Combined Stress Scenarios for Market and Credit Risks

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Generation of Random Stress Scenarios
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A Company's Value in Respect to the Merton Model/Credit Metrics
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1. Multi-firm Merton model

1.1 The asset value model

Portfolio of n firms with time horizon T.

- $S_t^{(i)}$: equity value of firm $i \in \{1, \ldots, n\}$ at time $t \in [0, T]$,
- $F_t^{(i)}$: value of the liabilities of firm i at time $t \in [0, T]$,
- $V_t^{(i)}$: value of firm i's assets at time $t \in [0, T]$.

Assumptions

- T_i: time to maturity of liabilities of firm i.
- $H^{(i)}$: face value of liabilities of firm i.
- $S_{T_i}^{(i)} := \max(V_{T_i}^{(i)} - H^{(i)}, 0)$ is the amount that the equity holders would receive at maturity T_i in case of liquidation of firm i.

- $F_{T_i}^{(i)} := \min(V_{T_i}^{(i)}, H^{(i)}) = H^{(i)} - \max(H^{(i)} - V_{T_i}^{(i)}, 0)$ value of the bond at T_i. Default if $V_{T_i}^{(i)} < H^{(i)}$.
- $V_t^{(i)} = S_t^{(i)} + F_t^{(i)}$ total value of firm i at time $t \in [0, T]$.
- Dynamics of asset value processes:

$$dV_t^{(i)} = \mu_{V_i} V_t^{(i)} dt + \sigma_{V_i} V_t^{(i)} dW_t^{(i)} \tag{1}$$

with μ_{V_i} instantaneous expected rate of return to the firm per time unit,
$\sigma_{V_i}^2$ instantaneous variance of the logarithmic firm value per time unit,
$W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$ n-dimensional Brownian motion.

- Complete market. Under unique martingale measure $\mathbb{Q}$

$$dV_t^{(i)} = r\, V_t^{(i)} dt + \sigma_{V_i} V_t^{(i)} dW_t^{*,(i)}, \quad i = 1, \ldots, n, \tag{2}$$

  where r risk-less interest rate and
  $W_t^* = (W_t^{*,(1)}, \ldots, W_t^{*,(n)})$ n-dimensional Brownian motion.

Prices $S_t^{(i)}$ and $F_t^{(i)}$ for equities and bonds

$$S_t^{(i)} = e^{r\,(t-T_i)}\, \mathsf{E}_{\mathbb{Q}}\Big[S_{T_i}^{(i)} \,|\, \mathcal{F}_t\Big] \tag{3}$$

$$F_t^{(i)} = e^{r\,(t-T_i)}\, \mathsf{E}_{\mathbb{Q}}\Big[F_{T_i}^{(i)} \,|\, \mathcal{F}_t\Big] \tag{4}$$

Yield spread:

$$Y_0^{(i)} \stackrel{\text{def}}{=} -\frac{1}{T_i} \log \frac{F_0^{(i)}}{H^{(i)}} - r.$$

1.2 The joint assets value distribution*

Substituting $\sum_{j=1}^{M} \sigma_{j,V_i} \, d\bar{W}_t^{(j)}$ for $dW_t^{(i)}$ into (1) yields:

$$V_t^{(i)} = V_0^{(i)} \exp\left(\mu_{V_i} t - \frac{(\sigma_{V_i})^2}{2} t + \sum_{j=1}^{M} \sigma_{j,V_i} \bar{W}_t^{(j)}\right), \quad i = 1, \ldots, n, \qquad (5)$$

where σ_{V_i} satisfies $(\sigma_{V_i})^2 := \sum_{j=1}^{M} (\sigma_{j,V_i})^2$,
if $(\bar{W}^{(j)})_{j=1,\ldots,M}$ independent Brownian motions.

