as solution of a deterministic partial integro-differential equation (PIDE).
Unless explicitly stated otherwise, we assume in the following that the price process has a non-zero diffusion component, i.e. $\sigma \neq 0$. Furthermore, we change to logarithmic price $x = \ln(S) \in \mathbb{R}$ and time to maturity $\tau = T - t$.

Theorem 3.2 *Assume that $u(\tau, x)$ is a sufficiently regular solution of the following parabolic partial integro-differential equation*

$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + (\frac{\sigma^2}{2} - r + c_{\exp})\frac{\partial u}{\partial x} + A[u] + ru = 0 \quad \text{in } (0,T)\times\mathbb{R} \tag{3.14}$$

where A denotes the following integro-differential operator

$$A[\varphi](x) := -\int_{\mathbb{R}} \{\varphi(x+y) - \varphi(x) - y\varphi'(x)\chi_{\{|y|\le 1\}}(y)\}k(y)\, dy$$

and $c_{\exp} \in \mathbb{R}$ is given by

$$c_{\exp} := -e^{-x}A[e^{\cdot}](x) = \int_{\mathbb{R}} \{e^y - 1 - y\chi_{\{|y|\le 1\}}(y)\}k(y)\, dy. \tag{3.15}$$

together with the initial condition

$$u|_{\tau=0} = h := (e^x - K)_+. \tag{3.16}$$

Then $f(t,S) := u(T-t, \ln(S))$ satisfies

$$f(t, S_t) = E_{\mathbb{Q}}[e^{r(t-T)}g(S_T)|\mathcal{F}_t]. \tag{3.17}$$

Conversely, if $f(t,S)$ in (3.17) is sufficiently regular, then $u(\tau,x) := f(T-\tau, e^x)$ is solution of (3.14), (3.16).

Proof. The process $f(t,S)$ can be written as $f(t, S_0 e^{(r+c-\sigma^2/2)t+X}) = g(t,X)$. By our assumption on the regularity, we may apply Ito's Lemma for semi-martingales (see e.g., Theorem 4.57 in [18]) to $g(t,X)$ which leads to

$$\begin{aligned} g(T,X_T) = g(0,X_0) &+ \int_0^T \frac{\partial g}{\partial t}(t, X_{t_-})dt + \frac{\sigma^2}{2}\int_0^T \frac{\partial^2 g}{\partial X^2}(t, X_{t_-})dt \\ &+ \int_0^T \int_{\mathbb{R}} \left\{ g(t, X_{t_-} + y) - g(t, X_{t_-}) - \frac{\partial g}{\partial X}(t, X_{t_-})y \right\}\mu(dy, dt), \end{aligned}$$

or equivalently

$$\begin{aligned} f(T,S_T) = f(0,S_0) &+ \int_0^T \left\{ \frac{\partial f}{\partial t}(t, S_{t_-}) + S_{t_-}(r + c - \sigma^2/2)\frac{\partial f}{\partial S}(t, S_{t_-}) \right\}dt \\ &+ \int_0^T S_{t_-}\frac{\partial f}{\partial S}(t, S_{t_-})dX_t + \int_0^T \frac{\sigma^2}{2}\left\{ S_{t_-}^2 \frac{\partial^2 f}{\partial S^2}(t, S_{t_-}) + S_{t_-}\frac{\partial f}{\partial S}(t, S_{t_-}) \right\}dt \\ &+ \int_0^T \int_{\mathbb{R}} \left\{ f(t, S_{t_-}e^y) - f(t, S_{t_-}) - S_{t_-}\frac{\partial f}{\partial S}(t, S_{t_-})y \right\}\mu(dy, dt). \end{aligned} \tag{3.18}$$

Using (3.13) in the previous equation and replacing $S_{t_-}\, dX_t$ by $dS_t - S_{t_-}\int_{\mathbb{R}}(e^y - 1 - y)\mu(dy,dt) - (r+c)dt$ we obtain

$$\begin{aligned} f(T,S_T) = f(0,S_0) &+ \int_0^T \left\{ \frac{\partial f}{\partial t}(t, S_{t_-}) + \frac{\sigma^2}{2}S_{t_-}^2\frac{\partial^2 f}{\partial S^2}(t, S_{t_-}) \right\}dt + \int_0^T \frac{\partial f}{\partial S}(t, S_{t_-})dS_t \\ &+ \int_0^T \int_{\mathbb{R}} \left\{ f(t, S_{t_-}e^y) - f(t, S_{t_-}) - S_{t_-}\frac{\partial f}{\partial S}(t, S_{t_-})(e^y - 1) \right\}\mu(dy, dt). \end{aligned}$$

The previous results applied to $e^{r(T-t)}f(t,S_t)$ lead to

$$\begin{aligned}f(T,S_T) = e^{rT}f(0,S_0) &+ \int_0^T e^{r(T-t)}\frac{\partial f}{\partial S}(t,S_{t_-})dS_t \\ &+ \int_0^T e^{r(T-t)}[\frac{\partial f}{\partial t}(t,S_{t_-}) - rf(t,S_{t_-}) + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}(t,S_{t_-})S_{t_-}^2]dt \\ &+ \int_0^T e^{r(T-t)}\int_{\mathbb{R}}[f(t,S_{t_-}e^x) - f(t,S_{t_-}) - \frac{\partial f}{\partial S}(t,S_{t_-})S_{t_-}(e^x-1)]\mu(dx,dt).\end{aligned}$$

Equivalently,

$$\begin{aligned}f(T,S_T) = e^{rT}f(0,S_0) &+ \int_0^T e^{r(T-t)}\frac{\partial f}{\partial S}(t,S_{t_-})[dS_t - rS_{t_-}dt] \\ &+ \int_0^T e^{r(T-t)}\int_{\mathbb{R}}[f(t,S_{t_-}e^x) - f(t,S_{t_-}) - \frac{\partial f}{\partial S}(t,S_{t_-})S_{t_-}(e^x-1)]\mu(dx,dt) \\ &+ \int_0^T e^{r(T-t)}[\frac{\partial f}{\partial t}(t,S_{t_-}) - rf(t,S_{t_-}) + rS_{t_-}\frac{\partial f}{\partial S}(t,S_{t_-}) + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}(t,S_{t_-})S_{t_-}^2]dt\end{aligned}$$

Adding and subtracting

$$\int_0^T e^{r(T-t)}\int_{\mathbb{R}}[f(t,S_{t_-}e^x) - f(t,S_{t_-}) - \frac{\partial f}{\partial S}(t,S_{t_-})S_{t_-}(e^x-1)]\nu_{\mathbb{Q}}(dx)dt$$

with $\nu_{\mathbb{Q}}(dx)$ being the Lévy measure we obtain

$$\begin{aligned}f(T,S_T) = e^{rT}f(0,S_0) &+ \int_0^T e^{r(T-t)}\frac{\partial f}{\partial S}(t,S_{t_-})[dS_t - rS_{t_-}dt] \\ &+ \int_0^T e^{r(T-t)}\int_{\mathbb{R}}[f(t,S_{t_-}e^x) - f(t,S_{t_-}) - \frac{\partial f}{\partial S}(t,S_{t_-})S_{t_-}(e^x-1)](\mu-\nu_{\mathbb{Q}})(dx,dt) \\ &+ \int_0^T e^{r(T-t)}\int_{\mathbb{R}}[f(t,S_{t_-}e^x) - f(t,S_{t_-}) - \frac{\partial f}{\partial S}(t,S_{t_-})S_{t_-}(e^x-1)]\nu_{\mathbb{Q}}(dx)dt \\ &+ \int_0^T e^{r(T-t)}[\frac{\partial f}{\partial t}(t,S_{t_-}) - rf(t,S_{t_-}) + rS_{t_-}\frac{\partial f}{\partial S}(t,S_{t_-}) + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}(t,S_{t_-})S_{t_-}^2]dt.\end{aligned}$$

Taking the expectation under $\mathbb{Q}$ we obtain

$$\begin{aligned}E_{\mathbb{Q}}\Bigg[\int_0^T e^{r(T-t)}\Bigg\{&\int_{\mathbb{R}}[f(t,S_{t_-}e^x) - f(t,S_{t_-}) - \frac{\partial f}{\partial S}(t,S_{t_-})S_{t_-}(e^x-1)]\nu_{\mathbb{Q}}(dx) \\ &+ \frac{\partial f}{\partial t}(t,S_{t_-}) - rf(t,S_{t_-}) + rS_{t_-}\frac{\partial f}{\partial S}(t,S_{t_-}) + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}(t,S_{t_-})S_{t_-}^2\Bigg\}dt\Bigg] = 0\end{aligned}$$

This implies that f has to satisfy the following Partial Integro-Differential Equation (PIDE):

$$\begin{aligned}&\frac{\partial f}{\partial t}(t,S_{t_-}) + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}(t,S_{t_-})S_{t_-}^2 + rS_{t_-}\frac{\partial f}{\partial S}(t,S_{t_-}) \\ &+ \int_{\mathbb{R}}[f(t,S_{t_-}e^x) - f(t,S_{t_-}) - \frac{\partial f}{\partial S}(t,S_{t_-})S_{t_-}(e^x-1)]\nu_{\mathbb{Q}}(dx) - rf(t,S_{t_-}) = 0. \qquad (3.19)\end{aligned}$$

Recall now that we assumed that the Lévy measure $\nu_{\mathbb{Q}}(dx)$ has a density $k(x)$ and perform the change the variables $x = \ln(S)$ and $\tau = T - t$ to obtain (3.14).

□

3.4 Variational setting of PIDE

Our pricing methodology is based on the numerical solution of the PIDE (3.14). Numerical solution of PIDEs for European vanillas by characteristic functions and FFT techniques has been advocated in [10]. Our solution algorithm aims at American put and Barrier options. It will be based on a variational formulation of the PIDE which we now give. Since we work in logarithmic asset price $x = \ln(S)$, the pay-off function grows exponentially at ∞. Thus, as in the Black-Scholes setting, the variational formulation of the PIDE must be based on weighted Sobolev spaces allowing exponential growth of the solution at ∞.

In the following, we restrict ourselves to the case that the risk-neutral mean rate of return of the asset is $r = 0$, since the change of variables

$$u(\tau, x) = e^{-r\tau}\tilde{u}(\tau, x + r\tau)$$

reduces the problem for $r \neq 0$ to the case when $r = 0$.

3.4.1 Weighted spaces

Let $\eta \in L^1_{\text{loc}}(\mathbb{R})$, $\eta' \in L^\infty(\mathbb{R})$. We denote by $H^1_\eta(\mathbb{R})$ the weighted Sobolev space given by

$$H^1_\eta(\mathbb{R}) := \{\varphi \in L^1_{\text{loc}}(\mathbb{R}) \ : \ e^\eta\varphi, e^\eta\varphi' \in L^2(\mathbb{R})\}.$$

We observe that $h \in H^1_{-\zeta}(\mathbb{R})$ for all ζ of the form

$$\zeta(x) = \begin{cases} \mu_1|x| & \text{if } x < 0 \\ \mu_2|x| & \text{if } x > 0 \end{cases} \tag{3.20}$$

for all $\mu_1 > 0$ and $\mu_2 > 1$. We will denote by $\mathcal{A}$ the spatial operator in (3.14) given by

$$\mathcal{A}[\varphi](x) := -\frac{\sigma^2}{2}\left(\frac{d^2\varphi}{dx^2}(x) - \frac{d\varphi}{dx}(x)\right) + A[\varphi](x) + c_{\text{exp}}\frac{d\varphi}{dx}. \tag{3.21}$$

For $\varphi, \psi \in C^\infty_0(\mathbb{R})$ we associate with operator $\mathcal{A}$ the following bilinear form

$$a^\eta(\varphi, \psi) := \int_{\mathbb{R}} \mathcal{A}[\varphi](x)\psi(x)e^{2\eta(x)}dx. \tag{3.22}$$

For a certain class of weighting functions $\eta \in L^1_{\text{loc}}(\mathbb{R})$, $\eta' \in L^\infty(\mathbb{R})$, the bilinear form $a^\eta(\cdot,\cdot)$ can be extended continuously to $H^1_\eta(\mathbb{R}) \times H^1_\eta(\mathbb{R})$. Moreover, under certain conditions on η this bilinear form is, up to a L^2_η-scalar product, coercive on $H^1_\eta(\mathbb{R}) \times H^1_\eta(\mathbb{R})$ in the sense that the following analogue of Proposition 2.1 holds.

