- Quality of Value-at-Risk Approximations by Numerical Solutions of SDEs -

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This work is a joint collaboration between the RiskLab and INRIA Sophia Antipolis. (Institut National de Recherche en Informatique et en Automatique)

- What is VaR? -

Value at Risk is the loss, which is exceeded with some given probability α, over a given horizon.

Mathematically, Value at Risk is the quantile of a random variable, which describes the possible loss in the future.

- A typical example -

A financial market consists in $N + M + 1$ securities: N stocks, M bonds and an instantaneous riskless saving account.

An investment activity is modelled by a $N + M + 2$ dimensional stochastic differential equation, whose solution represents the prices of N stocks, the prices of M bonds, the saving account and the wealth process $X(\cdot)$ of the trader's portfolio.

The $VaR(\alpha)$ for this investment is the largest value such that

$$\mathbb{P}\left(X(T) - X(0) \leq VaR(\alpha)\right) \leq \alpha.$$

- Calculation of VaR -

The best way to calculate VaR, of course, would be to use an analytic formula.

For a model whose coefficients are complex functions or a high dimensional model, it is impossible to calculate the VaR explicitly.

Solution: Approximate the VaR by numerical methods.

- Evaluating existing methods -

When one chooses a numerical method for VaR calculation, one has to make a tradeoff between accuracy, numerical cost and generality. We give some comments on some standard methods.

Delta method, Delta-gamma method and related methods are fast but not very accurate, and they require a simple model (which decreases the global accuracy).

Full Monte Carlo method gives a better accuracy, but it requires exact pricing formula (which restricts the choice of the models), and is time-consuming.

- Evaluating existing methods (cont.) -

The Grid Monte Carlo method gives good accuracy and can be applied to general nonlinear models, but we recommend it for low dimensional problems only, for a portfolio with a large number of securities, the numerical cost is very expensive.

- What is the suitable numerical method to calculate VaR? -

May we find a new method, which is accurate, can be applied to general nonlinear and high dimensional models, and to long-term risk measurement problems?

We emphasize that, in the stochastic model as above,

The VaR under interest is the quantile of the marginal distribution of the solution at horizon T to a stochastic differential equation.

Thus the problem is: find a suitable numerical method to approximate such quantiles.

- The answer -

The Monte Carlo method.

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The Euler scheme.

- What is the Euler scheme? -

Let (X_t) be a real valued diffusion process, solution to

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

where (W_t) is a r-dimensional Brownian motion.

The Euler scheme is a time discretization of the SDE:

$$X^n_{(p+1)T/n} = X^n_{pT/n} + b(X^n_{pT/n})\frac{T}{n} + \sigma(X^n_{pT/n})(W_{(p+1)T/n} - W_{pT/n}).$$

n is the number of steps of the discretization and $p = 0,1,\ldots,n-1$.

X^n_T is a good approximation of X_T.

- What is the Monte Carlo method? -

The law of X_T^n is too complex to calculate explicitly.

We use N i.i.d. copies of X_T^n to approximate $\mathbb{E}\left[f(X_T)\right]$.

$$\frac{1}{N}\sum_{i=1}^{N} f(X_T^{n,i}) \to \mathbb{E}\left[f(X_T)\right], \text{ as } n,N \to \infty.$$

The left hand side can be easily simulated on a computer: one simply has to simulate the independent Gaussian increments of the Brownian motion.

- The discretization error -

Under hypotheses of uniformly hypoellipticity type, X_T has a smooth density p_{X_T}. Therefore given a positive real $0 < \delta < 1$, there exists a quantile $\rho(\delta)$ such that

$$\mathbb{P}\left[X_T \leq \rho(\delta)\right] = \delta.$$

For the mollified Euler scheme, X_T^n also has a smooth density, thus there exists a quantile $\rho^n(\delta)$ such that

$$\mathbb{P}\left[X_T^n \leq \rho^n(\delta)\right] = \delta.$$

- The discretization error (cont.) -

We have proved that the discretization error on the quantile satisfies

$$|\rho^n(\delta) - \rho(\delta)| \leq \frac{C(T)}{q_T(\delta)n},$$

where

$$q_T(\delta) = \inf_{y \in [\rho(\delta)-1, \rho(\delta)+1]} p_{X_T}(y) \simeq p_{X_T}(\rho(\delta)).$$

- The statistical error -

We can sort the simulated $\left(X_T^{n,i}, i = 1,\dots,N\right)$, and thus get the empirical quantile $\rho_N^n(\delta)$.

