# Combined Stress Scenarios for Market and Credit Risks

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Generation of Random Stress Scenarios
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A Company's Value in Respect to the Merton Model/Credit Metrics
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# 1. Multi-firm Merton model

## 1.1 The asset value model

Portfolio of $n$ firms with time horizon $T$.

- $S_t^{(i)}$: equity value of firm $i \in \{1, \ldots, n\}$ at time $t \in [0, T]$,
- $F_t^{(i)}$: value of the liabilities of firm $i$ at time $t \in [0, T]$,
- $V_t^{(i)}$: value of firm $i$'s assets at time $t \in [0, T]$.

## Assumptions

- $T_i$: time to maturity of liabilities of firm $i$.
- $H^{(i)}$: face value of liabilities of firm $i$.
- $S_{T_i}^{(i)} := \max(V_{T_i}^{(i)} - H^{(i)}, 0)$ is the amount that the equity holders would receive at maturity $T_i$ in case of liquidation of firm $i$.

- $F_{T_i}^{(i)} := \min(V_{T_i}^{(i)}, H^{(i)}) = H^{(i)} - \max(H^{(i)} - V_{T_i}^{(i)}, 0)$ value of the bond at $T_i$. Default if $V_{T_i}^{(i)} < H^{(i)}$.
- $V_t^{(i)} = S_t^{(i)} + F_t^{(i)}$ total value of firm $i$ at time $t \in [0, T]$.
- Dynamics of asset value processes:

$$dV_t^{(i)} = \mu_{V_i} V_t^{(i)} dt + \sigma_{V_i} V_t^{(i)} dW_t^{(i)} \tag{1}$$

with $\mu_{V_i}$ instantaneous expected rate of return to the firm per time unit,
$\sigma_{V_i}^2$ instantaneous variance of the logarithmic firm value per time unit,
$W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$ $n$-dimensional Brownian motion.

- Complete market. Under unique martingale measure $\mathbb{Q}$

$$dV_t^{(i)} = r\,V_t^{(i)} dt + \sigma_{V_i} V_t^{(i)} dW_t^{*,(i)}, \quad i = 1, \ldots, n, \tag{2}$$

  where $r$ risk-less interest rate and
  $W_t^* = (W_t^{*,(1)}, \ldots, W_t^{*,(n)})$ $n$-dimensional Brownian motion.

Prices $S_t^{(i)}$ and $F_t^{(i)}$ for equities and bonds

$$S_t^{(i)} = e^{r\,(t-T_i)}\, \mathsf{E}_{\mathbb{Q}}\Big[S_{T_i}^{(i)}\,|\,\mathcal{F}_t\Big] \tag{3}$$

$$F_t^{(i)} = e^{r\,(t-T_i)}\, \mathsf{E}_{\mathbb{Q}}\Big[F_{T_i}^{(i)}\,|\,\mathcal{F}_t\Big] \tag{4}$$

Yield spread:

$$Y_0^{(i)} \overset{\text{def}}{=} -\frac{1}{T_i} \log \frac{F_0^{(i)}}{H^{(i)}} - r.$$

## 1.2 The joint assets value distribution*

Substituting $\sum_{j=1}^{M} \sigma_{j,V_i} \, d\bar{W}_t^{(j)}$ for $dW_t^{(i)}$ into (1) yields:

$$V_t^{(i)} = V_0^{(i)} \exp\left( \mu_{V_i} t - \frac{(\sigma_{V_i})^2}{2} t + \sum_{j=1}^{M} \sigma_{j,V_i} \bar{W}_t^{(j)} \right), \quad i = 1, \ldots, n, \tag{5}$$

where $\sigma_{V_i}$ satisfies $(\sigma_{V_i})^2 := \sum_{j=1}^{M} (\sigma_{j,V_i})^2$,
if $(\bar{W}^{(j)})_{j=1,\ldots,M}$ independent Brownian motions.