Theorem 3.3 *Let $\eta \in L^1_{\text{loc}}(\mathbb{R})$ such that $\eta' \in L^\infty(\mathbb{R})$.*

1. *Assume that*

$$\eta(x + \theta y) - \eta(x) \leq \eta(y) \quad \forall\, x, y \in \mathbb{R} \quad \forall\, \theta \in [0, 1] \tag{3.23}$$

and

$$C(\eta) := \int_{\mathbb{R}} e^{\eta(y)}|y|\chi_{\{|y|\geq 1\}}(y)k(y)\, dy < +\infty \tag{3.24}$$

hold. Then, there exist α_η, $\beta_\eta > 0$ and $C_\eta > 0$ such that

$$|a^{-\eta}(\varphi, \psi)| \leq C_\eta\|\varphi\|_{H^1_{-\eta}(\mathbb{R})}\|\psi\|_{H^1_{-\eta}(\mathbb{R})} \quad \forall\, \varphi, \psi \in H^1_{-\eta}(\mathbb{R})$$
$$a^{-\eta}(\varphi, \varphi) \geq \alpha_\eta\|\varphi\|^2_{H^1_{-\eta}(\mathbb{R})} - \beta_\eta\|\varphi\|^2_{L^2_{-\eta}(\mathbb{R})} \quad \forall\, \varphi \in H^1_{-\eta}(\mathbb{R}).$$

2. *Let η be such that*

$$-\eta(x+\theta y)+\eta(x)\le\eta(-y)\quad\forall\, x,y\in\mathbb{R}\quad\forall\,\theta\in[0,1] \tag{3.25}$$

and

$$\tilde{C}(-\eta):=\int_{\mathbb{R}} e^{\eta(-y)}|y|\chi_{\{|y|\ge 1\}}(y)k(y)dy<+\infty \tag{3.26}$$

hold. Then, there exist α'_η, $\beta'_\eta>0$ and $C'_\eta>0$ such that

$$|a^\eta(\varphi,\psi)|\le C'_\eta\|\varphi\|_{H^1_\eta(\mathbb{R})}\|\psi\|_{H^1_\eta(\mathbb{R})}\quad\forall\,\varphi,\psi\in H^1_\eta(\mathbb{R})$$
$$a^\eta(\varphi,\varphi)\ge\alpha'_\eta\|\varphi\|^2_{H^1_\eta(\mathbb{R})}-\beta'_\eta\|\varphi\|^2_{L^2_\eta(\mathbb{R})}\quad\forall\,\varphi\in H^1_\eta(\mathbb{R}).$$

The proof of this theorem is given in Appendix A.

3.4.2 Reduction to homogeneous initial condition

We come back to problem (3.14)–(3.16). Since $h\in H^1_{-\zeta}(\mathbb{R})$ for all ζ as in (3.20), we can cast (3.14)–(3.16) in the following weak form:

Find $u\in L^2((0,T);H^1_{-\zeta}(\mathbb{R}))\cap H^1((0,T);(H^1_{-\zeta}(\mathbb{R}))^*)$ such that

$$\frac{d}{d\tau}(u(\tau,\cdot),v)_{L^2_{-\zeta}(\mathbb{R})}+a^{-\zeta}(u(\tau,\cdot),v)=0\quad\forall\, v\in H^1_{-\zeta}(\mathbb{R})$$
$$u(0,\cdot)=h. \tag{3.27}$$

By Theorem 3.3, Item 1., (3.27) admits a unique weak solution $u\in L^2((0,T);H^1_{-\zeta}(\mathbb{R}))\cap H^1((0,T);(H^1_{-\zeta}(\mathbb{R}))^*)$.

For numerical computations it will be more convenient to work on a bounded domain with homogeneous initial and artificial boundary conditions. We realize this by removing the inhomogeneous initial condition by a particular solution and by analyzing the image of the pay-off function under the operator $\mathcal{A}$.

To this end, we write the space operator $\mathcal{A}$ as $\mathcal{A}=-\frac{\sigma^2}{2}\frac{d^2}{dx^2}+\frac{\sigma^2}{2}\frac{d}{dx}+\hat{A}$, with

$$\hat{A}[\phi](x):=-\int_{\mathbb{R}}[\phi(x+z)-\phi(x)-z\phi'(x)\chi_{\{|z|\le 1\}}]k(z)dz+c_{\exp}\phi'(x), \tag{3.28}$$

with density function $k(z)$ satisfying the integrability conditions (3.3) and $\int_{|z|\ge 1}e^z k(z)dz<\infty$, see also Remark 3.1.

The constant $c_{\exp}$ is chosen by (3.15) such that $\hat{A}[e^x]=0$. Note that $\hat{A}$ corresponds exactly to the integral operator in (3.19) in logarithmic price $x=\ln(S)$. In the following we will assume $k(z)$ to be the Lévy density of the CGMY process in (3.9), but we emphasize that the result in Theorem 3.4 below is not restricted to this case. We assume thus

$$k(z)=\begin{cases}\dfrac{e^{Gz}}{|z|^{1+Y}}, & z<0\\[2ex] \dfrac{e^{-Mz}}{|z|^{1+Y}}, & z>0\end{cases}$$

with $Y<2$ due to (3.3) and $G>0$, $M>1$ due to $\int_{|z|\ge 1}e^z k(z)dz<\infty$.

The operator $\hat{A}$ in (3.28) satisfies a strong pseudo-local property: singular support is preserved and we have an exponential decay at ∞.

Theorem 3.4 *Let the pay-off function h be given by $h(x) = (e^x - K)_+$ and let $\psi := -\hat{A}[h]$. Then $\psi \in C^\infty(\mathbb{R}\backslash\{\ln(K)\}) \cap L^1_{\mathrm{loc}}(\mathbb{R})$ and ψ decays exponentially at $\pm\infty$: there exist $C_1, C_2 > 0$ such that $0 \le \psi(x) \le C_1 e^{-Gx}$ for $x > 0$ sufficiently large and $0 \le \psi(x) \le C_2 e^{Mx}$ for $x < 0$ and $|x|$ sufficiently large. Hence, $\psi \in (H^1_\eta(\mathbb{R}))^*$ for all $\eta \ge 0$ satisfying (3.25) and (3.26), in particular, for $\eta = 0$.*

Proof. Let $x > \ln(K)$. Then there holds

$$\begin{aligned}\psi(x) &= \int_{\mathbb{R}} \left[(e^{x+z} - K)_+ - (e^x - K)_+ - z((e^x - K)_+)'\chi_{\{|z|\le 1\}})\right] k(z)dz - c_{\exp}((e^x - K)_+)' \\ &= \int_{-\infty}^{\ln(K)-x} \left[0 - (e^x - K) - ze^x\chi_{\{|z|\le 1\}}\right] k(z)dz \\ &+ \int_{\ln(K)-x}^{\infty} \left[e^{x+z} - e^x - ze^x\chi_{\{|z|\le 1\}}\right] k(z)dz - c_{\exp}e^x.\end{aligned}$$

By the choice of $c_{\exp}$ in (3.15) we obtain that

$$\begin{aligned}\psi(x) &= \int_{-\infty}^{\ln(K)-x} \left[K - e^x - ze^x\chi_{\{|z|\le 1\}}\right] k(z)dz - \int_{-\infty}^{\ln(K)-x} \left[e^{x+z} - e^x - ze^x\chi_{\{|z|\le 1\}}\right] k(z)dz \\ &= \int_{-\infty}^{\ln(K)-x} (K - e^{x+z})k(z)dz, \quad \ln(K) - x < 0.\end{aligned}$$

Analogously, for $x < \ln(K)$ we obtain that

$$\psi(x) = \int_{\ln(K)-x}^{\infty} (e^{x+z} - K)k(z)dz, \quad \ln(K) - x > 0.$$

With k as in (3.9) and

$$\psi(x) = \begin{cases} \displaystyle\int_{-\infty}^{\ln(K)-x} (K - e^{x+z})|z|^{-1-Y}e^{-G|z|}dz, & x > \ln(K) \\ \displaystyle\int_{\ln(K)-x}^{\infty} (e^{x+z} - K)|z|^{-1-Y}e^{-M|z|}dz, & x < \ln(K), \end{cases}$$

we obtain that $\psi \in C^\infty(\mathbb{R}\backslash\{\ln(K)\}$, i.e. $\operatorname{sing\,supp}\psi = \{\ln(K)\}$.

We claim that $|\ln(K) - x|^p\psi(x) \in L^\infty(\mathbb{R})$ for $p = Y - 1$. Indeed,

$$\begin{aligned}\lim_{x\downarrow\ln(K)} |\ln(K) - x|^p\psi(x) &= \lim_{\varepsilon\downarrow 0} \varepsilon^p \int_{-\infty}^{-\varepsilon} (K - Ke^\varepsilon e^z)|z|^{-Y-1}e^{-G|z|}dz \\ &= \lim_{\varepsilon\downarrow 0} K\varepsilon^p \int_{\varepsilon}^{\infty} (1 - e^\varepsilon e^{-z})z^{-Y-1}e^{-Gz}dz \\ &= \lim_{\varepsilon\downarrow 0} K\frac{\int_\varepsilon^\infty z^{-Y-1}e^{-(G+1)z}dz}{p\varepsilon^{-p-1}} \\ &= K\frac{1}{p(p+1)}\lim_{\varepsilon\downarrow 0}\varepsilon^{p+2-Y-1} = K\frac{1}{p(p+1)},\end{aligned}$$

due to $p = Y - 1$. Since $Y < 2$ it holds that $p = Y - 1 < 1$, therefore $\psi \in C^\infty(\mathbb{R}\backslash\{\ln(K)\}) \cap L^1_{\mathrm{loc}}(\mathbb{R})$. Moreover, ψ decays exponentially at $\pm\infty$. More precisely, for $x > \max\{\ln(K) + 1, 0\}$, $\psi(x) \le CK^{G+1}e^{-Gx}$, and for $x \le \min\{\ln(K) - 1, 0\}$, $\psi(x) \le CK^{1-M}e^{Mx}$. Consequently, $\psi \in (H^1_\eta(\mathbb{R}))^*$ for all $\eta \ge 0$ satisfying (3.25) and (3.26), in particular, for $\eta = 0$.

□

Proposition 3.5 *$-\mathcal{A}[h] \in (H^1_\eta(\mathbb{R}))^*$ for all η as in (3.25) and (3.26). In particular, for $\eta = 0$.*

Proof. We recall that $\mathcal{A} = -\frac{\sigma^2}{2}\frac{d^2}{dx^2} + \frac{\sigma^2}{2}\frac{d}{dx} + \hat{A}$. Therefore, $-\mathcal{A}[h] = \frac{\sigma^2}{2}K\delta_{\ln(K)} - \hat{A}[h]$. By Theorem 3.4 it follows that $-\mathcal{A}[h] \in (H^1_\eta(\mathbb{R}))^*$ for all η as in (3.25) and (3.26). In particular, for $\eta = 0$. □

Let u denote the solution of the parabolic evolution problem (3.14)–(3.16) and denote by $\mathcal{A}$ the spatial operator given by (3.21). By Proposition 3.5 we have that $f := -\mathcal{A}[h] \in (H^1_\eta(\mathbb{R}))^*$ for all η satisfying (3.26) and (3.25).

The difference $U := u - h$ between the option price and the pay-off function h solves the following parabolic problem

$$\frac{\partial U}{\partial \tau} + \mathcal{A}[U] = -\mathcal{A}[h] =: f \quad \text{in} \quad (0,T)\times\mathbb{R} \tag{3.29}$$

$$U|_{\tau=0} = 0 \quad \text{in} \quad \mathbb{R}, \tag{3.30}$$

i.e., in variational form: Find $U \in L^2((0,T); H^1_\eta(\mathbb{R})) \cap H^1(0,T; \left(H^1_\eta(\mathbb{R})\right)^*)$ such that

$$(\frac{d}{d\tau}U(\tau,\cdot), v)_{L^2_\eta(\mathbb{R})} + a^\eta(U(\tau,\cdot), v) = \langle f, v\rangle_{(H^1_\eta(\mathbb{R}))^*\times H^1_\eta(\mathbb{R})} \quad \forall\, v \in H^1_\eta(\mathbb{R}) \tag{3.31}$$

$$U|_{\tau=0} = 0. \tag{3.32}$$

Let $X := H^1_\eta(\mathbb{R})$ and $H := L^2_\eta(\mathbb{R})$. We have the following Gelfand triple with dense embeddings

$$X \hookrightarrow H \cong H^* \hookrightarrow X^*.$$

By Theorem 3.3 and Theorem 2.3, applied to $\mathcal{A} \in \mathcal{L}(X, X^*)$, $X = H^1_\eta(\mathbb{R})$, given $f \in \left(H^1_\eta(\mathbb{R})\right)^*$, there exists a unique weak solution $U \in L^2((0,T); H^1_\eta(\mathbb{R})) \cap H^1(0,T; \left(H^1_\eta(\mathbb{R})\right)^*)$ of (3.31)–(3.32). Indeed, by Theorem 3.3, item *2.*, there exists $\lambda > 0$ such that the shifted operator $\mathcal{A} + \lambda\cdot\mathrm{id}$ induces a coercive bilinear form on $H^1_\eta(\mathbb{R}) \times H^1_\eta(\mathbb{R})$. For the case $\eta = 0$ we denote by $a(\cdot,\cdot) = a^0(\cdot,\cdot)$

$$a(\varphi,\psi) := \langle \mathcal{A}[\varphi], \psi\rangle_{(H^1(\mathbb{R}))^*\times H^1(\mathbb{R})} \quad \forall\, \varphi,\psi \in H^1(\mathbb{R}). \tag{3.33}$$

We prove an *a-priori* estimate for the weak solution U of (3.31)–(3.32). To this end, let us denote by $T^{\mathcal{A}+\lambda\cdot\mathrm{id}}(\cdot)$ the analytic semi-group induced by the operator $\mathcal{A} + \lambda\cdot\mathrm{id}$ in $(H^1_\eta(\mathbb{R}))^*$ and let $f := -\mathcal{A}[h]$. Then U admits the Duhamel's representation in $(H^1_\eta(\mathbb{R}))^*$, see e.g. [2], Proposition III.1.3.1,

$$U(\tau) = \int_0^\tau T^{\mathcal{A}+\lambda\cdot\mathrm{id}}(s)[f]e^{-\lambda s}ds. \tag{3.34}$$

Recall that by Theorem 3.4 and Proposition 3.5, $f = \sigma^2/2K\delta_{\ln(K)} + \psi$, with $\psi \in C^\infty(\mathbb{R}\setminus\ln(K)) \cap L^1_{\mathrm{loc}}(\mathbb{R})$ decaying exponentially at $\pm\infty$. Therefore, $f \in (H^{1/2+\varepsilon}_\eta(\mathbb{R}))^*$ for all $\varepsilon > 0$.

