Long-Term Financial Risks: the One-Dimensional Case

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joint work with Pierre Patie

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Aim of the Project

One-dimensional:

- Model one-year returns for stock indices, 10-year bonds and foreign exchange rates.

Multi-dimensional:

- Measurement of one-year financial risk of investment portfolios.

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Risk Measures

Definition 1. *The value-at-risk VaR_p at level p of the return R is*

$$VaR_p(R) = -\inf\{x \in \mathbb{R} \mid \mathbb{P}[R \leq x] \geq p\},$$

i.e. VaR is the negative of the p-quantile of R.

Definition 2. *The expected shortfall ES_p at a level p is defined by*

$$ES_p(R) = -\mathbb{E}[R \,|\, R < -VaR_p(R)].$$

We consider as risk measure the expected shortfall for the level $p = 1\%$. The expected shortfall is a *coherent* risk measure in the sense of Artzner, Delbaen, Eber and Heath. In general, value-at-risk is not!

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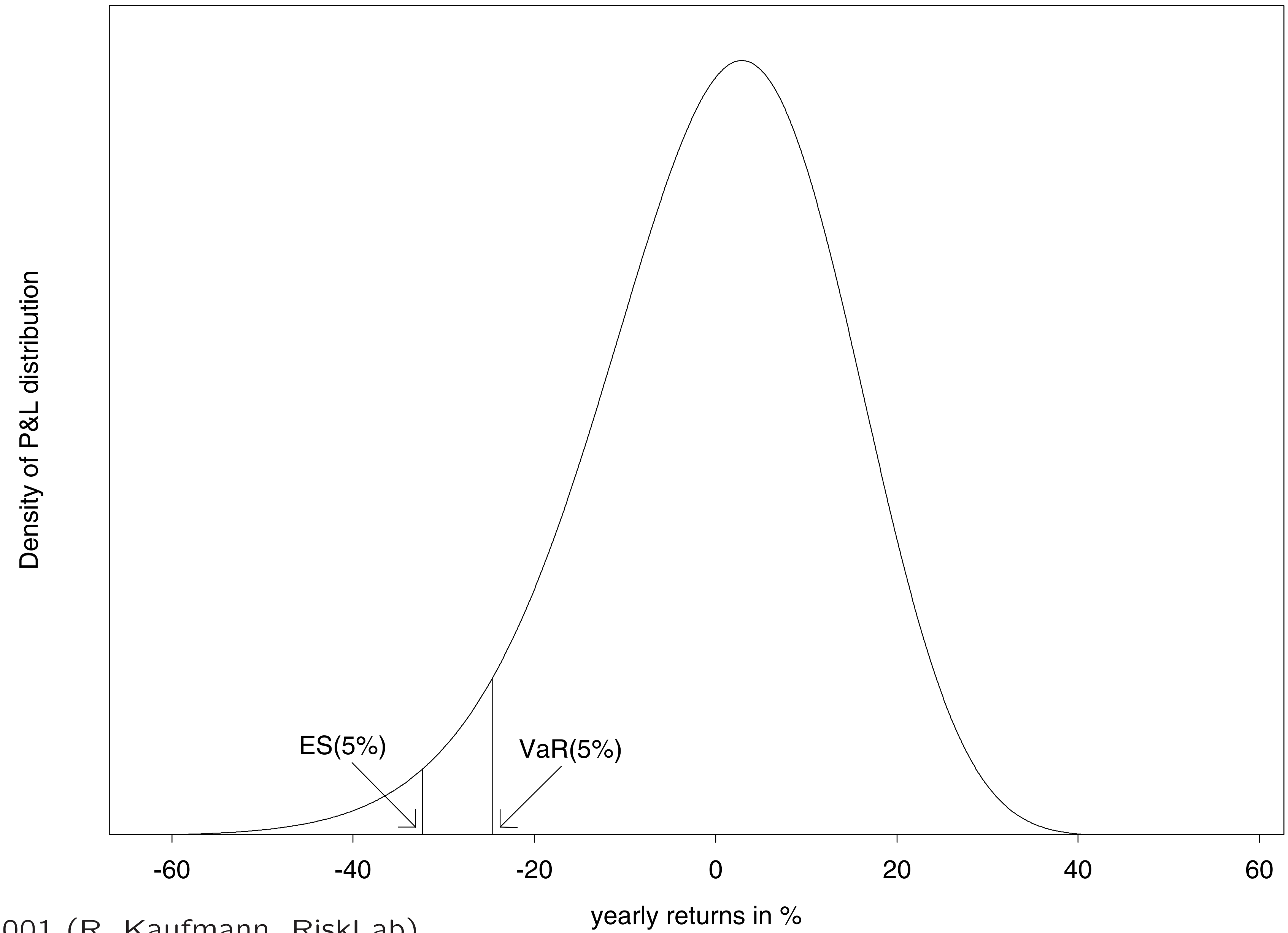
Value-at-Risk and Expected Shortfall