Note $\underline{Z} \sim \mathcal{N}(\underline{0}, \Sigma)$ for $Z_i := \frac{\sum_{j=1}^{M} \sigma_{j,V_i} \bar{W}_T^{(j)}}{\sigma_{V_i} \sqrt{T}}$ with

$$[\Sigma]_{ij} = \frac{\sum_{s=1}^{M} \sigma_{s,V_i} \sigma_{s,V_j}}{\sigma_{V_i} \sigma_{V_j}}. \tag{6}$$

In particular $\mathrm{var}(Z_i) = 1$.

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*See M. Nyfeler's Diploma thesis

Estimation of $\Sigma$:

$$\mathsf{Corr}\left(\log\left(\frac{V_T^{(i)}}{V_0^{(i)}}\right), \log\left(\frac{V_T^{(j)}}{V_0^{(j)}}\right)\right) = [\Sigma]_{ij}.$$

Number of parameters to estimate: $2n + \frac{(n-1)n}{2}$.

Reduction of dimension by factor model:
$\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T + \mathbf{D}$ with

$$[\mathbf{A}]_{ij} = \frac{\sigma_{j,V_i}}{\sigma_{V_i}}, \qquad [\mathbf{D}]_{ij} = \begin{cases} \left(\frac{\sigma_{K+i,V_i}}{\sigma_{V_i}}\right)^2 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Conclusion: $Z$ can be represented by the factor model

$$Z_i \stackrel{d}{=} \sum_{j=1}^{K} [\mathbf{A}]_{ij} R_j + \mathcal{E}_i \qquad i = 1, \ldots, n \tag{7}$$

where

$$\begin{gathered} M = K + n \\ \{(R_j)_j, (\mathcal{E}_i)_i\} \text{ independent} \\ R_j \sim \mathcal{N}(0,1), \qquad j = 1, \ldots, K \\ \mathcal{E}_i \sim \mathcal{N}(0, [\mathbf{D}]_{ii}), \qquad i = 1, \ldots, n. \end{gathered} \tag{8}$$

## 1.3 The loan-loss distribution

- Intention: determining distribution of *loan portfolio value*

$$\sum_{i=1}^{n} F_T^{(i)} \tag{9}$$

  Portfolio loss: difference of benchmark value and loan portfolio value

- Analytical approximation: under simplifying assumptions for large portfolio

- Monte Carlo simulation by using a factor model

## 1.4 Stress Scenarios

(i) Provide data for every firm $i \in \{1, \ldots, n\}$:

| | |
|---|---|
| $r$: | risk-less interest rate (from market) |
| $T_i$: | average maturity of outstanding debt (book value) |
| $H^{(i)}$: | face value of outstanding debt (book value) |
| $S_0^{(i)}$: | equity value (from market) |
| $\mathbb{P}\left(V_T^{(i)} < H^{(i)}\right)$: | physical default probability (by rating) |
| $Y_0^{(i)}$: | yield spread (by rating from market) |
| $V_0^{(i)}, \sigma_{V_i}$: | assets value and assets volatility (by numerical computation from Merton model) |

(ii) Value of outstanding debt for firm $i$ at time $T$:

$$F_T^{(i)} = \begin{cases} e^{r(T-T_i)} H^{(i)} \mathbb{I}_{\{V_{T_i}^{(i)} \geq H^{(i)}\}} & , \quad T \geq T_i \\ e^{r(T-T_i)} H^{(i)} \left( 1 - \mathrm{N} \left( \frac{\log \frac{H^{(i)}}{v} + (\sigma_{V_i}^2/2 - r)(T_i - T)}{\sigma_{V_i} \sqrt{T_i - T}} \right) \right) \Bigg|_{v = V_T^{(i)}} & , \quad T < T_i \end{cases} \tag{10}$$

(iii) Value of bond portfolio: $\sum_{i=1}^n F_T^{(i)}$
Monte Carlo simulation with (10).
Simulate $V_{T_i}^{(i)}$ if $T \geq T_i$, $V_T^{(i)}$ if $T < T_i$.

(iv) Shock scenarios:

Equity, interest: $S_0^{i,\text{new}} = (S_0^{i,\text{old}} + \Delta S)\, e^{-\Delta r\, T}$

Volatility: $\sigma_{V_i}^{\text{new}} = \sigma_{V_i}^{\text{old}}(1 + \Delta\sigma)$

# 2 Integrating foreign exchange risk into the credit default model

## 2.1 Model based on three risky assets

### Assumptions

- Constant domestic and foreign risk-free interest rates.