We denote by $X_\theta := [X^*, X]_{\theta,2}$ the interpolation space for $0 \le \theta \le 1$ between X^* and X ($X_0 = X^*$ and $X_1 = X$). Then there exists $\theta > 0$ such that $f \in X_\theta$ and there exist $C, d > 0$ such that for all $t > 0$

$$\|(T^{\mathcal{A}+\lambda\cdot\mathrm{id}})^{(k)}(\tau)\|_{\mathcal{L}(X_\theta, X)} \le Cd^{k+1/2-\theta}\sqrt{\Gamma(2k+2-2\theta)}\tau^{-(k+1)+\theta}.$$

The proof of this result can be found e.g. in [32], Theorem 1. By the representation (3.34) we obtain that

$$\|U(\tau,\cdot)\|_X + \tau\|\frac{\partial U}{\partial\tau}(\tau,\cdot)\|_X \le C\tau^\theta e^{\lambda\tau}\|f\|_{X_\theta}. \tag{3.35}$$

3.5 Positivity of the integro-differential operator

For the numerical solution below, it will be important to have information on the spectrum of the integral operator A.

Remark 3.6 $A + A^* \geq 0$. More precisely, for all φ, $\psi \in H^1(\mathbb{R})$ there holds

$$(A[\varphi], \psi)_{L^2(\mathbb{R})} + (A[\psi], \varphi)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} (\varphi(x+y) - \varphi(x))(\psi(x+y) - \psi(x))k(y)dydx. \quad (3.36)$$

Proof. A density argument allows us to check (3.36) only for $\varphi, \psi \in C_0^\infty(\mathbb{R})$. Elementary considerations lead to

$$\begin{aligned}
(A[\varphi], \psi)_{L^2(\mathbb{R})} = \\
&- \int_{\mathbb{R}} \int_{\mathbb{R}} \{\varphi(x+y) - \varphi(x) - y\chi_{\{|y|\leq 1\}}(y)\frac{d\varphi}{dx}(x)\}k(y)\{\psi(x) - \psi(x+y) + \psi(x+y)\}\, dxdy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} (\varphi(x+y) - \varphi(x)) \cdot (\psi(x+y) - \psi(x))k(y)dydx \\
&+ \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x)\{\psi(x+y) - \psi(x) - y\chi_{\{|y|\leq 1\}}(y)\frac{d\psi}{dx}(x)\}k(y)dxdy \\
&= -(A[\psi], \varphi)_{L^2(\mathbb{R})} + \int_{\mathbb{R}} \int_{\mathbb{R}} (\varphi(x+y) - \varphi(x)) \cdot (\psi(x+y) - \psi(x))k(y)dydx.
\end{aligned}$$

□

4 Localization

Numerical solution of (3.14)–(3.16) will require truncation of $\Omega = \mathbb{R}$ to a bounded computational domain $\Omega_R = (-R, R)$. Likewise, certain types of contracts (no-touch, touch-and-out) directly lead to the PIDE on the bounded domain. Here, we formulate the PIDE on the bounded domain Ω_R and investigate its solution as $R \to \infty$.
In the Black-Scholes case, the localization error can be estimated by local considerations near $\partial\Omega_R$ and a maximum principle (see, e.g., [20]). For the PIDE, such local arguments do not apply and we must resort to the weighted norm estimates for the PIDE to control the domain truncation error.

4.1 PIDE on bounded domain

Instead of solving (3.29)–(3.30) in $J \times \mathbb{R}$, where we denote by J the time interval $J = (0, T)$, we solve the following problem in $J \times \Omega_R$:

$$\frac{\partial U_R}{\partial \tau} + \mathcal{A}_R[U_R] = -\mathcal{A}[h]|_{\Omega_R} \quad \text{in} \quad J \times \Omega_R \quad (4.1)$$

$$U_R(\tau, \cdot)|_{\partial\Omega_R} = 0 \quad \text{on} \quad \partial\Omega_R \quad \forall\, 0 < \tau \leq T \quad (4.2)$$

$$U_R|_{\tau=0} = 0 \quad \text{in} \quad \Omega_R, \quad (4.3)$$

with $\mathcal{A}_R$ being the restriction of $\mathcal{A}$ to Ω_R. Note that, unlike in the Black-Scholes case, the non-local operator A forces to specify the pay-off function h also outside of Ω_R.
For a variational formulation of (4.1)–(4.3) we denote by $a_R(\cdot, \cdot) : H_0^1(\Omega_R) \times H_0^1(\Omega_R) \mapsto \mathbb{R}$ the bilinear form induced by $\mathcal{A}_R$

$$a_R(\varphi, \psi) := \langle \mathcal{A}_R[\varphi], \psi \rangle_{H^{-1}(\Omega_R) \times H_0^1(\Omega_R)} = \langle \mathcal{A}[\varphi], \psi \rangle_{(H^1(\mathbb{R}))^* \times H^1(\mathbb{R})} \quad \forall\, \varphi, \psi \in H_0^1(\Omega_R). \quad (4.4)$$

We denote by $V := H_0^1(\Omega_R)$ and we identify $L^2(\Omega_R)$ with its dual. Then

$$V \overset{d}{\hookrightarrow} L^2(\Omega_R) \overset{d}{\hookrightarrow} V^* \tag{4.5}$$

with dense embeddings and $V^* = H^{-1}(\Omega_R)$. The variational formulation of (4.1)–(4.3) reads: Given $f := -\mathcal{A}[h]|_{\Omega_R} \in H^{-1}(\Omega_R)$, find $U_R \in L^2(J,V) \cap H^1(J,V^*)$ such that $U_R(0) = 0$ and such that for every $v \in V$ and every $\varphi \in C_0^\infty(J)$

$$-\int_J (U_R(\tau,\cdot),v)_{L^2(\Omega_R)}\varphi'(\tau)d\tau + \int_J a_R(U_R(\tau,\cdot),v)\varphi(\tau)d\tau = \langle f, v\rangle_{V^*\times V}, \tag{4.6}$$

where by $\langle\cdot,\cdot\rangle_{V^*\times V}$ we denote the extension of $(\cdot,\cdot)_{L^2(\Omega_R)}$ as duality pairing in $V^* \times V$. By Theorem 3.3, the bilinear form $a_R(\cdot,\cdot)$ obtained as restriction of the bilinear form $a(\cdot,\cdot)$ to $V \times V$ is continuous and satisfies a **Gårding inequality**: there exist $C > 0$ and $\alpha > 0$, $\beta \geq 0$ such that

$$\forall\, \varphi,\psi \in V:\ |a_R(\varphi,\psi)| \leq C\|\varphi\|_V\|\psi\|_V \tag{4.7}$$

$$\forall \varphi \in V:\ a_R(\varphi,\varphi) + \beta\|\varphi\|^2 \geq \alpha\,\|\varphi\|_V^2. \tag{4.8}$$

Without loss of generality we may assume from now on that the bilinear form a_R is coercive on $V \times V$, since by the substitution $V_R = e^{-\beta\tau}U_R$, V_R solves the problem

$$\frac{d}{d\tau}V_R + (\mathcal{A}_R + \beta\cdot\mathrm{id})V_R = e^{-\beta\tau}f \quad \text{in } J,$$

and the operator $\mathcal{A}_R + \beta\cdot\mathrm{id}$ is, by (4.8), coercive. By Theorem 2.3, applied to the triple (4.5), there exists a unique solution.
Note that the initial condition (4.3) is well defined since

$$L^2(J,V) \cap H^1(J,V^*) \subset C^0([0,T];\, H). \tag{4.9}$$

4.2 Localization error estimates

The restriction of U from $\mathbb{R}$ to Ω_R introduces a localization error $e_R := U_R - U$ (where U_R is understood as zero extension to $\mathbb{R}$).

Theorem 4.1 *Let $\Omega_{R/2} := \{|x| \leq R/2\}$. Then there exist positive constants $C = C(T), \alpha > 0$ independent of R such that the $e_R = U_R - U$ satisfies the following estimate:*

$$\|e_R(\tau,\cdot)\|^2_{L^2(\Omega_{R/2})} + \int_0^\tau \|e_R(s,\cdot)\|^2_{H^1(\Omega_{R/2})}ds \leq Ce^{-\alpha R}. \tag{4.10}$$

Proof. Take the weighting exponent $\eta > 0$ as in (3.25)–(3.26). Inserting $v = U(\tau)$ in (3.31)–(3.32) and integrating from 0 to τ implies the following a-priori estimate

$$\|U(\tau)\|^2_{L^2_\eta(\mathbb{R})} + \int_0^\tau \|U(s)\|^2_{H^1_\eta(\mathbb{R})}ds \leq C\|f\|^2_{(H^1_\eta(\mathbb{R}))^*} \quad \forall\,\tau \in (0,T) \tag{4.11}$$

for some constant $C = C(T) > 0$ independent $C = C(T) > 0$ of R. Likewise, with U_R being understood as the zero extension of the localized solution U_R to $\mathbb{R}$ it holds

$$\|U_R(\tau)\|^2_{L^2_\eta(\mathbb{R})} + \int_0^\tau \|U_R(s)\|^2_{H^1_\eta(\mathbb{R})}ds \leq C\|f\|^2_{(H^1_\eta(\mathbb{R}))^*} \quad \forall\,\tau \in (0,T) \tag{4.12}$$

with same constant C as in (4.11). In particular, C is independent of R. Note also that the error e_R satisfies

$$\left(\frac{d}{d\tau}e_R(\tau), v\right)_{L^2(\mathbb{R})} + a(e_R(\tau), v) = 0 \quad \forall\, v \in H_0^1(\Omega_R). \tag{4.13}$$

Denote by ϕ a cut-off function with the properties: $\phi \in C_0^\infty(\Omega_R)$, $\phi \equiv 1$ on $\Omega_{R/2}$ and $\|\phi'\|_{L^\infty(\Omega_R)} \leq C$ for some constant $C > 0$ independent of R.
Inserting $v = \phi^2(x)e_R(\tau, x)$ in (4.13) we obtain

$$\frac{1}{2}\frac{d}{d\tau}\|\phi e_R(\tau)\|^2_{L^2(\Omega_R)} + a_R(\phi e_R(\tau), \phi e_R(\tau)) = \rho_R(\tau), \tag{4.14}$$

where the residual $\rho_R(\tau)$ is given by $\rho_R(\tau) := a_R(\phi e_R(\tau), \phi e_R(\tau)) - a(e_R(\tau), \phi^2 e_R(\tau))$. We observe that

$$\rho_R(\tau) = \frac{\sigma^2}{2}\int_{\mathbb{R}} |\phi'(x)|^2 |e_R(\tau, x)|^2 dx + \left(\frac{\sigma^2}{2} + c'_{\exp}\right)\int_{\mathbb{R}} \phi'(x)\phi(x)|e_R(\tau, x)|^2 dx + \bar{\rho}_R(\tau) \tag{4.15}$$

where we denote by $\bar{\rho}_R(\tau)$ the residual

$$\bar{\rho}_R(\tau) = \bar{a}(\phi e_R(\tau), \phi e_R(\tau)) - \bar{a}(e_R(\tau), \phi^2 e_R(\tau)) \tag{4.16}$$

from the bilinear form $\bar{a}(\varphi, \phi) := \int_{\mathbb{R}} \bar{A}[\varphi](x)\phi(x)dx$ with $\bar{A}$ defined by

$$\varphi \mapsto \bar{A}[\varphi](x) := -\int_{\mathbb{R}} \{\varphi(x + y) - \varphi(x) - y\varphi'(x)\}k(y)dy \tag{4.17}$$

and by $c'_{\exp}$ the constant

$$c'_{\exp} = \int_{\mathbb{R}} \{e^y - 1 - y\}k(y)dy.$$

The first two integral terms in the expression of the residual $\rho_R(\tau)$ in (4.15) are supported by $\Omega_R \backslash \Omega_{R/2}$ and can be estimated by

$$\begin{aligned}\left|\frac{\sigma^2}{2}\int_{\mathbb{R}} |\phi'(x)|^2 |e_R(\tau, x)|^2 dx + \left(\frac{\sigma^2}{2} + c'_{\exp}\right)\int_{\mathbb{R}} \phi'(x)\phi(x)|e_R(\tau, x)|^2 dx\right| \\ \leq C\int_{\Omega_R\backslash\Omega_{R/2}} |e_R(\tau, x)|^2 e^{\eta(x)}e^{-\eta(x)}dx \leq Ce^{-\alpha R}\|e_R(\tau)\|^2_{L^2_\eta(\mathbb{R})},\end{aligned} \tag{4.18}$$

for some positive constants C, α independent of R.