Note $\underline{Z} \sim \mathcal{N}(\underline{0}, \Sigma)$ for $Z_i := \dfrac{\sum_{j=1}^{M} \sigma_{j,V_i} \bar{W}_T^{(j)}}{\sigma_{V_i} \sqrt{T}}$ with

$$[\Sigma]_{ij} = \frac{\sum_{s=1}^{M} \sigma_{s,V_i} \sigma_{s,V_j}}{\sigma_{V_i} \sigma_{V_j}}. \qquad (6)$$

In particular $\mathrm{var}(Z_i) = 1$.

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*See M. Nyfeler's Diploma thesis

Estimation of Σ:

$$\mathrm{Corr}\left(\log\left(\frac{V_T^{(i)}}{V_0^{(i)}}\right), \log\left(\frac{V_T^{(j)}}{V_0^{(j)}}\right)\right) = [\Sigma]_{ij}.$$

Number of parameters to estimate: $2n + \frac{(n-1)n}{2}$.

Reduction of dimension by factor model:
$\Sigma = \mathbf{A}\mathbf{A}^T + \mathbf{D}$ with

$$[\mathbf{A}]_{ij} = \frac{\sigma_{j,V_i}}{\sigma_{V_i}}, \qquad [\mathbf{D}]_{ij} = \begin{cases} \left(\frac{\sigma_{K+i,V_i}}{\sigma_{V_i}}\right)^2 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Conclusion: Z can be represented by the factor model

$$Z_i \stackrel{d}{=} \sum_{j=1}^{K} [\mathbf{A}]_{ij} R_j + \mathcal{E}_i \qquad i = 1, \ldots, n \tag{7}$$

where

$$\begin{gathered} M = K + n \\ \{(R_j)_j, (\mathcal{E}_i)_i\} \text{ independent} \\ R_j \sim \mathcal{N}(0, 1), \qquad j = 1, \ldots, K \\ \mathcal{E}_i \sim \mathcal{N}(0, [\mathbf{D}]_{ii}), \qquad i = 1, \ldots, n. \end{gathered} \tag{8}$$

1.3 The loan-loss distribution

- Intention: determining distribution of *loan portfolio value*

$$\sum_{i=1}^{n} F_T^{(i)} \tag{9}$$

  Portfolio loss: difference of benchmark value and loan portfolio value

- Analytical approximation: under simplifying assumptions for large portfolio

- Monte Carlo simulation by using a factor model

1.4 Stress Scenarios

(i) Provide data for every firm $i \in \{1, \ldots, n\}$:

r:	risk-less interest rate (from market)
T_i:	average maturity of outstanding debt (book value)
$H^{(i)}$:	face value of outstanding debt (book value)
$S_0^{(i)}$:	equity value (from market)
$\mathbb{P}\left(V_T^{(i)} < H^{(i)}\right)$:	physical default probability (by rating)
$Y_0^{(i)}$:	yield spread (by rating from market)
$V_0^{(i)}, \sigma_{V_i}$:	assets value and assets volatility (by numerical computation from Merton model)

(ii) Value of outstanding debt for firm i at time T:

$$F_T^{(i)} = \begin{cases} e^{r(T-T_i)} H^{(i)} \mathbb{I}_{\{V_{T_i}^{(i)} \geq H^{(i)}\}} & , \quad T \geq T_i \\ e^{r(T-T_i)} H^{(i)} \left(1 - \mathsf{N} \left(\dfrac{\log \frac{H^{(i)}}{v} + (\sigma_{V_i}^2/2 - r)(T_i - T)}{\sigma_{V_i} \sqrt{T_i - T}} \right) \right) \Big|_{v = V_T^{(i)}} & , \quad T < T_i \end{cases} \tag{10}$$

(iii) Value of bond portfolio: $\sum_{i=1}^{n} F_T^{(i)}$
Monte Carlo simulation with (10).
Simulate $V_{T_i}^{(i)}$ if $T \geq T_i$, $V_T^{(i)}$ if $T < T_i$.