Problems

1. Which frequency do we use to fit models?
   - Are long datasets stationary?
   - What are the statistical restrictions? (lack of yearly returns)
   - How can we keep as much information as possible?

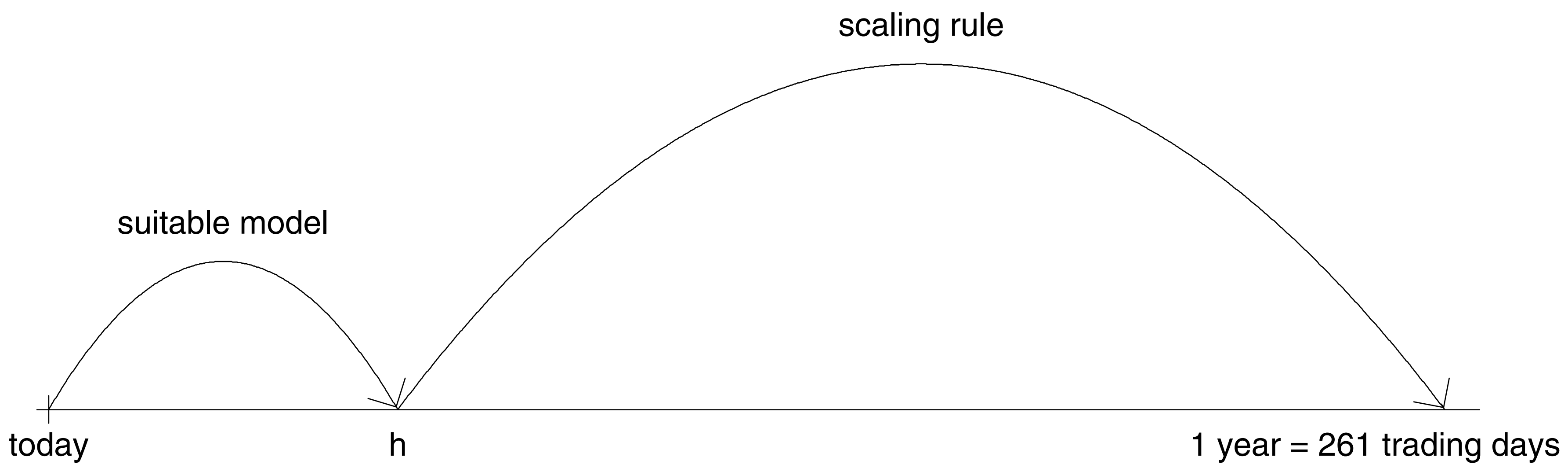
2. Do the properties of financial data change when we choose another time horizon?

3. What is the reliability of the time aggregation rule of each model if there is any such rule?

4. How can we compare different time horizons and models?

Model Comparison

We fix a horizon $h < 1$ year, for which we can use our models. For the gap between h and 1 year, we use a scaling rule.