It remains to estimate the residual $\bar{\rho}_R(\tau)$. To this end, let us denote by $k^{(-1)}$ the first antiderivative of the Lévy kernel k vanishing at $|x| \to \infty$

$$k^{(-1)}(x) = \begin{cases} -\displaystyle\int_x^\infty k(y)dy & \text{if } x > 0 \\ \displaystyle\int_{-\infty}^x k(y)dy & \text{if } x < 0. \end{cases}$$

Observe that for k being the CGMY Lévy density as in (3.9) with $Y < 2$, $G > 0$ and $M > 1$ the first antiderivative kernel $k^{(-1)}$ has the same rate of exponential decay as k as $\pm x \to \infty$ and $yk^{(-1)}(y)$ is in $L^1(\mathbb{R})$.

Integration by parts implies for $\bar{A}$ as in (4.17) the representation

$$\bar{A}[\varphi](x) = \int_{\mathbb{R}} \{\varphi'(x+y) - \varphi'(x)\} k^{(-1)}(y) dy \quad \forall\, \varphi \in H^1(\mathbb{R}).$$

With these notations and from (4.16),

$$\begin{aligned}
\bar{\rho}_R(\tau) = \iint_{\mathbb{R}^2} \Big(& \frac{\partial e_R}{\partial x}(\tau, x+y)\phi(x+y) + e_R(\tau, x+y)\phi'(x+y) \\
& - \frac{\partial e_R}{\partial x}(\tau, x)\phi(x) - e_R(\tau, x)\phi'(x) \Big) k^{(-1)}(y) e_R(\tau, x)\phi(x) dy dx \\
& - \iint_{\mathbb{R}^2} \Big(\frac{\partial e_R}{\partial x}(\tau, x+y) - \frac{\partial e_R}{\partial x}(\tau, x) \Big) k^{(-1)}(y) e_R(\tau, x)\phi^2(x) dy dx
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
\bar{\rho}_R(\tau) = & \iint_{\mathbb{R}^2} \frac{\partial e_R}{\partial x}(\tau, x+y)(\phi(x+y) - \phi(x)) k^{(-1)}(y) e_R(\tau, x)\phi(x) dy dx \\
& + \iint_{\mathbb{R}^2} (e_R(\tau, x+y)\phi'(x+y) - e_R(\tau, x)\phi'(x)) k^{(-1)}(y) e_R(\tau, x)\phi(x) dy dx \\
= & \, I_1 + I_2.
\end{aligned} \tag{4.19}$$

We observe that the integrand in the first term of $\rho_R(\tau)$ in (4.19) is supported by $|x+y| \geq R/2$ or $|x| \geq R/2$ (otherwise $\phi(x+y) - \phi(x) = 1 - 1 = 0$), i.e.,

$$I_1 = \iint_{\{|x+y| \geq R/2\} \cup \{|x| \geq R/2\}} \frac{\partial e_R}{\partial x}(\tau, x+y)(\phi(x+y) - \phi(x)) k^{(-1)}(y) e_R(\tau, x)\phi(x) dy dx.$$

It implies that

$$\begin{aligned}
|I_1| \leq & \, C \iint_{|x+y| \geq R/2} |\frac{\partial e_R}{\partial x}(\tau, x+y)| e^{\eta(x+y)} e^{-\eta(x+y)} |y| |k^{(-1)}(y)| |e_R(\tau, x)| \phi(x) dy dx \\
& + \iint_{|x| \geq R/2} |\frac{\partial e_R}{\partial x}(\tau, x+y)| |y| |k^{(-1)}(y)| |e_R(\tau, x)| e^{\eta(x)} e^{-\eta(x)} dy dx \\
\leq & \, C e^{-\alpha R} \Big(\|\frac{\partial e_R}{\partial x}(\tau)\|_{L^2_\eta(\mathbb{R})} \|e_R(\tau)\|_{L^2(\mathbb{R})} + \|\frac{\partial e_R}{\partial x}(\tau)\|_{L^2(\mathbb{R})} \|e_R(\tau)\|_{L^2_\eta(\mathbb{R})} \Big),
\end{aligned} \tag{4.20}$$

for some positive constants C, α independent of R. Analogous reasoning applies to the second integral term in (4.19) after we split it into

$$\begin{aligned}
I_2 = & \iint_{\mathbb{R}^2} (e_R(\tau, x+y) - e_R(\tau, x)) \phi'(x) k^{(-1)}(y) e_R(\tau, x)\phi(x) dy dx \\
& + \iint_{\mathbb{R}^2} e_R(\tau, x+y)(\phi'(x+y) - \phi'(x)) k^{(-1)}(y) e_R(\tau, x)\phi(x) dy dx = I_{21} + I_{22}.
\end{aligned} \tag{4.21}$$

The integral I_{21} in (4.21) is supported on $\{|x| \geq R/2\}$ and is estimated as follows

$$\begin{aligned}
|I_{21}| = & \left| \iint_{|x| \geq R/2} (e_R(\tau, x+y) - e_R(\tau, x)) \phi'(x) k^{(-1)}(y) e_R(\tau, x) e^{\eta(x)} e^{-\eta(x)} \phi(x) dy dx \right| \\
\leq & \, C e^{-\alpha R} \|e_R(\tau)\|_{H^1(\mathbb{R})} \|e_R(\tau)\|_{H^1_\eta(\mathbb{R})}.
\end{aligned} \tag{4.22}$$

The term I_{22} in (4.21) can be treated similar to I_1 and satisfies the following estimate

$$|I_{22}| \leq Ce^{-\alpha R}\|e_R(\tau)\|_{L^2_\eta(\mathbb{R})}\|e_R(\tau)\|_{L^2(\mathbb{R})}. \tag{4.23}$$

Integrating (4.14) from 0 to τ and using the estimates for $\rho_R(\tau)$ from (4.18), (4.20), (4.22) and (4.23) together with the a-priori estimates (4.11)–(4.12) yields the error estimate (4.10).

□

5 Numerical solution

For the pricing, we discretize (4.1)–(4.3) in time using the so-called θ-scheme and in Ω_R by a wavelet finite element method.

In the time interval $J = (0, T)$ with $T > 0$, we consider the parabolic evolution problem (4.1)–(4.3) where $\mathcal{A}_R$ is a second order nonlocal operator as in (3.21), $\Omega_R = (-R, R)$, $R > 0$, and $f \in V^* := H^{-1}(\Omega_R)$. We denote $H := L^2(\Omega_R)$ and by $H^s(\Omega_R)$, $s \geq 0$, the corresponding Sobolev spaces (see, e.g., [1]). Further, for $s \geq 0$, we define the space

$$\tilde{H}^s(\Omega_R) = \left\{ u|_{\Omega_R} \mid u \in H^s(\mathbb{R}),\, u|_{\mathbb{R}\setminus\Omega_R} = 0 \right\}. \tag{5.1}$$

If $s + 1/2 \notin \mathbb{N}$, then $\tilde{H}^s(\Omega_R)$ coincides with $H^s_0(\Omega_R)$, the closure of $C^\infty_0(\Omega_R)$ with respect to the norm in $H^s(\Omega_R)$.

By $\|\cdot\|$, $\|\cdot\|_V$, $\|\cdot\|_{V^*}$ we denote the norms in $L^2(\Omega_R)$, V, V^*, resp. The bilinear form $a_R(\cdot,\cdot)$: $V \times V \mapsto \mathbb{R}$ associated to $\mathcal{A}_R$ is given by (4.4).

5.1 Discretization

To discretize the parabolic problem (4.1)–(4.3) in space, we use an elliptic projection onto a family $\{V_h\}_h \subset V$ of finite dimensional subspaces of V, based on piecewise polynomials of degree $p \geq 0$ on a uniform family of triangulations $\{\mathcal{T}_h\}_h$ of meshwidth h on Ω_R. Let $\mathcal{T}^0$ be a fixed coarse discretization of Ω_R. We then define the mesh $\mathcal{T}^l$ for $l > 0$ by bisection of each interval in $\mathcal{T}^{l-1}$. We assume that the mesh $\{\mathcal{T}_h\}$ is obtained in this way as $\mathcal{T}^L$, for some $L > 0$ so that $h = 2R2^{-L}$.
The space V_h is defined as the space of continuous piecewise polynomials of degree $p \geq 1$ on $\{\mathcal{T}_h\}$ with zero values on $\partial\Omega_R$. In the same way we define the spaces V^l corresponding to the triangulation $\mathcal{T}^l$, so that we have

$$V^0 \subset V^1 \subset \cdots \subset V^L = V_h.$$

Let $N^l = \dim V^l$ and let $N = \dim V_h = N^L = C2^L$. The semi-discrete problem reads: given $f \in V^*$, find $U_h \in H^1(J, V_h)$ such that

$$U_h(0) = 0 \tag{5.2}$$

and

$$\frac{d}{d\tau}(U_h, v_h) + a_R(U_h, v_h) = \langle f, v_h\rangle_{V^*\times V} \quad \forall\, v_h \in V_h. \tag{5.3}$$

For $T < \infty$ and $M \in \mathbb{N}$, define the time step

$$k = T/M$$

and $t^m = mk$, $m = 0, \dots, M$. The fully discrete θ-scheme reads as follows: find $U_h^m \in V_h$ satisfying

$$U_h^0 = 0, \tag{5.4}$$

and, for $m = 0, 1, \dots, M-1$, find $U_h^{m+1} \in V_h$ such that for all $v_h \in V_h$

$$\Big(\frac{U_h^{m+1} - U_h^m}{k}, v_h\Big) + a_R\big(U_h^{m+\theta}, v_h\big) = \langle f, v_h\rangle_{V^*\times V} \tag{5.5}$$

holds. Here $U_h^{m+\theta} := \theta U_h^{m+1} + (1-\theta)U_h^m$ and $t^{m+\theta} = \theta t^{m+1} + (1-\theta)\, t^m = (m+\theta)\, k$. In matrix form, (5.5) reads

$$(k^{-1}\mathbf{K} + \theta\mathbf{A})\underline{U}^{m+1} = k^{-1}\mathbf{K}\underline{U}^m - (1-\theta)\mathbf{A}\underline{U}^m + \underline{f}, \quad m = 0, 1, ..., M-1.$$

where $\underline{U}^m$ is the coefficient vector of U_h^m with respect to a basis of V_h. The matrices $\mathbf{K}$, $\mathbf{A}$ denote the mass- and stiffness matrix, respectively, with respect to a basis of V^L. By Remark 3.6, all eigenvalues of $\mathbf{A}$ have positive real part.

5.2 Wavelet Compression

Matrix $\mathbf{A}$ is, due to the nonlocal operator $\mathcal{A}_R$, fully populated, increasing the complexity of the algorithm. Perturbed bilinear forms $\widetilde{a}_R$ are obtained by various matrix compression techniques which reduce the dense matrices $\mathbf{A}$ to sparse ones which can be manipulated in linear complexity.

5.2.1 Wavelet basis

By choosing a suitable basis for V_h we will be able to represent the bilinear form $a_R(\cdot,\cdot)$ as a matrix where most elements are small and can be neglected, yielding the approximate bilinear form $\tilde{a}_R(\cdot,\cdot)$. The basis will also allow optimal preconditioning. We will use so-called biorthogonal wavelets (note that the dual wavelets described below will not be used in the computation).

We will use a wavelet basis of functions ψ_j^l with $j = 1, \dots, M^l$ and $l = 0, 1, \dots$ with the following properties: We have $V^l = \operatorname{span}\{\psi_j^k \mid 0 \le k \le l, 1 \le j \le M^k\}$. The function ψ_j^l has support $S_j^l := \operatorname{supp}\psi_j^l$ of diameter bounded by $C\,2^{-l}$. Wavelets ψ_j^l with $\bar{S}_j^l \cap \partial\Omega_R = \emptyset$ have vanishing moments up to order p, i.e., $(\psi_j^l, q) = 0$ for all polynomials q of total degree p or less. The functions ψ_j^l for $l \ge l_0$ are obtained by scaling and translation of the functions $\psi_j^{l_0}$.