(iv) Shock scenarios:

Equity, interest: $S_0^{i,\mathsf{new}} = (S_0^{i,\mathsf{old}} + \Delta S)\, e^{-\Delta r\, T}$

Volatility: $\sigma_{V_i}^{\mathsf{new}} = \sigma_{V_i}^{\mathsf{old}} (1 + \Delta\sigma)$

2 Integrating foreign exchange risk into the credit default model

2.1 Model based on three risky assets

Assumptions

- Constant domestic and foreign risk-free interest rates.
- Stochastic processes as geometric Brownian motions.
- Dependence by driving multidimensional Brownian motion.

Approximate Black–Scholes valuation formula for basket options

Basket of tradable assets:

$$w_1 A_t^{(1)} + w_2 A_t^{(2)}, \qquad w_1 + w_2 = 1, \quad w_1, w_2 \geq 0$$

Strike price: C

Call option:

$$C_T \quad = \quad \left(w_1 A_t^{(1)} + w_2 A_t^{(2)} - C \right)^+ \tag{11}$$

In general: $w_1 A_t^{(1)} + w_2 A_t^{(2)}$ not log-normal

$\Rightarrow$

Approximation by geometric mean $(\widehat{A}_t^{(1)})^{\widehat{w}_1} (\widehat{A}_t^{(2)})^{\widehat{w}_2}$ of appropriately transformed $\widehat{A}_t^{(i)}$, $\widehat{w}_i$ (Gentle (1993))

$\Rightarrow$

Black–Scholes formula applicable, if market complete

Notation for credit default model

X_t: Exchange rate process representing the domestic price at time t of one unit of the foreign currency.

V_t^d: Value of the domestic part of the firm's assets at time t.

V_t^f: Value of the foreign part of the firm's assets at time t in foreign currency.

r_d: Domestic risk-free interest rate.

r_f: Foreign risk-free interest rate.

B_t^d: Value of risk-free domestic bond at time t.

B_t^f: Value of risk-free foreign bond at time t.

Dynamics under physical probability $\mathbb{P}$:

$$\begin{aligned} dX_t &= X_t[\mu_X\,dt + \sigma_1 \cdot dU_t] \\ dV_t^d &= V_t^d[\mu_{V^d}\,dt + \sigma_2 \cdot dU_t] \\ dV_t^f &= V_t^f[\mu_{V^f}\,dt + \sigma_3 \cdot dU_t] \\ dB_t^d &= r_d B_t^d\,dt \\ dB_t^f &= r_f B_t^f\,dt \end{aligned}$$

where

$$U_t = (U_t^1, U_t^2, U_t^3) \text{ is a standard BM,}$$
$$\sigma_i = (\sigma_{i1}, \sigma_{i2}, \sigma_{i3}) \in \mathbb{R}^3, \quad i = 1,2,3,$$
$$\sigma_i \cdot dU_t = \textstyle\sum_{j=1}^3 \sigma_{ij}\,dU_t^j.$$

Unique domestic martingale measure ($\Rightarrow$ complete market):

Girsanov theorem:
probability $\mathbb{Q}$ such that $\frac{V^d}{B^d}$, $\frac{XV^f}{B^d}$, $\frac{XB^f}{B^d}$ are martingales.

Modeling assets with foreign exchange risk in practice

"Total assets = domestic assets + foreign assets"

$$V_t = V_t^d + \bar{V}_t^f, \qquad \bar{V}_t^f = X_t\, V_t^f$$

Decomposition of V_t:

$$V_0^d = (1-\alpha)\, V_0, \qquad \bar{V}_0^f = \alpha\, V_0 \tag{12}$$

Take determination coefficient of regression of stock returns versus exchange rate returns as α.

Take correlation of stock returns and foreign currency returns as approximation for correlation of returns to V_t^d and $\bar{V}_t^f$.

Substitute V_t^d and $\bar{V}_t^f$ for $A_t^{(1)}$ and $A_t^{(2)}$ in Black–Scholes formula for basket option.