II Models

- Random walk
- GARCH(1,1)
- Vector Error Correction Model $\leadsto$ Auto-regressive
- Heavy-tailed distribution

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Random Walk Model

We assume that h-day log-returns are independent and normally distributed:

$$r_h(t) \overset{\text{iid}}{\sim} \mathcal{N}(\mu_h, \sigma_h^2) \quad \text{for } t \in h\mathbb{N}.$$

We estimate the one-year expected shortfall at a level p by

$$\widehat{\mathrm{ES}}_t^p = -\left(\exp\left(\widehat{\mu}(t) + \frac{\widehat{\sigma}^2(t)}{2}\right)\frac{\Phi(x_p - \widehat{\sigma}(t))}{p} - 1\right)$$

for $t \geq nh$, where:

x_p : p-quantile of the standard normal distribution,

Φ : cumulative standard normal distribution function,

$\widehat{\mu}(t) = \frac{261}{h}\widehat{\mu}_h(t),\ \widehat{\mu}_h(t) = \frac{1}{n}\sum_{i=0}^{n-1} r_h(t-ih),$

$\widehat{\sigma}(t) = \sqrt{\frac{261}{h}}\widehat{\sigma}_h(t),\ \widehat{\sigma}_h^2(t) = \frac{1}{n-1}\sum_{i=0}^{n-1}(r_h(t-ih) - \widehat{\mu}_h(t-ih))^2.$

Generalized Autoregressive Conditional Heteroskedastic Model

A GARCH(1,1) model with a Student-t distributed innovation process for centered h-day log-returns $X_h(t) = r_h(t) - \mu_h$ is defined by

$$X_h(t) = \sigma_h(t)\, \epsilon_h(t) \quad \text{for } t \in h\mathbb{N},$$

$$\sigma_h^2(t) = \alpha_{0,h} + \alpha_{1,h} X_h^2(t-h) + \beta_{1,h}\sigma_h^2(t-h),$$

$$\text{where} \quad \epsilon_h(t) \overset{\text{iid}}{\sim} t_{\nu_h}, \quad \mathbb{E}[\epsilon_h(t)] = 0, \quad \mathbb{E}[\epsilon_h^2(t)] = 1.$$

We estimate the one-year expected shortfall at a level p in 4 steps:

1. Fit the GARCH(1,1) process to the h-day log-returns using quasi maximum likelihood estimators (QMLE).

2. Apply the Drost–Nijman scaling rule to get the parameters $\widehat{\alpha}_0, \widehat{\alpha}_1, \widehat{\beta}_1, \widehat{\nu}$ of the weak GARCH process for centered yearly log-returns $X(t)$.

3. Forecast the yearly volatility $\widehat{\sigma}(t)$ using the following recursive relation:

$$\widehat{\sigma}^2(s+261,t) = \widehat{\alpha}_0+\widehat{\alpha}_1(r(s)-\widehat{\mu}(t))^2+\widehat{\beta}_1\widehat{\sigma}^2(s,t), \quad s = t_0,\ldots,t-261,t$$

starting with

$$\widehat{\sigma}^2(t_0,t) = \frac{261}{h}\frac{1}{n-1}\sum_{i=0}^{n-1}(r_h(t-ih)-\widehat{\mu}_h(t))^2,$$

$\widehat{\mu}_h(t)$: QMLE for mean h-day log-return at time t,

$\widehat{\mu}(t) = \frac{261}{h}\widehat{\mu}_h(t)$,

$\widehat{\sigma}(t) = \widehat{\sigma}(t+261,t)$.

4. $$\widehat{\mathrm{ES}}_t^p = -\left(\frac{1}{p}\int_0^p \exp\left(\widehat{\mu}(t)+\widehat{\sigma}(t)x_{\widehat{\nu},q}\right)dq - 1\right),$$

where $x_{\widehat{\nu},q}$ is the q-quantile of a $t_{\widehat{\nu}}$-distributed random variable with mean zero and variance one.