One example of wavelets are the so-called prewavelets: these are functions in V^l with small support which are orthogonal on V^{l-1}. Another possibility are so-called biorthogonal wavelets which need not be orthogonal on V^{l-1}. For piecewise linears the functions ψ_j^l in the interior of the interval have values $0, \dots, 0, -1, 2, -1, 0, \dots, 0$. In the case of Neumann boundary conditions the wavelet at the left boundary has values $-2, 2, -1, 0, \dots, 0$; in the case of Dirichlet conditions the values are $0, 2, -1, 0, \dots, 0$ (and similarly at the right boundary), see Figure 1. Note that the boundary wavelets have fewer vanishing moments in general.

Any function $v \in V_h$ has the representation

$$v = \sum_{l=0}^{L} \sum_{j=1}^{M^l} v_j^l \psi_j^l$$

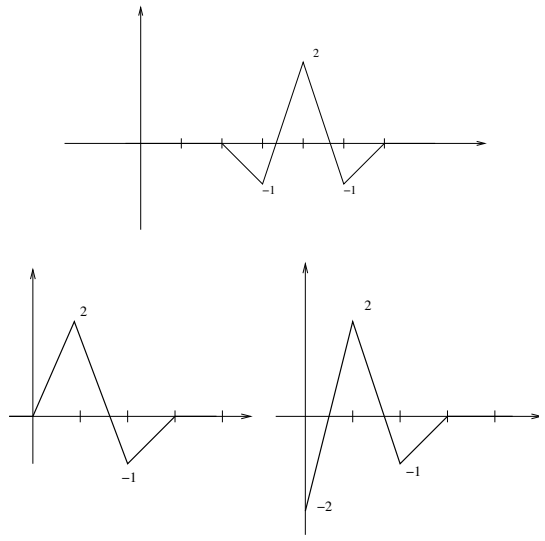

Figure 1: Generating wavelets: interior wavelets (top) and boundary wavelets for Dirichlet boundary conditions (bottom left) and Neumann boundary conditions (bottom right).

with $v_j^l = (v, \tilde{\psi}_j^l)$ where $\tilde{\psi}_j^l$ are the so-called dual wavelets.
For $v \in V$ one obtains an infinite series

$$v = \sum_{l=0}^{\infty} \sum_{j=1}^{M^l} v_j^l \psi_j^l$$

with $v_j^l = (v, \tilde{\psi}_j^l)$ which converges in $\tilde{H}^s$ for $0 \le s \le 1$.
For $v \in V$ we can define a projection $P_h : V \to V_h$ by truncating the wavelet expansion:

$$P_h v := \sum_{l=0}^{L} \sum_{j=1}^{M^l} v_j^l \psi_j^l. \tag{5.6}$$

5.2.2 Matrix compression

The bilinear form a_R on $V_h \times V_h$ corresponds to a matrix $\mathbf{A}$ with elements $A_{(l,j),(l',j')} = a_R(\psi_j^l, \psi_{j'}^{l'})$.

The kernel of the operator satisfies the estimates (5.7) below $\forall\, \alpha \in \mathbb{N}_0^n$, $\forall\, x \neq 0$:

$$|\partial_x^\alpha\, k(x)| \le C(\alpha)|x|^{-(1+Y+|\alpha|)}\,. \tag{5.7}$$

This implies a decay of the matrix elements with increasing distance of their supports.
We define the compressed matrix $\tilde{\mathbf{A}}$ and the corresponding bilinear form $\tilde{a}_R$ by replacing certain small matrix elements in $\mathbf{A}$ with zero:

$$\tilde{A}_{(j,l),(j',l')} := \begin{cases} A_{(j,l),(j',l')} & \text{if } \operatorname{dist}(S_j^l, S_{j'}^{l'}) \le \delta_{l,l'} \text{ or } S_j^l \cap \partial\Omega_R \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

Here the truncation parameters $\delta_{l,l'}$ are given by

$$\delta_{l,l'} := c \max\{2^{-L+\hat{\alpha}(2L-l-l')}, 2^{-l}, 2^{-l'}\} \tag{5.9}$$

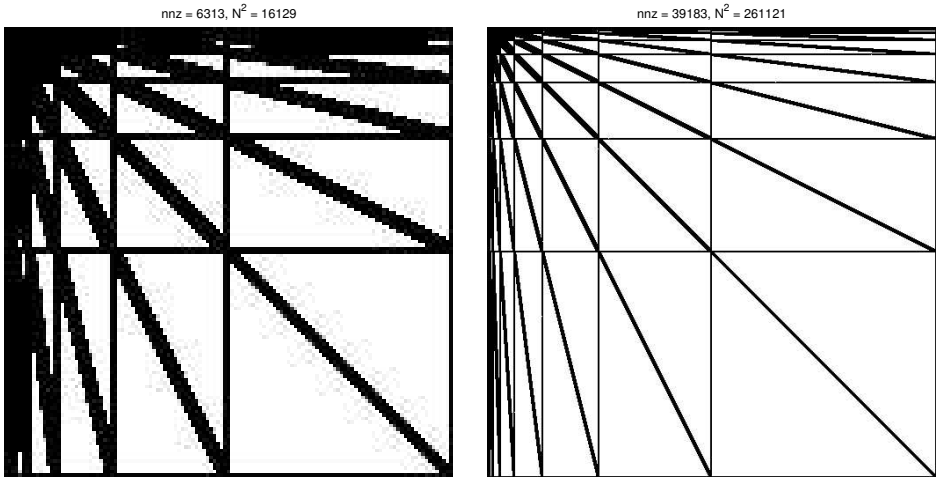

Figure 2: Sparsity pattern of the compressed matrix in wavelet basis; CGMY parameters: $C = 1.0$, $Y = 1.5$, $G = 0.6$, $M = 2.8$; $N = 127$ (left) and $N = 511$ (right).

with some parameters $c > 0$ and $\hat{\alpha} > 0$.
By continuity and coercivity of the bilinear form a_R we can define on $H_0^1(\Omega_R)$ an equivalent norm by

$$\|u\|_a := (a_R(u,u))^{1/2} \sim \|u\|_{H^1}.$$

In the following we need to consider functions in V which have additional regularity and introduce for this purpose the spaces $\mathcal{H}^s(\Omega_R)$ which are defined as

$$\mathcal{H}^s(\Omega_R) = \begin{cases} V & \text{for } s = 1, \\ V \cap H^s(\Omega_R) & \text{for } s > 1. \end{cases}$$

Proposition 5.1 *If c in (5.9) is chosen sufficiently large then there exists $0 < \delta < 1$ independent of L such that for all $L > 0$ condition*

$$|a_R(u_h, v_h) - \widetilde{a}_R(u_h, v_h)| \le \delta \, \|u_h\|_a \, \|v_h\|_a \quad \forall u_h, v_h \in V_h \tag{5.10}$$

holds. If additionally

$$\hat{\alpha} \ge \frac{2p+2}{2p+2+Y}, \tag{5.11}$$

then

$$|a_R(P_h u, v_h) - \widetilde{a}_R(P_h u, v_h)| \le C h^p |\log h|^\nu \|u\|_{\mathcal{H}^{p+1}(\Omega_R)} \, \|v_h\|_{H_0^1(\Omega_R)} \quad \forall u \in \mathcal{H}^{p+1}(\Omega_R), v_h \in V_h \tag{5.12}$$

holds with $\nu = \frac{3}{2}$ if equality holds in (5.11), and $\nu = \frac{1}{2}$ otherwise.

The matrix compression (5.8) reduces the number of nonzero elements from N^2 in $\mathbf{A}$ to N times a logarithmic term in $\tilde{\mathbf{A}}$, see Figure 2 and [26].

Proposition 5.2 *We can choose $\hat{\alpha}$ such that $\nu = \frac{1}{2}$ in (5.12) and the number of nonzero elements in $\tilde{\mathbf{A}}$ is $O(N \log N)$.*

Using $\tilde{a}_R(\cdot,\cdot)$ in place of $a_R(\cdot,\cdot)$ in (5.5) gives **perturbed θ-schemes**

$$\widetilde{U}_h^0 = 0\,, \tag{5.13a}$$

$$\Big(\frac{\widetilde{U}_h^{m+1}-\widetilde{U}_h^m}{k}\,, v_h\Big) + \widetilde{a}_R\big(\widetilde{U}_h^{m+\theta}, v_h\big) = \langle f, v_h\rangle_{V^*\times V} \tag{5.13b}$$

for $m = 0, 1, 2, \ldots, M-1$ and every $v_h \in V_h$, where again $\widetilde{U}_h^{m+\theta} := \theta\widetilde{U}_h^{m+1} + (1-\theta)\,\widetilde{U}_h^m$. In matrix form, (5.13b) reads

$$(k^{-1}\mathbf{K} + \theta\tilde{\mathbf{A}})\underline{\tilde{U}}^{m+1} = k^{-1}\mathbf{K}\underline{\tilde{U}}^m - (1-\theta)\tilde{\mathbf{A}}\underline{\tilde{U}}^m + \underline{f}, \quad m = 0, 1, ..., M-1$$

where $\underline{\tilde{U}}^m$ is the coefficient vector of $\tilde{U}_h^m$ with respect to a basis of V_h.

5.3 Stability and Convergence

Consider now the sequence $\{\widetilde{U}_h^m\}_{m=0}^M$ of solutions to the perturbed θ-scheme (5.13a), (5.13b). These solutions are stable and converge with optimal order as $h \to 0$, regardless of the wavelet compression.
We define for $v_h \in V_h$ and $f \in V_h^*$

$$\|v_h\|_{\tilde{a}} := (\tilde{a}_R(v_h, v_h))^{1/2}, \qquad \|f\|_{\tilde{*}} := \sup_{v_h\in V_h} \frac{(f, v_h)}{\|v_h\|_{\tilde{a}}}, \qquad \lambda_{\tilde{A}} := \sup_{v_h\in V_h} \frac{\|v_h\|^2}{\|v_h\|_{\tilde{*}}^2}. \tag{5.14}$$

Proposition 5.3 *Assume that (5.10) holds with $\delta < 1$. In the case of $\frac{1}{2} \le \theta \le 1$ assume that*

$$0 < C_1 < 2, \qquad C_2 \ge \frac{1}{2 - C_1}.$$

In the case of $0 \le \theta < \frac{1}{2}$ assume the time-step restriction

$$\sigma := k(1-2\theta)\lambda_{\tilde{A}} < 2 \tag{5.15}$$

and that

$$0 < C_1 < 2 - \sigma, \qquad C_2 \ge \frac{1 + (4 - C_1)\sigma}{2 - \sigma - C_1}. \tag{5.16}$$

Then the sequence $\{\tilde{U}_h^m\}_{m=0}^M$ of solutions of the perturbed θ-scheme (5.13a), (5.13b) satisfies the stability estimate

$$\|\tilde{U}_h^M\|^2 + C_1 k \sum_{m=0}^{M-1} \|\tilde{U}_h^{m+\theta}\|_{\tilde{a}}^2 \le C_2 T \|f\|_{\tilde{*}}^2. \tag{5.17}$$

Assume that the consistency conditions (5.10), (5.12) hold. For $\theta \in [0, \frac{1}{2})$ assume the stability condition (5.15). If the solution $U_R(\tau, x)$ is sufficiently smooth, there holds the following error estimate for the perturbed θ-scheme with $\theta \in [0, 1]$

$$\Big\|U^M - \widetilde{U}_h^M\Big\|^2 + k \sum_{m=0}^{M-1} \|U^{m+\theta} - \widetilde{U}_h^{m+\theta}\|_a^2 \le C\{h^{2p}|\log h|^{2\nu} + k^{2\mu}\}, \tag{5.18}$$

where $C > 0$ depends on R, and $\mu = 1$ if $\theta \ne \frac{1}{2}$ and $\mu = 2$ else and ν is as in (5.12). The convergence result (5.18) is proved in [27], Theorem 5.4.

Remark 5.4 For any $g \in V^*$, let

$$\|g\|_{V^*} := \sup_{v \in V} \frac{(g,v)}{\|v\|_a}.$$

Then holds the inverse estimate

$$\lambda_A^{1/2} = \sup_{v_h \in V_h} \frac{\|v_h\|}{\|v_h\|_*} \leq Ch^{-1}. \tag{5.19}$$

Hence there exists a positive constant C_* independent of h and θ such that the time-step restriction

$$k \leq C_* \frac{h^2}{1-2\theta} \tag{5.20}$$

is sufficient for stability. For $\theta < \frac{1}{2}$ (e.g., forward Euler and the heat equation) this reduces to the well-known time-step restriction $k \leq C_\theta h^2$ for explicit schemes.

Remark 5.5 If $\sigma \neq 0$, the price process contains a diffusion component and the order of the operator $\mathcal{A}$ is 2. In the pure jump case, $\sigma = 0$ and the order of $\mathcal{A}$ is, in general, $\max\{1, Y\}$ and the CFL-condition (5.20) becomes

$$k \leq C_* \frac{h^{\max\{1,Y\}}}{1-2\theta}. \tag{5.21}$$

Note that for $1 < Y < 2$ the PIDE is of parabolic type, whereas for $0 \leq Y < 1$ it is, in general, hyperbolic. If $Y = 1$, its formal type is neither parabolic nor hyperbolic.
If $\sigma = 0$ and $Y < 1$, discontinuous spatial approximations are admissible. Then the advection term $\frac{\partial u}{\partial x}$ can be stably discretized by an upwinding Finite Volume Method (FVM).