The classical theory tell us that the statistical error is of order

$$|\rho^n(\delta) - \rho_N^n(\delta)| \sim \frac{1}{q_T(\delta)\sqrt{N}}.$$

- The global error of the method -

Thus the global error of the simulated quantile satisfies

$$\mathbb{E}\left|\rho(\delta)-\rho_N^n(\delta)\right| \leq \frac{C(T)}{q_T(\delta)\sqrt{N}}+\frac{C(T)}{q_T(\delta)n}.$$

For a practical use, one needs an accurate lower bound of the density of X_T in order to choose n,N in terms of the desired accuracy. For strictly uniform elliptic generators, see, e.g., Azencott. In degenerate case, under restrictive assumption on the drift b, see Kusuoka & Stroock.

- The case of high dimension and marginal law -

Let (X_t) be a $\mathbb{R}^d$ valued process, solution to

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t.$$

Denote by (X_t^i) the coordinate process of (X_t). Suppose the generator of (X_t) is uniformly hypoelliptic, then there exists a quantile $\rho^i(\delta)$ such that

$$\mathbb{P}\left[X_T^i \leq \rho^i(\delta)\right] = \delta.$$

and a quantile $\rho^{n,i}(\delta)$ of the Euler scheme such that

$$\mathbb{P}\left[X_T^{n,i} \leq \rho^{n,i}(\delta)\right] = \delta.$$

- The case of a marginal law (cont.) -

We have proved

$$|\rho^{n,i}(\delta) - \rho^{i}(\delta)| \leq \frac{C(T)}{q^{i}_{T}(\delta) n}.$$

$$q^{i}_{T}(\delta) = \inf_{y \in [\rho^{i}(\delta)-1, \rho^{i}(\delta)+1]} p^{i}_{X_T}(y) \simeq p^{i}_{X_T}(\rho^{i}(\delta)),$$

where $p^{i}_{X_T}$ is the i-th marginal distribution of X_T.

We recall that the situation here is exactly the one we gave in the example for the portfolio of $N + M + 1$ securities.

- Conclusion -

We conclude by listing some properties of our numerical approach:

1. A desired accuracy can be achieved by choosing N and n properly.

2.Our error estimates hold for hypoelliptic SDE, which is a common situation in finance.

3. The numerical cost grows only linearly w.r.t the dimension. Moreover, the numerical cost can be reduced by using parallel architectures.

4. It can be applied to long term risk measurement problems.

- An application to Model Risk measurement -

Given two maturities: $T^O < T$.

The trader wants to hedge a European option with maturity T^O written on a discount bond $B(t,T)$. The payoff at maturity is denoted by $\Phi(B(T^O,T))$.

The trader uses two bonds to hedge the option: the bond with maturity T^O and the bond of maturity T.

- The Heath-Jarrow-Morton model -

Instantaneous forward rate under the spot martingale measure:

$$f(t,T^*) = f(0,T^*) + \int_0^t \sigma(s,T^*)\sigma^*(s,T^*)ds + \int_0^t \sigma(s,T^*)dW_s,$$

with

$$\sigma^*(s,T^*) := \int_s^{T^*} \sigma(s,u)du.$$

Discount bond price $B(t,T)$:

$$B(t,T) = 1 - \int_t^T r(s)B(s,T)ds + \int_t^T \sigma^*(s,T)B(s,T)dW_s,\ 0 \leq t \leq T.$$

- The portfolio -

The portfolio

$$V_t = H_t B(t,T) + H_t^O B(t,T^O).$$

Assumption: The portfolio is self-financing, i.e.,

$$V_t = V_0 + \int_0^t H_s dB(s,T) + \int_0^t H_s^O dB(s,T^O).$$

- The forward prices of the bond and of the portfolio -

Forward price of the bond:

$$B^F(t,T) := \frac{B(t,T)}{B(t,T^O)}.$$

Forward price of the portfolio:

$$V_t^F := \frac{V_t}{B(t,T^O)} = \frac{H_t B(t,T) + H_t^O B(t,T^O)}{B(t,T^O)}.$$