Stress scenarios for foreign exchange risk:

$$\bar{V}_0^{f,\text{new}} \quad = \quad \bar{V}_0^{f,\text{old}}\,(1+\Delta)$$

Problems with three-assets approach

- No obvious decomposition of assets value into domestic and foreign parts

- Estimation from regression of stock returns versus exchange rate returns does not seem reliable

- Alternative: estimate weight α from balance sheet. Impractical. Does not yield correlation of domestic and foreign returns.

2.2 Model based on two risky assets

Modeling directly foreign exchange shocks difficult
$\Rightarrow$
indirect modeling by damped volatility shocks

Notation

U_t: 2-dimensional Brownian motion (U_t^1, U_t^2).

X_t: Exchange rate process representing the domestic price at time t of one unit of the foreign currency.

V_t: Value of the firm's assets at time t in domestic currency.

r_d: Domestic risk-free interest rate.

r_f: Foreign risk-free interest rate.

B_t^d: Value of risk-free domestic bond at time t.

B_t^f: Value of risk-free foreign bond at time t.

Dynamics under physical probability $\mathbb{P}$:

$$\begin{aligned} dX_t &= X_t[\mu_X\, dt + \sigma_1 \cdot dU_t] \\ dV_t &= V_t[\mu_V\, dt + \sigma_2 \cdot dU_t] \\ dB_t^d &= r_d B_t^d\, dt \\ dB_t^f &= r_f B_t^f\, dt \end{aligned}$$

where

$$\begin{aligned} U_t &= (U_t^1, U_t^2) \quad \text{standard BM} \\ \sigma_i &= (\sigma_{i1}, \sigma_{i2}) \in \mathbb{R}^2, \quad i = 1, 2, \\ \sigma_i \cdot dU_t &= \sigma_{i1} dU_t^1 + \sigma_{i2} dU_t^2. \end{aligned}$$

Unique domestic martingale measure ($\Rightarrow$ complete market):

Girsanov theorem:

probability $\mathbb{Q}$ such that $\frac{V^d}{B^d}$ and $\frac{XB^f}{B^d}$ are martingales.

$\Rightarrow$ "classical" Merton formula for value of equity.

Alternative representation for X_t, V_t:

$$\begin{aligned} dX_t &= X_t[\mu_X\,dt + \sigma_X\,dW_t^1] \\ dV_t &= V_t[\mu_V\,dt + \sigma_V\,dW_t^2] \end{aligned}$$

with

$$W_t^1 = \sigma_{11}\,U_t^1 + \sigma_{12}\,U_t^2, \quad W_t^2 = \sigma_{21}\,U_t^1 + \sigma_{22}\,U_t^2\,.$$

Here

$$\mathsf{corr}(W_t^1, W_t^2) = \sigma_{11}\,\sigma_{21} + \sigma_{12}\,\sigma_{22} \stackrel{\mathsf{def}}{=} \rho_{XV}$$

and

$$\sigma_X = \sqrt{\sigma_{11}^2 + \sigma_{12}^2}\,, \quad \sigma_V = \sqrt{\sigma_{21}^2 + \sigma_{22}^2}\,.$$

Generating stress scenarios

Assumption: $\sigma_{12} = \sigma_{21}$

Then

$$\begin{aligned} \sigma_{11}^2 + \sigma_{12}^2 &= \sigma_X^2 \,, \\ \sigma_{12}^2 + \sigma_{22}^2 &= \sigma_V^2 \end{aligned}$$

Interpret foreign exchange shock as shock on σ_X:

$$\sigma_X^{\text{new}} = \sigma_X^{\text{old}}(1 + \Delta\sigma_X).$$

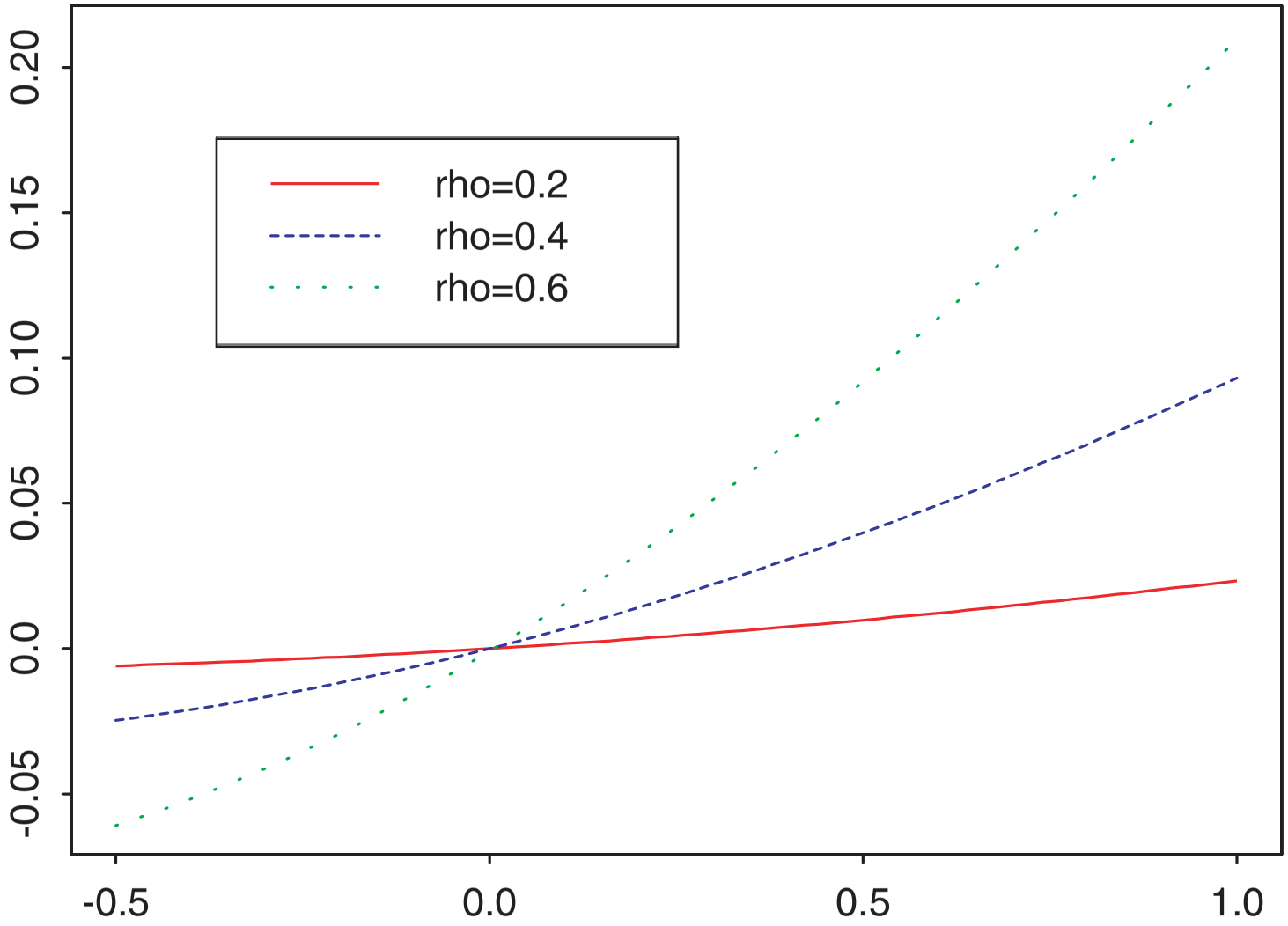
Influence on σ_V can be modeled by

$$(\sigma_V^{\text{new}})^2 = \sigma_{22}^2 + (1 + \Delta\sigma_X)^2 \sigma_{12}^2 \,.$$

Relative difference $\Delta\sigma_V$ in asset volatility:

$$\begin{aligned} \Delta\sigma_V &= \frac{\sigma_V^{\text{new}}}{\sigma_V^{\text{old}}} - 1 \\ &= (\sigma_V^{\text{old}})^{-1} \sqrt{\sigma_{22}^2 + (1 + \Delta\sigma_X)^2 \, \sigma_{12}^2} - 1 \,. \end{aligned}$$

Relative difference in asset volatility as function of relative difference in foreign exchange rate volatility.