5.4 Approximate Solution of Linear Equations and Complexity

In order to compute the approximate solution $\tilde{U}_h^m$ in (5.13) for $m = 1, \ldots, M$ we proceed as follows:
We first compute the mass matrix $\mathbf{K}$ in the wavelet basis with elements $K_{(l,j),(l',j')}$ where $O(N \log N)$ elements are nonzero.
Then we compute the compressed stiffness matrix $\tilde{\mathbf{A}}$ where $O(N(\log N))$ elements are nonzero, see Proposition 5.2. If explicit antiderivatives of the kernel function are available (as is often the case), the total cost for computing the stiffness matrix $\tilde{\mathbf{A}}$ is $O(N(\log N))$ operations. In other cases quadratures can be used. This preserves the consistency conditions (5.10),(5.12) and the total cost of computing $\tilde{\mathbf{A}}$ is $O(N(\log N)^2)$.
For each time step we have to solve (5.13b): We have to find $\tilde{w}_h^m := \tilde{U}_h^{m+1} - \tilde{U}_h^m \in V_h$ satisfying

$$k^{-1}(\tilde{w}_h^m, v_h) + \theta \tilde{a}_R(\tilde{w}_h^m, v_h) = (f^{m+\theta}, v_h) - \tilde{a}_R(\tilde{U}_h^m, v_h) \quad \forall v_h \in V_h \tag{5.22}$$

and then update $\tilde{U}_h^{m+1} := \tilde{U}_h^m + \tilde{w}_h^m$. Let $\underline{\tilde{w}}^m \in \mathbb{R}^N$ denote the coefficient vectors of $\tilde{w}_h^m$ with respect to the wavelet basis, and $\mathbf{K}$, $\tilde{\mathbf{A}} \in \mathbb{R}^{N \times N}$ the mass and stiffness matrices corresponding to $(\cdot,\cdot)$ and $\tilde{a}_R(\cdot,\cdot)$ in this basis. Then we obtain for $\underline{\tilde{w}}^m$ a linear system $\mathbf{B}\underline{\tilde{w}}^m = \underline{\tilde{b}}^m$ with the matrix $\mathbf{B} = k^{-1}\mathbf{K} + \theta\tilde{\mathbf{A}}$ and a known right-hand side vector $\underline{\tilde{b}}^m$.

For a standard finite element basis, the matrix $\mathbf{B}$ has a condition number of order h^{-2} for small h and fixed k. For the matrix $\mathbf{B}$ in the wavelet basis we can achieve a uniformly bounded

condition number if we scale the rows and columns of $\mathbf{B}$ as follows: let $\mu_l := (k^{-1} + \theta 2^{2l})^{1/2}$ and let $\hat{B}_{(l,j),(l',j')} := \mu_l^{-1}\mu_{l'}^{-1} B_{(l,j),(l',j')}$. Let in what follows $\|\cdot\|$ denote the 2-norm of a vector, or the 2-norm of a matrix.
Let $\mathbf{D}$ denote the diagonal matrix with entries $D_{(l,j),(l,j)} = 2^l$. Scaling with the diagonal matrix $\mathbf{S} := (k^{-1}\mathbf{I} + \theta\mathbf{D}^2)^{1/2}$ yields with $\hat{\mathbf{B}} = \mathbf{S}^{-1}\mathbf{B}\mathbf{S}^{-1}$

$$\lambda_{\min}\big((\hat{\mathbf{B}} + \hat{\mathbf{B}}^\top)/2\big) \geq C_1, \qquad \left\|\hat{\mathbf{B}}\right\| \leq C_2$$

for some C_1, $C_2 > 0$ independent of h and k which implies convergence of the GMRES with rate independent of L (see [27]).
For a function $v_h \in V_h$ with coefficient vector $\underline{v}$ and scaled coefficient vector $\hat{\underline{v}} = \mathbf{S}\underline{v}$ we have that with $b(u,v) := k^{-1}(u,v) + \theta\tilde{a}_R(u,v)$ and $\|v\|_b^2 := b(v,v)$

$$\|\underline{\hat{v}}\|^2 \sim \hat{\underline{v}}^\top \hat{\mathbf{B}} \hat{\underline{v}} = \|v_h\|_b^2 .$$

A functional $g_h \in V_h^*$ corresponds to a coefficient vector $\underline{g}$ so that $(g_h, v_h) = \underline{g}^\top \underline{v}$, and a scaled vector $\hat{\underline{g}} = \mathbf{S}^{-1}\underline{g}$ so that $(g_h, v_h) = \hat{\underline{g}}^\top \hat{\underline{v}}$.

We now define the **perturbed θ-scheme with GMRES approximation** as follows: Pick a value $m_0 \geq 1$ for the restart number, e.g., $m_0 = 1$, and a value n_G for the number of iterations. At each time step we want to find an approximation of $w_{h,*}^m$ satisfying

$$b(w_{h,*}^m, v_h) = (f^{m+\theta}, v_h) - \tilde{a}_R(\check{U}_h^m, v_h) \quad \text{for all } v_h \in V_h, \quad \check{U}_h^0 = 0,$$

which corresponds to a scaled linear system $\hat{\mathbf{B}}\hat{\underline{w}}_*^m = \hat{\underline{b}}^m$. We solve this system approximately with n_G steps of GMRES(m_0), using zero as initial guess, yielding an approximation $\hat{\underline{w}}^m$ of the exact solution $\hat{\underline{w}}_*^m$. We then let $\check{U}_h^{m+1} := \check{U}_h^m + w_h^m$, where $w_h^m \in V_h$ is the function corresponding to the scaled vector $\hat{\underline{w}}^m$. Then we have, see Theorem 6.3 in [27]

Proposition 5.6 *Assume that the consistency conditions (5.10), (5.12) hold. For $\theta \in [0, \frac{1}{2})$ assume $\sigma := k(1-2\theta)\lambda_A < 2$. Then the solution $\check{U}_h^m$ of the θ-scheme with wavelet compression and approximate GMRES solution satisfies the same error bound as $\tilde{U}_h^m$ in (5.18) if $n_G \geq C\,|\log h|$. Given the compressed stiffness matrix $\tilde{\mathbf{A}}$, the work for computing $\check{U}_h^1, \ldots, \check{U}_h^M$ is bounded by $CMN(\log N)^2$ floating point operations.*

5.5 Numerical results

We restrict the numerical experiments to the case when the risk-neutral interest rate $r = 0$, see also Section 2.3.2. In Figure 3 we present the option prices versus the stock price S for the case of an European call contract on Lévy driven assets. We use different maturities (top) and different strike prices K (bottom) for an extended CGMY process [9] with $\sigma = 0.1$, $C = 1$, $G = 1.8$, $M = 2.5$ and $Y = 0.2$. We plot for each case (top right and top bottom, respectively) the difference between the option prices in the jump-diffusion case and the prices obtained by the standard Black-Scholes formula (only diffusion) with $\sigma = 0.1$.
In Figure 4 we plot the option prices versus the stock price S for the case of an European call contract on pure jump Lévy driven assets ($\sigma = 0$) at different maturities (left and right (zoom)); CGMY parameters are: $Y = 0.1430$, $C = 9.61$, $G = 9.97$ and $M = 16.51$ (see [9]).
In the next set of numerical experiments we consider the variance gamma process as particular case of the CGMY process with $Y = 0$. Here explicit formulas for the prices of European

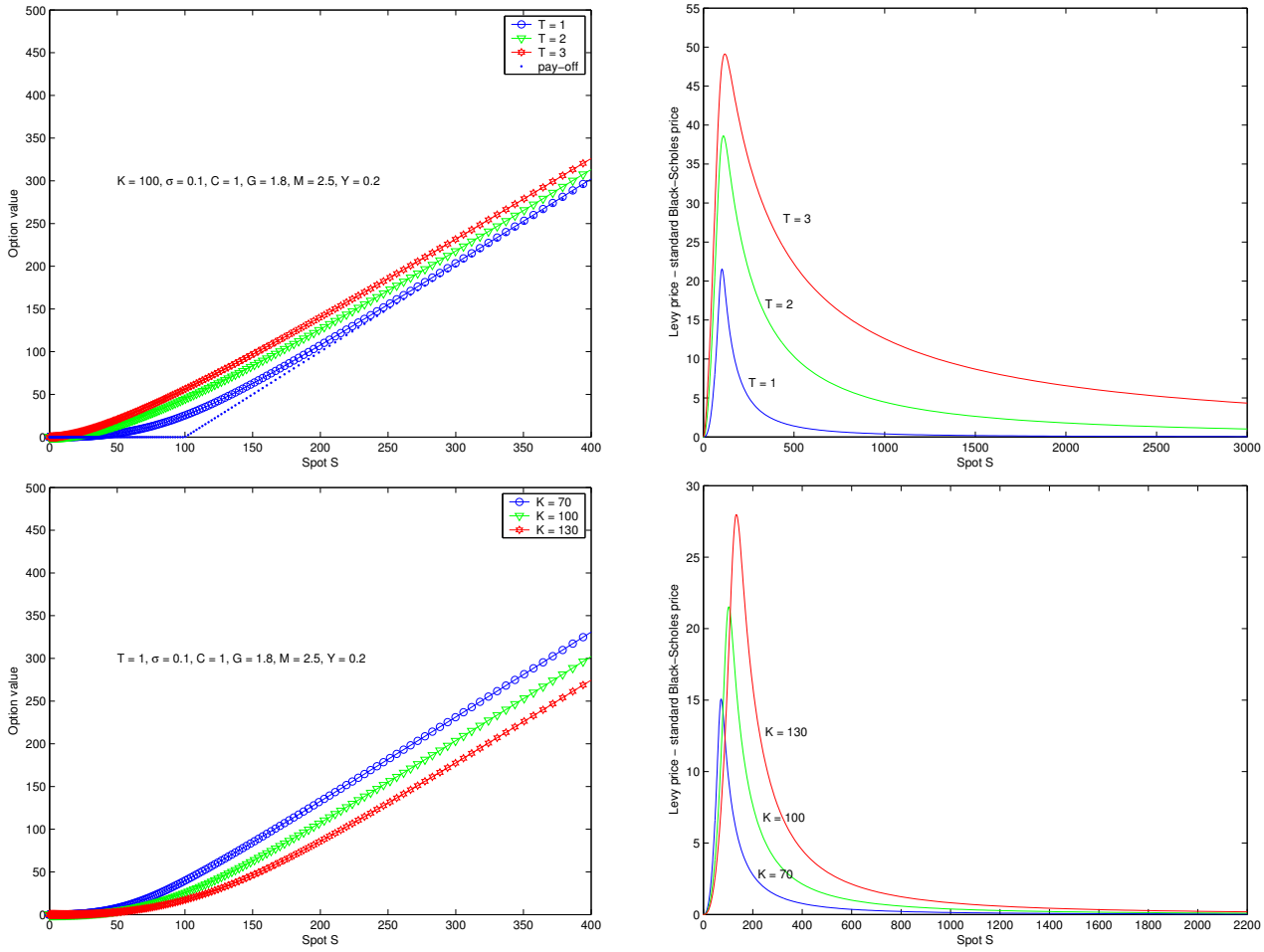

Figure 3: Option prices versus the stock price S for the case of an European call contract on Lévy driven assets as compared to the Black-Scholes prices; different maturities (top) and different strike prices K (bottom) for the case of an extended CGMY process with $\sigma = 0.1$, $Y = 0.2$, $G = 1.8$ and $M = 2.5$.

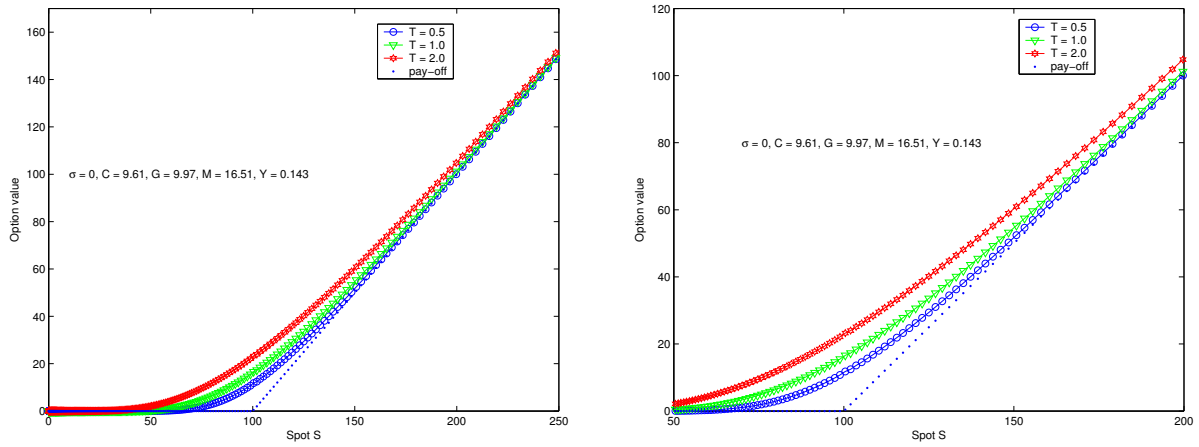

Figure 4: Option prices versus the stock price S for the case of an European call contract on pure jump Lévy driven assets ($\sigma = 0$) at different maturities (left and right (zoom)); CGMY parameters are: $Y = 0.1430$, $C = 9.61$, $G = 9.97$ and $M = 16.51$.