From the self-financing condition

$$dV_t^F = H_t dB^F(t,T).$$

- Exact hedging strategy in the case: $\sigma(t,T)$ is deterministic -

A self-financing exact hedging strategy is a pair (H_t^0, H_t)

$$H_t = \frac{\partial \pi_\sigma}{\partial x}(t, B^F(t,T)),$$

where the forward price of the option π_σ is the solution to a Black-Scholes type equation,

$$\begin{cases} \frac{\partial \pi_\sigma}{\partial t}(t,x) + \frac{1}{2} x^2 (\sigma^*(t,T^O) - \sigma^*(t,T))^2 \frac{\partial^2 \pi_\sigma}{\partial x^2}(t,x) = 0, \\ \pi_\sigma(T,x) = \Phi(x). \end{cases}$$

From the self-financing condition,

$$H_t^0 = V_0^F + \int_0^t H_s dB^F(s,T) - H_t B^F(t,T).$$

- Model risk measurement -

A bad choice or estimate $\overline{\sigma}$ of σ leads to a wrong hedging strategy

$$\overline{H}_t = \frac{\partial \pi_{\overline{\sigma}}}{\partial x}(t, B^F(t,T)).$$

The portfolio using the misspecified hedging strategy:

$$\overline{V}_t^F = V_0^F + \int_0^t \overline{H}_t dB^F(t,T).$$

Forward price of Profit & Loss:

$$P\&L_t^F := \overline{V}_t^F - V_t^F.$$

Model risk measurement: Quantile, $\mathbb{E}\left[U(P\&L_{TO})\right]$, (see Artzner, Delbaen, Eber and Heath.), etc.

- Quantile of model risk -

$$\mathbb{P}\left[P\&L_{T^O} \leq \rho(\delta)\right] = \delta.$$

Set

$$\begin{cases} u_1(t) := \sigma^*(t,T^O), \\ u_2(t) := (\sigma^*(t,T^O) - \sigma^*(t,T)), \\ \varphi(t,x) := \dfrac{\partial \pi_{\overline{\sigma}}}{\partial x}(t,x) - \dfrac{\partial \pi_{\sigma}}{\partial x}(t,x). \end{cases}$$

Thus $P\&L_t^F$ is the solution to

$$\begin{cases} dB^F(t,T) = B^F(t,T)u_1(t)u_2(t)dt + B^F(t,T)u_2(t)dW_t, \\ dP\&L_t^F = \varphi(t,B^F(t,T))dB^F(t,T). \end{cases}$$

- Existence of a smooth density of $P\&L_{T^O}$. -

Suppose that $|u_2(t)| \geq a > 0$, for any t in $[0,T^O]$, and $\varphi(0,B^F(0,T)) \neq 0$, then the law of $P\&L_{T^O}$ has a smooth density p_{T^O} (marginal law of the Markov process $(B^F(t,T),P\&L^F_t)$). We also have the strict positivity of p_{T^O} on its support. Moreover, we can obtain upper bounds on the derivatives of p_{T^O}.

Proof. We study the Malliavin variance $\langle D(P\&L^F_{T^O}),D(P\&L^F_{T^O})\rangle$.

- The lower bound. -

By adding one technical assumption, we obtain

$$p_t(y_0,y) \geq \frac{1}{\sqrt{2\pi \int_0^{T^O} u_2^2(s)ds}}$$

$$\exp(-\frac{(\ln(\Upsilon^{-1}(T^O,y - y_0 + \Upsilon(0,x_0)) - \ln x_0) - \int_0^{T^O}(u_1(s) - \frac{1}{2}u_2^2(s))ds)^2}{4\int_0^{T^O} u_2^2(s)ds} - C),$$

where $\Upsilon(t,y) := \int_0^y \varphi(t,x)dx$. and $\Upsilon^{-1}(t,\cdot)$ is the inverse function of $\Upsilon(t,\cdot)$.

Proof. We apply the technique of Girsanov transformation.

- Conclusion -

From our study on the density of the Profit & Loss, we obtain a precise estimate on the global error of the Monte Carlo method for the approximation of the quantiles of the Profit & Loss from misspecified hedging strategies.

Our methodology can be applied to wide class of problems from Risk measurement and VaR analysis...