- Stochastic processes as geometric Brownian motions.

- Dependence by driving multidimensional Brownian motion.

# Approximate Black–Scholes valuation formula for basket options

Basket of tradable assets:

$$w_1 \, A_t^{(1)} + w_2 \, A_t^{(2)}, \qquad w_1 + w_2 = 1, \quad w_1, w_2 \geq 0$$

Strike price: $C$

Call option:

$$C_T \quad = \quad \left( w_1 \, A_t^{(1)} + w_2 \, A_t^{(2)} - C \right)^+ \tag{11}$$

In general: $w_1 \, A_t^{(1)} + w_2 \, A_t^{(2)}$ not log-normal

$\Rightarrow$

Approximation by geometric mean $(\widehat{A}_t^{(1)})^{\widehat{w}_1} \, (\widehat{A}_t^{(2)})^{\widehat{w}_2}$ of appropriately transformed $\widehat{A}_t^{(i)}$, $\widehat{w}_i$ (Gentle (1993))

$\Rightarrow$

Black–Scholes formula applicable, if market complete

# Notation for credit default model

$X_t$: Exchange rate process representing the domestic price at time $t$ of one unit of the foreign currency.

$V_t^d$: Value of the domestic part of the firm's assets at time $t$.

$V_t^f$: Value of the foreign part of the firm's assets at time $t$ in foreign currency.

$r_d$: Domestic risk-free interest rate.

$r_f$: Foreign risk-free interest rate.

$B_t^d$: Value of risk-free domestic bond at time $t$.

$B_t^f$: Value of risk-free foreign bond at time $t$.

Dynamics under physical probability $\mathbb{P}$:

$$\begin{aligned} dX_t &= X_t[\mu_X\,dt + \sigma_1 \cdot dU_t] \\ dV_t^d &= V_t^d[\mu_{V^d}\,dt + \sigma_2 \cdot dU_t] \\ dV_t^f &= V_t^f[\mu_{V^f}\,dt + \sigma_3 \cdot dU_t] \\ dB_t^d &= r_d B_t^d\,dt \\ dB_t^f &= r_f B_t^f\,dt \end{aligned}$$

where

$$U_t = (U_t^1, U_t^2, U_t^3) \text{ is a standard BM,}$$
$$\sigma_i = (\sigma_{i1}, \sigma_{i2}, \sigma_{i3}) \in \mathbb{R}^3, \quad i = 1, 2, 3,$$
$$\sigma_i \cdot dU_t = \textstyle\sum_{j=1}^3 \sigma_{ij}\,dU_t^j.$$

Unique domestic martingale measure ($\Rightarrow$ complete market):

Girsanov theorem:
probability $\mathbb{Q}$ such that $\frac{V^d}{B^d}$, $\frac{XV^f}{B^d}$, $\frac{XB^f}{B^d}$ are martingales.

## Modeling assets with foreign exchange risk in practice

"Total assets = domestic assets + foreign assets"

$$V_t = V_t^d + \bar{V}_t^f, \qquad \bar{V}_t^f = X_t V_t^f$$

Decomposition of $V_t$:

$$V_0^d = (1-\alpha) V_0, \qquad \bar{V}_0^f = \alpha V_0 \tag{12}$$

Take determination coefficient of regression of stock returns versus exchange rate returns as $\alpha$.

Take correlation of stock returns and foreign currency returns as approximation for correlation of returns to $V_t^d$ and $\bar{V}_t^f$.

Substitute $V_t^d$ and $\bar{V}_t^f$ for $A_t^{(1)}$ and $A_t^{(2)}$ in Black–Scholes formula for basket option.

Stress scenarios for foreign exchange risk:

$$\bar{V}_0^{f,\text{new}} \quad = \quad \bar{V}_0^{f,\text{old}} (1+\Delta)$$

## Problems with three-assets approach

- No obvious decomposition of assets value into domestic and foreign parts

- Estimation from regression of stock returns versus exchange rate returns does not seem reliable

- Alternative: estimate weight $\alpha$ from balance sheet. Impractical. Does not yield correlation of domestic and foreign returns.