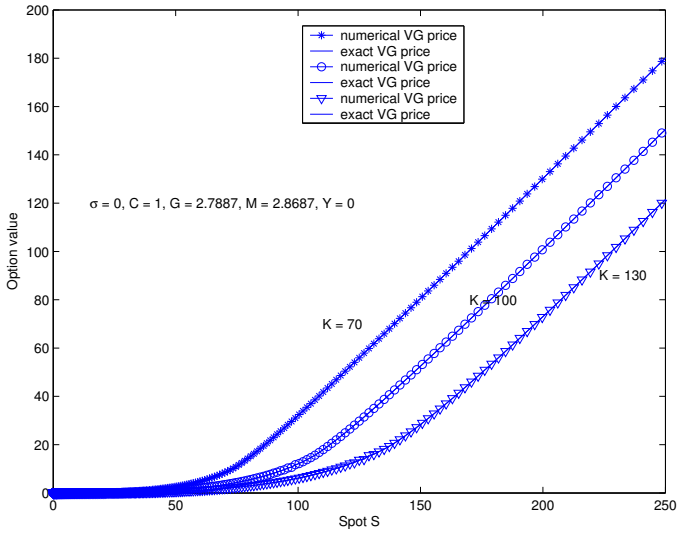

Figure 5: Option prices versus the stock price S for the case of an European call contract on VG (pure jump) driven assets for different strike prices K at maturity $T = 0.5$; VG parameters: $\sigma_{VG} = 0.5$, $\nu_{VG} = 1.0$, $\theta_{VG} = -0.01 \Rightarrow$ CGMY parameters: $Y = 0$, $G = 2.78$ and $M = 2.86$.

options are available [23]. The VG parameters from (3.4) are here $\sigma_{VG} = 0.5$, $\nu_{VG} = 1.0$, $\theta_{VG} = -0.01$, corresponding to the CGMY parameters $Y = 0$, $G = 2.78$, $M = 2.86$ and $C = 1.0$. In Figure 5 we compare our numerical results obtained with the exact VG prices obtained by the explicit formulae in [23] for different strike prices K and maturity $T = 0.5$. The computed values are at the top of the exact price values obtained by the explicit formula in [23].

Figure 6 shows pricing of options with the forward Euler scheme, i.e. with $\theta = 0$. We clearly see the impact of the CFL-condition (5.20) – if it is violated, instability results. In the jump-diffusion case, the time-step restriction (5.20) renders the explicit scheme inefficient. In the pure jump case, however, CFL-condition (5.21) yields a competitive scheme for $Y \leq 1$; again

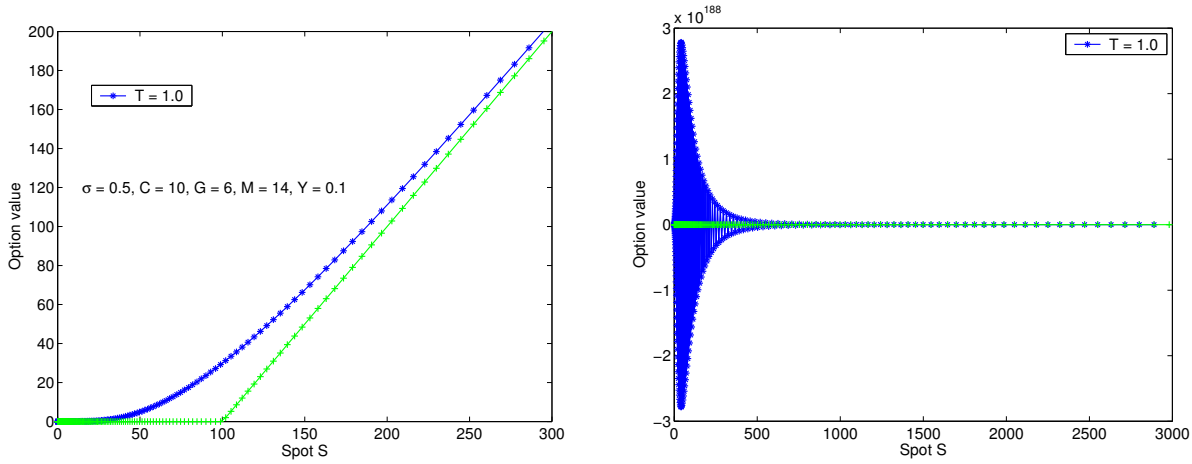

Figure 6: Explicit Euler scheme ($\theta = 0.0$), $h = 0.0312$ ($L = 8$, $R = 8$), $T = 1.0$; CGMY parameters: $C = 10.0$, $G = 6.0$, $M = 14.0$, $Y = 0.1$, $\sigma = 0.5$; stable $k = h^2$ (left), unstable: $k = 2h^2$

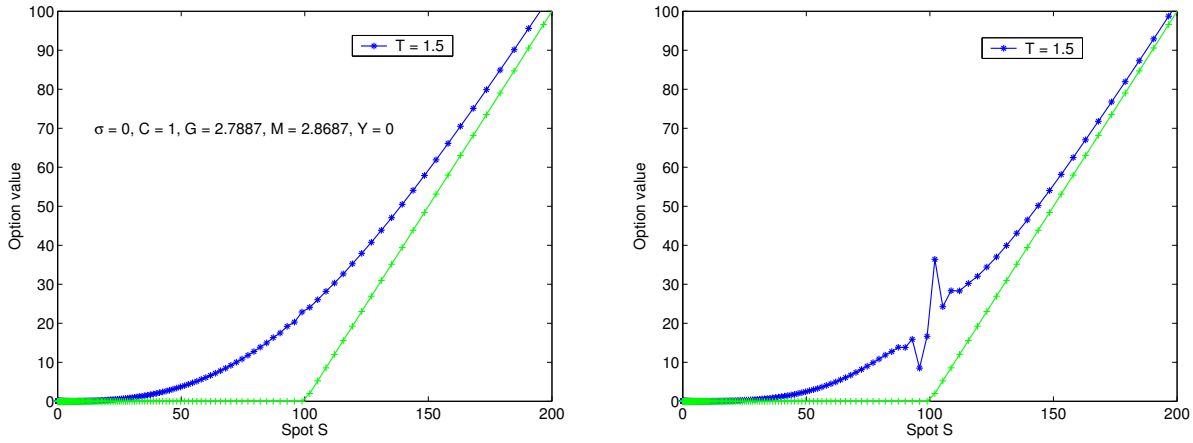

Figure 7: Explicit Euler scheme, pure jump VG process: $\theta = 0.0$, $L = 8$; α is the coefficient in the convection term $\alpha \frac{\partial u}{\partial x}$. $|\alpha|\frac{k}{h} \leq 1$ stable; $|\alpha|\frac{k}{h} > 1$ unstable VG parameters: $\sigma_{VG} = 0.5$, $\nu_{VG} = 1.0$, $\theta_{VG} = -0.01 \Rightarrow$ CGMY parameters: $Y = 0$, $G = 2.78$ and $M = 2.86$.

the condition (5.21) is sharp, as is evidenced by Figure 7 right and Figure 8.

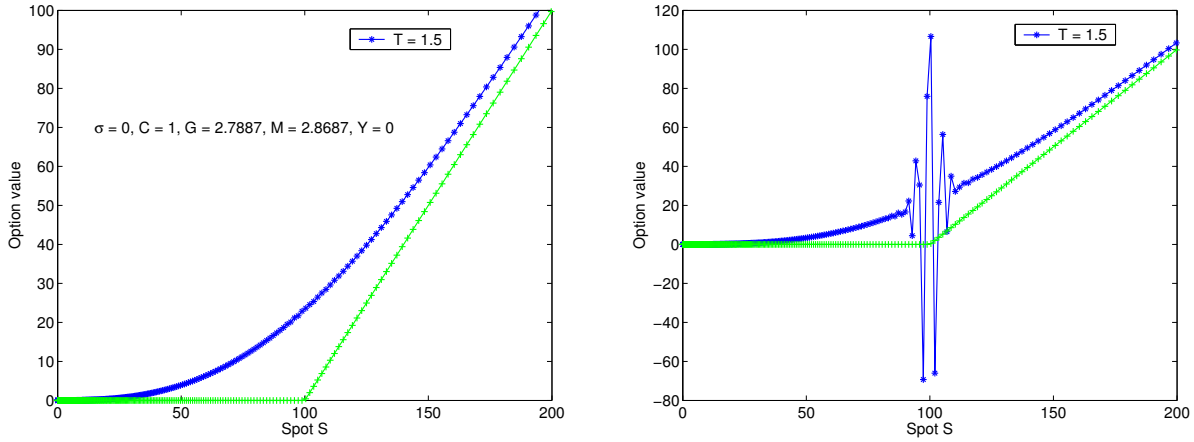

Figure 8: Explicit Euler scheme, pure jump VG process: $\theta = 0.0$, $L = 9$; $|\alpha|\frac{k}{h} \leq 1$ (left) stable, $|\alpha|\frac{k}{h} > 1$ (right) unstable; VG parameters: $\sigma_{VG} = 0.5$, $\nu_{VG} = 1.0$, $\theta_{VG} = -0.01 \Rightarrow$ CGMY parameters: $Y = 0$, $G = 2.78$ and $M = 2.86$.

A Proof of Theorem 3.3

We consider φ, $\psi \in C_0^\infty(\mathbb{R})$. Integration by parts in the definition of $a^{-\eta}$ in (3.22) gives

$$
\begin{aligned}
a^{-\eta}(\varphi,\psi) &= -\frac{\sigma^2}{2}\int_{\mathbb{R}}\left(\frac{d^2\varphi}{dx^2}(x) - \frac{d\varphi}{dx}(x)\right)\psi(x)e^{-2\eta(x)}dx \\
&\quad - \int_{\mathbb{R}}\int_{\mathbb{R}}\left\{\varphi(x+y) - \varphi(x) - y\frac{d\varphi}{dx}(x)\chi_{\{|y|\leq 1\}}(y)\right\}k(y)\psi(x)e^{-2\eta(x)}dydx \\
&\quad + c_{\exp}\int_{\mathbb{R}}\frac{d\varphi}{dx}(x)\psi(x)e^{-2\eta(x)}dx \\
&= \frac{\sigma^2}{2}\int_{\mathbb{R}}\frac{d\varphi}{dx}\frac{d\psi}{dx}(x)e^{-2\eta(x)}dx + \frac{\sigma^2}{2}\int_{\mathbb{R}}\frac{d\varphi}{dx}(x)\psi(x)(-2\eta'(x)+1)e^{-2\eta(x)}dx \\
&\quad - \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1 d\theta\frac{d\varphi}{dx}(x+\theta y)y\chi_{\{|y|\geq 1\}}(y)k(y)\psi(x)e^{-2\eta(x)}dxdy \\
&\quad - \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1 d\theta\int_0^\theta d\theta'\frac{d^2\varphi}{dx^2}(x+\theta' y)y^2\chi_{\{|y|\leq 1\}}(y)k(y)\psi(x)e^{-2\eta(x)}dxdy \\
&\quad + c_{\exp}\int_{\mathbb{R}}\frac{d\varphi}{dx}(x)\psi(x)e^{-2\eta(x)}dx.
\end{aligned}
$$

We write therefore $a^{-\eta}(\varphi,\psi) = a_1^{-\eta}(\varphi,\psi) + a_2^{-\eta}(\varphi,\psi) + a_3^{-\eta}(\varphi,\psi)$, where

$$
\begin{aligned}
a_1^{-\eta}(\varphi,\psi) &= \frac{\sigma^2}{2}\int_{\mathbb{R}}\frac{d\varphi}{dx}\frac{d\psi}{dx}(x)e^{-2\eta(x)}dx + \frac{\sigma^2}{2}\int_{\mathbb{R}}\frac{d\varphi}{dx}(x)\psi(x)(-2\eta'(x)+1)e^{-2\eta(x)}dx, \\
a_2^{-\eta}(\varphi,\psi) &= -\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1 d\theta\frac{d\varphi}{dx}(x+\theta y)y\chi_{\{|y|\geq 1\}}(y)k(y)\psi(x)e^{-2\eta(x)}dxdy, \\
a_3^{-\eta}(\varphi,\psi) &= -\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1 d\theta\int_0^\theta d\theta'\frac{d^2\varphi}{dx^2}(x+\theta' y)y^2\chi_{\{|y|\leq 1\}}(y)k(y)\psi(x)e^{-2\eta(x)}dxdy \\
&\quad + c_{\exp}\int_{\mathbb{R}}\frac{d\varphi}{dx}(x)\psi(x)e^{-2\eta(x)}dx.
\end{aligned}
$$