3 Comparison of different portfolio structures

Alternative factorization of the driving Brownian motion W in (1).

Special case of the correlation structure in (6):

Assume $0 < \rho_* < \rho_1, \ldots, \rho_K \leq 1$.
Hence $W_t \sim \mathcal{N}(0, R(n))$ with correlation matrix $R(n) = \rho_{ij}(n)$

$$\rho_{ij}(n) = \begin{cases} 1, & \text{if } i = j, \\ \rho_h, & \text{if } i \text{ and } j \text{ are both in sector } h, \\ \rho_*, & \text{if } i \text{ and } j \text{ are in different sectors.} \end{cases}$$

ρ_h gives the default correlation of counter-parties in sector h and ρ_* is the default correlation which is due to the global economic situation.
$\rho_h - \rho_*$ is the additional correlation in sector h.

Suppose that $\rho_1 = \cdots = \rho_K$.
Aim: Comparison of two different portfolio structures s and s'

$$s = (s_1, \ldots, s_r, 0, \ldots, 0), \quad s' = (s'_1, \ldots, s'_l, 0, \ldots, 0), \qquad 1 \leq r, l \leq n$$

r, l: number of sectors in these two portfolio structures,
$s_i, s'_i \in \mathbb{N}$: number of firms in the two structures for all i,
$\sum_{i=1}^n s_i = \sum_{i=1}^n s'_i = n$.

Definition of a ordering of portfolio structures:
Denote by $s_{[1]} \geq \cdots \geq s_{[n]}$ the decreasing rearrangement of s, analogously for s'. s' majorizes s ($s \prec s'$) if and only if

$$\sum_{i=1}^m s_{[i]} \leq \sum_{i=1}^m s'_{[i]}, \quad m = 1, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^n s_{[i]} = \sum_{i=1}^n s'_{[i]}.$$

Theorem from Shaked, Tong (1992).

If $s \prec s'$, then $E[L-c]^+ \leq E[L'-c]^+$ for all thresholds $c \geq 0$,
where L and L' describe the total loss of the different portfolios.
(analogous to (9))

This means that well-balanced portfolios are less risky in the sense that the *expected loss shortfall* over a threshold c (c arbitrary) is smaller.

Example* (Effect of dependencies)

Suppose that the individual losses Y_i have a two-point distribution on 0 and 4, where the value 4 occurs with probability 0.06. The portfolio consists of $n = 20$ risks.

scenario m	$s^{(m)}$
1	$(1, 1, 1, \ldots, 1, 1, 1)$
2	$(4, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1)$
3	$(8, 2, 2, 2, 2, 2, 2)$
4	$(4, 4, 4, 3, 3, 2)$
5	$(15, 2, 1, 1, 1)$
6	$(5, 5, 5, 5)$
7	$(10, 5, 5)$
8	(20)

Table 1

*See Bäuerle/Müller (1998)

Relative expected loss excess (divided by the independent case $m = 1$) multiplied by 100 for several thresholds for the 8 different scenarios in Table 1:

$$(100E[\sum_{i=1}^{20} Y_i^{(m)} - c]^+ / E[\sum_{i=1}^{20} Y_i^{(1)} - c]^+).$$

Note that the expectation of the total loss equals 4.8 and the outcomes range between 0 and 80.

	scenario							
threshold	$s^{(1)}$	$s^{(2)}$	$s^{(3)}$	$s^{(4)}$	$s^{(5)}$	$s^{(6)}$	$s^{(7)}$	$s^{(8)}$
0	100	100	100	100	100	100	100	100
1	100	105	109	110	111	112	113	116
2	100	113	121	124	126	129	132	139
3	100	124	140	145	150	155	161	173
4	100	144	173	182	191	200	210	233
6	100	174	210	229	272	272	295	347
8	100	270	330	385	537	506	572	717
10	100	327	478	480	830	700	834	1128

Table 2

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