## 2.2 Model based on two risky assets

Modeling directly foreign exchange shocks difficult
$\Rightarrow$
indirect modeling by damped volatility shocks

### Notation

$U_t$: 2-dimensional Brownian motion $(U_t^1, U_t^2)$.

$X_t$: Exchange rate process representing the domestic price at time $t$ of one unit of the foreign currency.

$V_t$: Value of the firm's assets at time $t$ in domestic currency.

$r_d$: Domestic risk-free interest rate.

$r_f$: Foreign risk-free interest rate.

$B_t^d$: Value of risk-free domestic bond at time $t$.

$B_t^f$: Value of risk-free foreign bond at time $t$.

Dynamics under physical probability $\mathbb{P}$:

$$
\begin{aligned}
dX_t &= X_t[\mu_X \, dt + \sigma_1 \cdot dU_t] \\
dV_t &= V_t[\mu_V \, dt + \sigma_2 \cdot dU_t] \\
dB_t^d &= r_d B_t^d \, dt \\
dB_t^f &= r_f B_t^f \, dt
\end{aligned}
$$

where

$$
\begin{aligned}
U_t &= (U_t^1, U_t^2) \quad \text{standard BM} \\
\sigma_i &= (\sigma_{i1}, \sigma_{i2}) \in \mathbb{R}^2, \quad i = 1, 2, \\
\sigma_i \cdot dU_t &= \sigma_{i1} dU_t^1 + \sigma_{i2} dU_t^2.
\end{aligned}
$$

Unique domestic martingale measure ($\Rightarrow$ complete market):

Girsanov theorem:

probability $\mathbb{Q}$ such that $\frac{V^d}{B^d}$ and $\frac{XB^f}{B^d}$ are martingales.

$\Rightarrow$ “classical” Merton formula for value of equity.

Alternative representation for $X_t$, $V_t$:

$$\begin{aligned} dX_t &= X_t[\mu_X \, dt + \sigma_X \, dW_t^1] \\ dV_t &= V_t[\mu_V \, dt + \sigma_V \, dW_t^2] \end{aligned}$$

with

$$W_t^1 = \sigma_{11} \, U_t^1 + \sigma_{12} \, U_t^2, \quad W_t^2 = \sigma_{21} \, U_t^1 + \sigma_{22} \, U_t^2 \, .$$

Here

$$\mathrm{corr}(W_t^1, W_t^2) = \sigma_{11} \, \sigma_{21} + \sigma_{12} \, \sigma_{22} \stackrel{\mathrm{def}}{=} \rho_{XV}$$

and

$$\sigma_X = \sqrt{\sigma_{11}^2 + \sigma_{12}^2}, \quad \sigma_V = \sqrt{\sigma_{21}^2 + \sigma_{22}^2} \, .$$

# Generating stress scenarios

Assumption: $\sigma_{12} = \sigma_{21}$

Then

$$\begin{aligned} \sigma_{11}^2 + \sigma_{12}^2 &= \sigma_X^2 , \\ \sigma_{12}^2 + \sigma_{22}^2 &= \sigma_V^2 \end{aligned}$$

Interpret foreign exchange shock as shock on $\sigma_X$:

$$\sigma_X^{\text{new}} = \sigma_X^{\text{old}}(1 + \Delta\sigma_X).$$

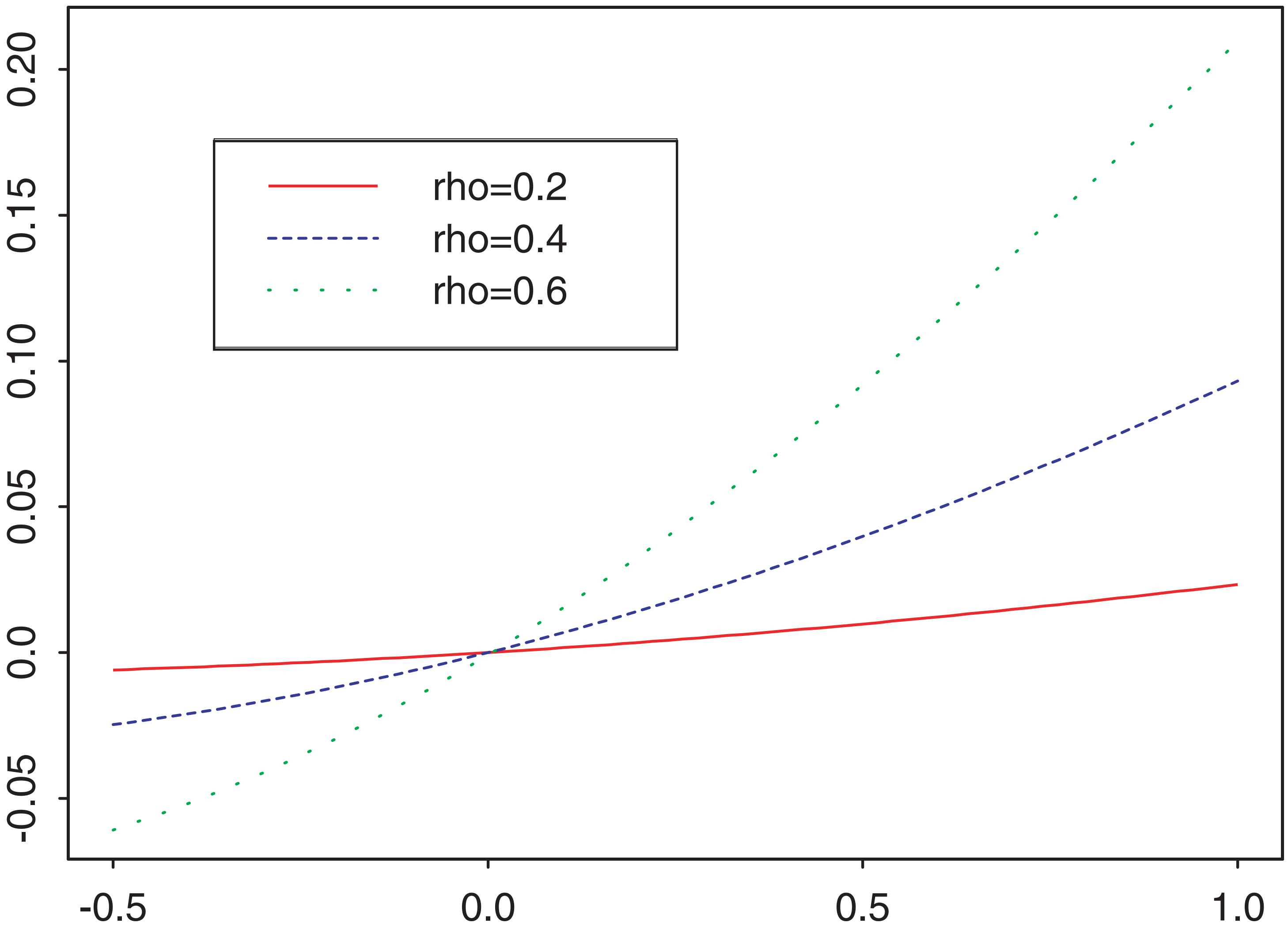
Influence on $\sigma_V$ can be modeled by

$$(\sigma_V^{\text{new}})^2 = \sigma_{22}^2 + (1 + \Delta\sigma_X)^2 \sigma_{12}^2 .$$

Relative difference $\Delta\sigma_V$ in asset volatility:

$$\begin{aligned} \Delta\sigma_V &= \frac{\sigma_V^{\text{new}}}{\sigma_V^{\text{old}}} - 1 \\ &= (\sigma_V^{\text{old}})^{-1} \sqrt{\sigma_{22}^2 + (1 + \Delta\sigma_X)^2 \, \sigma_{12}^2} - 1 . \end{aligned}$$

Relative difference in asset volatility as function of relative difference in foreign exchange rate volatility.



# 3 Comparison of different portfolio structures

Alternative factorization of the driving Brownian motion $W$ in (1).