Auto-regressive Model

An AR(q) model with a normally distributed innovation process for the drift-free h-day log-prices $\widetilde{s}(t) = s(t) - \mu t$ $(t \in h\mathbb{N})$ is defined by

$$\widetilde{s}(t) = \sum_{i=1}^{q_h} a_{h,i}\, \widetilde{s}(t - ih) + \epsilon_h(t) \quad \text{for } t \in h\mathbb{N},\ t \geq q_n h,$$

$$\text{where} \quad \epsilon_h(t) \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_h^2).$$

We estimate the one-year expected shortfall at a level p as follows:

1. Subtract the linear trend from h-day log-prices $s_h(t)$:

$$\widetilde{s}(t) = s(t) - \widehat{\mu} t, \quad \widehat{\mu} = \frac{s(t) - s(t - (n-1)h)}{(n-1)h}.$$

2. Fit the AR-process to the drift-free h-day log-prices: maximum likelihood estimation (MLE) $\rightsquigarrow \widehat{q}_h, \widehat{a}_{h,i}\,(i = 1, \ldots, \widehat{q}_h), \widehat{\sigma}_h$.

3. Forecast the one-year log-return using the following relation:

$$\widehat{\mu}(t) = 261\widehat{\mu} + \widehat{m}(t),$$

$$\widehat{m}(t) = \widetilde{\widetilde{s}}(t+kh) - \widetilde{s}(t), \qquad k = \left\lfloor \frac{261}{h} + \frac{1}{2} \right\rfloor,$$

where $\widetilde{\widetilde{s}}(t+jh)$ $(j = 1, \ldots, k)$ are defined recursively:

$$\widetilde{\widetilde{s}}(t+jh) = \sum_{i=1}^{\widehat{q}_h} \widehat{a}_{h,i} \widetilde{\widetilde{s}}(t+(j-i)h), \qquad \widetilde{\widetilde{s}}(u) = \widetilde{s}(u) \ \text{ for } u \le t.$$

4. Forecast the yearly volatility using the following formulas:

$$\delta_0 = 1,$$

$$\delta_j = \sum_{i=1}^{j} \widehat{\alpha}_{h,i} \delta_{j-i}, \qquad \widehat{\alpha}_{h,i} = 0 \ \forall i > \widehat{q}_h,$$

$$\widehat{\sigma}(t) = \widehat{\sigma}_h(t) \sqrt{\sum_{j=0}^{k-1} \delta_j^2}.$$

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5.

$$\widehat{\mathsf{ES}}_t^p = -\left(\exp\left(\widehat{\mu}(t) + \frac{\widehat{\sigma}^2(t)}{2}\right)\frac{\Phi(x_p - \widehat{\sigma}(t))}{p} - 1\right),$$

where:

x_p : p-quantile of the standard normal distribution,

Φ : cumulative standard normal distribution function.

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Heavy-Tailed Distribution

We assume the h-day log-returns to be independent and identically distributed, further

$$\mathbb{P}[r_h < -x] = x^{-\alpha} L(x) \quad \text{as } x \to \infty,$$

where $\alpha \in \mathbb{R}^+$ and L is a slowly varying function.

By inverting this formula and using the scaling rule for heavy-tailed distributions (Feller's theorem) we can derive estimates for the one-year expected shortfall at a level p:

$$\widehat{\mathrm{ES}}_t^p = -\left(\frac{1}{p}\int_0^p \exp\left(\left(\frac{261\, k(n,p)}{h\, n\, q}\right)^{1/\widehat{\alpha}_{k(n,p)}} r_{k(n,p),n}\right) dq - 1\right),$$

where

$k(n,p) = \lfloor n(p + 4.5\% + \frac{h}{2} 1\%)\rfloor, \quad h = 1, 5, 22, \quad$ (heuristic choice)

$\widehat{\alpha}_{k,n} = 1 \Big/ \frac{1}{k} \sum_{i=1}^{k} \log\left(\frac{r_{i,n}}{r_{k,n}}\right) \quad$ (Hill estimator),

$r_{k,n}$ is the kth order statistics, i.e. $r_{1,n} \leq r_{2,n} \leq \cdots \leq r_{n,n}$.