We analyse each $a_j^\eta(\cdot,\cdot)$ $(j=1,2,3)$ separately. For the bilinear form $a_1^{-\eta}$ we obtain

$$|a_1^{-\eta}(\varphi,\psi)| \le C_1(\|\eta'\|_{L^\infty(\mathbb{R})})\|\varphi\|_{H^1_{-\eta}(\mathbb{R})}\|\psi\|_{H^1_{-\eta}(\mathbb{R})}$$

and

$$a_1^{-\eta}(\varphi,\varphi) \ge \frac{\sigma^2}{2}\|\varphi'\|^2_{L^2_{-\eta}(\mathbb{R})} - \frac{\sigma^2}{2}(2\|\eta'\|_{L^\infty(\mathbb{R})}+1)\|\varphi'\|_{L^2_{-\eta}(\mathbb{R})}\|\varphi\|_{L^2_{-\eta}(\mathbb{R})} \tag{A.1}$$

$$\ge \frac{\sigma^2}{4}\|\varphi'\|^2_{L^2_{-\eta}(\mathbb{R})} - c_1(\|\eta'\|_{L^\infty(\mathbb{R})})\|\varphi\|^2_{L^2_{-\eta}(\mathbb{R})}. \tag{A.2}$$

In order to estimate $a_2^{-\eta}(\cdot,\cdot)$ we write it first in the following form

$$a_2^{-\eta}(\varphi,\psi) = -\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1 d\theta \frac{d\varphi}{dx}(x+\theta y)e^{-\eta(x+\theta y)}e^{\eta(x+\theta y)-\eta(x)}y\chi_{\{|y|\ge 1\}}(y)k(y)\psi(x)e^{-\eta(x)}\,dxdy.$$

By (3.23) we obtain

$$|a_2^{-\eta}(\varphi,\psi)| \le \int_{\mathbb{R}} e^{\eta(y)}|y|\chi_{\{|y|\ge 1\}}(y)k(y)\,dy\cdot\|\phi'\|_{L^2_{-\eta}(\mathbb{R})}\|\psi\|_{L^2_{-\eta}(\mathbb{R})}.$$

Hence, since $C(\eta) := \int_{\mathbb{R}} e^{\eta(y)}|y|\chi_{\{|y|\ge 1\}}(y)k(y)\,dy < \infty$ by assumption (3.24), $a_2^{-\eta}(\cdot,\cdot)$ is a bounded bilinear form on $H^1_{-\eta}(\mathbb{R})\times H^1_{-\eta}(\mathbb{R})$.

Finally, we analyse $a_3^{-\eta}(\cdot,\cdot)$ and we show that for all $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$|a_3^{-\eta}(\varphi,\psi)| \le \|\varphi\|_{H^1_{-\eta}(\mathbb{R})}(\varepsilon\|\psi\|_{H^1_{-\eta}(\mathbb{R})} + C_\varepsilon\|\psi\|_{L^2_{-\eta}(\mathbb{R})}). \tag{A.3}$$

To prove (A.3), let $\delta\in(0,1)$ be arbitrary, but fixed. We start by writing $a_3^{-\eta}(\cdot,\cdot)$ in the following form:

$$\begin{aligned}
a_3^{-\eta}(\varphi,\psi) &= -\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1 d\theta\int_0^\theta d\theta'\frac{d^2\varphi}{dx^2}(x+\theta' y)y^2\chi_{\{|y|\le\delta\}}(y)k(y)\psi(x)e^{-2\eta(x)}dxdy\\
&\quad -\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1 d\theta\frac{d\varphi}{dx}(x+\theta y)y\chi_{\{\delta\le|y|\le 1\}}(y)k(y)\psi(x)e^{-2\eta(x)}dx\\
&\quad +\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{d\varphi}{dx}(x)\psi(x)y\chi_{\{\delta\le|y|\le 1\}}(y)k(y)\psi(x)e^{-2\eta(x)}dx\\
&\quad +c_{\exp}\int_{\mathbb{R}}\frac{d\varphi}{dx}(x)\psi(x)e^{-2\eta(x)}dx\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1 d\theta\int_0^\theta d\theta'\frac{d\varphi}{dx}(x+\theta' y)e^{-\eta(x+\theta' y)}y^2\chi_{\{|y|\le\delta\}}(y)k(y)\cdot\\
&\quad \cdot\{\frac{d\psi}{dx}(x)-2\eta'(x)\psi(x)\}e^{\eta(x+\theta' y)-\eta(x)}e^{-\eta(x)}\,dxdy\\
&\quad -\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1 d\theta\frac{d\varphi}{dx}(x+\theta y)e^{-\eta(x+\theta y)}y\chi_{\{\delta\le|y|\le 1\}}(y)k(y)e^{\eta(x+\theta y)-\eta(x)}\psi(x)e^{-\eta(x)}dxdy\\
&\quad +\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{d\varphi}{dx}(x)\psi(x)y\chi_{\{\delta\le|y|\le 1\}}(y)k(y)\psi(x)e^{-2\eta(x)}dx\\
&\quad +c_{\exp}\int_{\mathbb{R}}\frac{d\varphi}{dx}(x)\psi(x)e^{-2\eta(x)}dx.
\end{aligned}$$

By(3.23), $e^{\eta(x+\theta y)-\eta(x)} \le e^{\eta(y)} \le C$ for all $|y| \le 1$. We obtain therefore the following estimate

$$|a_3^{-\eta}(\varphi,\psi)| \le C_3(\|\eta'\|_{L^\infty(\mathbb{R})}) \int_{\mathbb{R}} y^2 \chi_{\{|y|\le\delta\}}(y)k(y)dy \cdot \|\varphi\|_{H^1_{-\eta}(\mathbb{R})}\|\psi\|_{H^1_{-\eta}(\mathbb{R})}$$
$$+ \left(C\int_{\mathbb{R}} |y|\chi_{\{\delta\le|y|\le 1\}}(y)k(y)dy + |c_{\exp}|\right) \cdot \|\varphi'\|_{L^2_{-\eta}(\mathbb{R})}\|\psi\|_{L^2_{-\eta}(\mathbb{R})}.$$

Since by (3.3) $\int_{\mathbb{R}} y^2\chi_{\{|y|\le\delta\}}(y)k(y)dy \to 0$ as $\delta \to 0$, for $\varepsilon > 0$ fixed we can choose $\delta = \delta(\varepsilon) \in (0,1)$ sufficiently small such that (A.3) holds.
The above calculations with η replaced by $-\eta$ lead to identical conclusions, if instead of (3.23) (3.25) holds and if condition (3.24) is replaced by (3.26).

□

Remark A.1 (3.23)–(3.24) hold for all η of the form

$$\eta(x) = \begin{cases} \nu_1|x| & \text{if } x < 0 \\ \nu_2|x| & \text{if } x > 0. \end{cases} \tag{A.4}$$

with $0 \le \nu_1 < G$ and $0 \le \nu_2 < M$. We distinguish 4 cases

Case 1. $(x+\theta y > 0,\ x > 0)$. Then, $\nu_2(x+\theta y) - \nu_2 x = \nu_2\theta y \le$ a) $-\nu_1 y$, if $y < 0$ b) $\nu_2 y$, if $y > 0$.

Case 2. $(x+\theta y > 0,\ x < 0)$. Here, $\nu_2(x+\theta y) + \nu_1 x = (\nu_1+\nu_2)x + \nu_2\theta y \le \nu_2\theta y \le \nu_2 y$ $(= \eta(y)$, since $y > 0$ in this case).

Case 3. $(x+\theta y < 0,\ x > 0)$. Then, $-\nu_1(x+\theta y) - \nu_2 x = -(\nu_1+\nu_2)x - \nu_1\theta y \le -\nu_1 y$ $(= \eta(y)$, since $y < 0$ here).

Case 4. $(x+\theta y < 0,\ x < 0)$. Here, $-\nu_1(x+\theta y) + \nu_1 x = -\nu_1\theta y \le$ a) $-\nu_1 y$ if $y < 0$ or b) $\nu_2 y$ if $y > 0$.

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York 1978.

[2] H. Amann, *Linear and Quasilinear Parabolic Problems*, Vol. I: Abstract Linear Theory, Monographs Math., Vol. 89, Birkhäuser, Basel 1995.

[3] O.E. Barndorff-Nielsen, *Exponentially decreasing distributions for the logarithm of particle size*, Proceedings of the Royal Society London A 353 (1977) 401–419.

[4] A. Bensoussan, J.-L. Lions, *Impulse control and quasi-variational inequalities*, Gauthier-Villars, Paris 1984.

[5] J. Bertoin, *Lévy processes*, Cambridge Univ. Press, 1996.

[6] F. Black, M. Scholes, *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy **81** (1973) 637–654.

[7] S. Boyarchenko, S. Levendorski, *Barrier Options and Touch-and-Out Options under Regular Lévy Processes of Exponential Type*, to appear in The Annals of Applied Probability, 2002.

[8] S. Boyarchenko, S. Levendorski, *Option pricing for truncated Lévy processes*, Int. Journ. Theor. and Appl. Finance (2000).

[9] P. Carr, H. Geman, D.B. Madan, *The Fine Structure of Asset Returns: An Empirical Investigation*, Journal of Business 2002.

[10] P. Carr and D. Madan, *Option valuation using the FFT*, J. Comp. Finance 2 (1999) 61–73.

[11] T. Chan, *Pricing Contingent Claims on Stocks Driven by Lévy Processes*, The Annals of Applied Probability, (1999) Vol. 9, No. 2, 504–528.

[12] A. Cohen, *Wavelet methods for operator equations*, to appear in "Handbook of Numerical Analysis", 2001.

[13] F. Delbaen, W. Schachermayer, *The Variance-Optimal Martingale Measure for Continuous Processes*, Bernoulli 2 (1996), no. 1, 81–105.

[14] F. Delbaen, P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer, C. Stricker, *Exponential Hedging and Entropic Penalties*, Mathematical Finance, Vol.12, No.2, 99–123.

[15] E. Eberlein, *Application of generalized hyperbolic Lévy motions to finance* in *Lévy Processes: Theory and Applications*, O.E. Barndorff-Nielsen, T. Mikosch, and S. Resnick (Eds.), Birkhäuser (2001), 319 – 337.

[16] E. Eberlein, U. Keller, *Hyperbolic distributions in finance*, Bernoulli **1** (1995) 281–299.

[17] H. Föllmer, M. Schweizer, *Hedging of Contingent Claims under incomplete information*, Applied Stochastic Analysis, M.H.A. Davis and R.J. Elliot, Eds., (1991) Gordon and Breach New York, 389–414.

[18] J. Jacod, A.N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin 1987.

[19] P. Jaillet, D. Lamberton, B. Lapeyre, *Variational Inequalities and the Pricing of American Options* Acta Applicandae Mathematicae **21** (1990) 263–289.

[20] R. Kangro, R. Nicolaides, *Far Field Boundary Conditions for Black-Scholes Equations*, SIAM J. Numer. Anal. **38** (2000) 1357–1368.

[21] I. Karatzas, S.E. Shreve *Methods of Mathematical Finance*, Springer Verlag 1999.

[22] D. Lamberton, B. Lapeyre, *Introduction to Stochastic Calculus Applied to Finance*, Chapman & Hall 1997.

[23] D.B. Madan, P. Carr, E. Chang, *The Variance Gamma Process and Option Pricing*, European Finance Review **2** (1998) 79–105.

[24] D.B. Madan, E. Seneta, *The variance gamma (V.G.) model for share market returns*, Journal of Business **63**(4) (1990) 511–524.

[25] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences **44** Springer Verlag, New-York 1983.

[26] T. von Petersdorff, R. Schneider, C. Schwab, *Multiwavelets for second kind integral equations*, SIAM J. Num. Anal. **34** (1997), 2212–2227.

[27] T. von Petersdorff, C. Schwab, *Wavelet-discretizations of parabolic integro-differential equations*, SIAM Journal on Numerical Analysis **41** (2003) Number 1, 159–180.

[28] K. Prause, *The Generalized Hyperbolic Model: Estimation, Financial Derivatives, and Risk Measures*, Ph.D. Thesis Albert-Ludwigs-Universität Freiburg i.Br. 1999.

[29] P. Protter, *Stochastic Integration and Differential Equations*, Springer Verlag 1990.

[30] S. Raible, *Lévy processes in Finance: Theory, Numerics, and Empirical Facts*, Ph.D. Thesis Albert-Ludwigs-Universität Freiburg i.Br. 2000.

[31] K.-I. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, 1999.

[32] D. Schötzau, C. Schwab, *hp-discontinuous Galerkin time-stepping for parabolic problems*, C.R.Acad. Sci. Paris, t. 333, Série I (2001) 1121–1126.

[33] L. Schwartz, *Theorie des distributions*, Hermann & Cie., Paris 1957.

[34] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer Verlag 1997.