Special case of the correlation structure in (6):

Assume $0 < \rho_* < \rho_1, \ldots, \rho_K \leq 1$.
Hence $W_t \sim \mathcal{N}(0, R(n))$ with correlation matrix $R(n) = \rho_{ij}(n)$

$$\rho_{ij}(n) = \begin{cases} 1, & \text{if } i = j, \\ \rho_h, & \text{if } i \text{ and } j \text{ are both in sector } h, \\ \rho_*, & \text{if } i \text{ and } j \text{ are in different sectors.} \end{cases}$$

$\rho_h$ gives the default correlation of counter-parties in sector $h$ and $\rho_*$ is the default correlation which is due to the global economic situation.
$\rho_h - \rho_*$ is the additional correlation in sector $h$.

Suppose that $\rho_1 = \cdots = \rho_K$.
Aim: Comparison of two different portfolio structures $s$ and $s'$

$$s = (s_1, \ldots, s_r, 0, \ldots, 0), \quad s' = (s'_1, \ldots, s'_l, 0, \ldots, 0), \qquad 1 \leq r, l \leq n$$

$r, l$: number of sectors in these two portfolio structures,
$s_i, s'_i \in \mathbb{N}$: number of firms in the two structures for all $i$,
$\sum_{i=1}^n s_i = \sum_{i=1}^n s'_i = n$.

## Definition of a ordering of portfolio structures:

Denote by $s_{[1]} \geq \cdots \geq s_{[n]}$ the decreasing rearrangement of $s$, analogously for $s'$. $s'$ majorizes $s$ ($s \prec s'$) if and only if

$$\sum_{i=1}^{m} s_{[i]} \leq \sum_{i=1}^{m} s'_{[i]}, \quad m = 1, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} s_{[i]} = \sum_{i=1}^{n} s'_{[i]}.$$

Theorem from Shaked, Tong (1992).

If $s \prec s'$, then $E[L - c]^+ \leq E[L' - c]^+$ for all thresholds $c \geq 0$, where $L$ and $L'$ describe the total loss of the different portfolios. (analogous to (9))

This means that well-balanced portfolios are less risky in the sense that the *expected loss shortfall* over a threshold $c$ ($c$ arbitrary) is smaller.

## Example*(Effect of dependencies)

Suppose that the individual losses $Y_i$ have a two-point distribution on 0 and 4, where the value 4 occurs with probability 0.06. The portfolio consists of $n = 20$ risks.

| scenario $m$ | $s^{(m)}$ |
|---|---|
| 1 | $(1, 1, 1, \ldots, 1, 1, 1)$ |
| 2 | $(4, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1)$ |
| 3 | $(8, 2, 2, 2, 2, 2, 2)$ |
| 4 | $(4, 4, 4, 3, 3, 2)$ |
| 5 | $(15, 2, 1, 1, 1)$ |
| 6 | $(5, 5, 5, 5)$ |
| 7 | $(10, 5, 5)$ |
| 8 | $(20)$ |

Table 1

*See Bäuerle/Müller (1998)

Relative expected loss excess (divided by the independent case $m = 1$) multiplied by 100 for several thresholds for the 8 different scenarios in Table 1:

$$(100E[\sum_{i=1}^{20} Y_i^{(m)} - c]^+/E[\sum_{i=1}^{20} Y_i^{(1)} - c]^+).$$

Note that the expectation of the total loss equals 4.8 and the outcomes range between 0 and 80.

| | scenario | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| threshold | $s^{(1)}$ | $s^{(2)}$ | $s^{(3)}$ | $s^{(4)}$ | $s^{(5)}$ | $s^{(6)}$ | $s^{(7)}$ | $s^{(8)}$ |
| 0 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| 1 | 100 | 105 | 109 | 110 | 111 | 112 | 113 | 116 |
| 2 | 100 | 113 | 121 | 124 | 126 | 129 | 132 | 139 |
| 3 | 100 | 124 | 140 | 145 | 150 | 155 | 161 | 173 |
| 4 | 100 | 144 | 173 | 182 | 191 | 200 | 210 | 233 |
| 6 | 100 | 174 | 210 | 229 | 272 | 272 | 295 | 347 |
| 8 | 100 | 270 | 330 | 385 | 537 | 506 | 572 | 717 |
| 10 | 100 | 327 | 478 | 480 | 830 | 700 | 834 | 1128 |

Table 2

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