III Backtesting

- Backtesting description

- Results:

    - foreign exchange rates

    - stock indices

    - 10-year bonds

Backtesting Measures

Measure 1: Use values below the negative of the estimated value-at-risk $\widehat{\mathrm{VaR}}_t^p$:

$$V_1^{\mathrm{ES}} = \frac{\sum_{t=t_0}^{t_1} \left(R_{t+1} - (-\widehat{\mathrm{ES}}_t^p) \right) 1_{\{R_{t+1} < -\widehat{\mathrm{VaR}}_t^p\}}}{\sum_{t=t_0}^{t_1} 1_{\{R_{t+1} < -\widehat{\mathrm{VaR}}_t^p\}}}.$$

Measure 2: Use values below the "1 in $1/p$ event" (for $p = 1\%$: *one in hundred* event):

$$V_2^{\mathrm{ES}} = \frac{\sum_{t=t_0}^{t_1} D_t \, 1_{\{D_t < D^p\}}}{\sum_{t=t_0}^{t_1} 1_{\{D_t < D^p\}}}, \qquad D_t = R_{t+1} - (-\widehat{\mathrm{ES}}_t^p),$$

where D^p is the p-quantile of $\{D_t\}_{t_0 \le t \le t_1}$.

Combined measure: $\quad V^{\mathrm{ES}} = (|V_1^{\mathrm{ES}}| + |V_2^{\mathrm{ES}}|)/2.$

Frequency of exceedance:

$$V^{\mathrm{freq}} = \left(\frac{1}{t_1 - t_0 + 1} \sum_{t=t_0}^{t_1} 1_{\{R_{t+1} < -\widehat{\mathrm{VaR}}_t^p\}} \right).$$

Backtesting Description

Problem: Not enough yearly data for estimating model parameters *and* proceeding the backtesting!

Solution: We use 4 stock samples (foreign exchange rates), each containing 16 years of data, we carry out the backtesting on each sample independently, then we aggregate the results.

For each model, each intermediate horizon h and each sample we proceed as follows:

1. We estimate the yearly expected shortfall $\widehat{\mathrm{ES}}_t^p$ on a window of size N (e.g. N=2000 daily data). We use the $n = \lfloor N/h \rfloor$ non-overlapping h-day log-returns for this estimation.
2. We compare the estimates with the following returns R_{t+1} using different measures.
3. We move the window by one, then we repeat steps 1 and 2 up to the end of the whole dataset.

Backtesting Results for the 1% One-Year Expected Shortfall

averaged over 4 foreign exchange rates

(DEM/CHF, GBP/CHF, USD/CHF, JPY/CHF)

Model	Freq days	ES			VaR
		V^{ES}	V_1^{ES}	V_2^{ES}	V^{freq}
Optimal		0%	0%	0%	1%
GARCH(1,1)	1	N/A	N/A	3.9	0.0
	5	2.2	1.9	2.6	0.3
	22	0.8	0.7	−0.9	0.7
	65	2.2	2.1	−2.4	1.2
	261	11.4	−6.0	−16.8	6.3

The 1% One-Year Expected Shortfall for Foreign Exchange Rates:

Model	Freq days	ES V^{ES}	V_1^{ES}	V_2^{ES}	VaR V^{freq}
Optimal		0%	0%	0%	1%
Random Walk	1	1.4	1.1	1.7	0.4
	5	1.1	0.8	1.4	0.5
	22	1.0	0.7	1.3	0.5
	65	0.9	0.5	1.4	0.5
	261	0.8	−0.4	−1.1	1.6
GARCH(1,1)	1	N/A	N/A	3.9	0.0
	5	2.2	1.9	2.6	0.3
	22	0.8	0.7	−0.9	0.7
	65	2.2	2.1	−2.4	1.2
	261	11.4	−6.0	−16.8	6.3
AR(p)	1	1.1	−0.2	−2.0	5.8
	5	1.1	−0.3	−2.0	5.1
	22	1.2	−0.3	−2.0	4.6
	65	1.1	−0.3	−1.9	4.2
	261	3.9	−1.6	−6.2	9.6
Heavy-Tailed Distribution *	1	5.1	−1.2	−8.9	15.2
	5	3.7	0.2	−7.1	9.3
	22	0.8	1.6	0.1	1.4

- DEM/CHF
- GBP/CHF
- USD/CHF
- JPY/CHF

* For 65 and 261 days we do not have enough data to estimate the tail index with the Hill estimator.

The 1% One-Year Expected Shortfall for Stock Indices:

Model	Freq days	ES V^{ES}	V_1^{ES}	V_2^{ES}	VaR V^{freq}
Optimal		0%	0%	0%	1%
Random Walk	1	0.8	0.3	1.3	0.8
	5	1.2	0.5	1.9	0.7
	22	0.7	0.2	1.1	0.8
	65	1.3	−1.2	−1.3	1.0
	261	10.5	−6.0	−15.0	2.5
GARCH(1,1)	1	0.6	0.2	−1.1	1.3
	5	3.7	2.6	4.9	0.5
	22	4.0	2.1	−6.0	1.3
	65	13.2	−4.5	−21.9	2.7
	261	16.4	−10.6	−22.3	5.2
AR(p)	1	6.4	−4.0	−8.9	2.4
	5	6.6	−3.9	−9.3	2.5
	22	7.3	−4.4	−10.1	2.4
	65	8.9	−4.6	−13.2	3.1
	261	13.5	−6.2	−20.9	12.2
Heavy-Tailed Distribution	1	3.0	4.1	1.9	2.0
	5	2.4	1.8	2.9	0.8
	22	1.7	−0.5	2.9	0.5

- SMI
- DAX
- FTSE
- SnP
- NIKKEI

The 1% One-Year Expected Shortfall for 10-Year Bonds:

Model	Freq days	ES V^{ES}	V_1^{ES}	V_2^{ES}	VaR V^{freq}
Optimal		0%	0%	0%	1%
Random Walk	1	1.0	0.3	1.8	0.6
	5	1.8	0.5	3.2	0.4
	22	2.4	−0.4	4.4	0.2
	65	3.6	1.1	6.1	0.1
	261	4.1	−4.7	−3.4	0.9
GARCH(1,1)	1	6.1	−1.8	10.4	0.0
	5	10.7	11.8	9.5	0.1
	22	2.6	−1.2	4.1	0.4
	65	8.2	−4.7	−11.7	1.9
	261	12.2	−8.4	−16.0	4.4
AR(p)	1	5.8	−2.4	−9.1	3.3
	5	5.7	−2.7	−8.7	2.8
	22	5.5	−2.8	−8.2	2.6
	65	5.8	−3.1	−8.4	2.9
	261	11.9	−4.7	−19.1	12.9
Heavy-Tailed Distribution	1	11.4	−2.2	−20.5	35.0
	5	8.4	−1.2	−15.6	25.2
	22	7.6	−1.4	−13.7	11.7

Government bonds from

- CH
- DE
- UK
- US
- JP

IV Conclusions

- The random walk approach gives good results for appropriate choices of the time horizon h. The optimal h depends on the kind of data the model is applied to. It varies from 1 day (10-year bonds) to 1 year (foreign exchange rates).

- GARCH underestimates the risk when used for low frequency data (lack of stationary data). For foreign exchange rates and stock indices the model performs about as well as the random walk approach.

- AR only performs reasonably well for foreign exchange rates. For less extreme quantiles AR models performs even poorer.

- Heavy-tailed distributions only perform reasonably well for foreign exchange rates. Lower frequency data (monthly data) provides better forecasts than higher frequency data (daily data).

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