Overview and Comparisons of Long-Term Financial Risk Models

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Overview and Comparisons of Long-Term Financial Risk Models

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I Introduction

- Aim of the project
- Key questions
- Risk measures

Aim of the project

- Measurement of long-term financial risk of investment portfolios.

First steps:

- Modelling the stochastic evolution of risk factors associated to portfolio positions.
- Test the goodness of such models for a long time horizon (e.g. 1 year).

Risk measure definition

We consider as risk measure the expected shortfall.

Definition 1 *The expected shortfall ES_α at a level α is defined by*

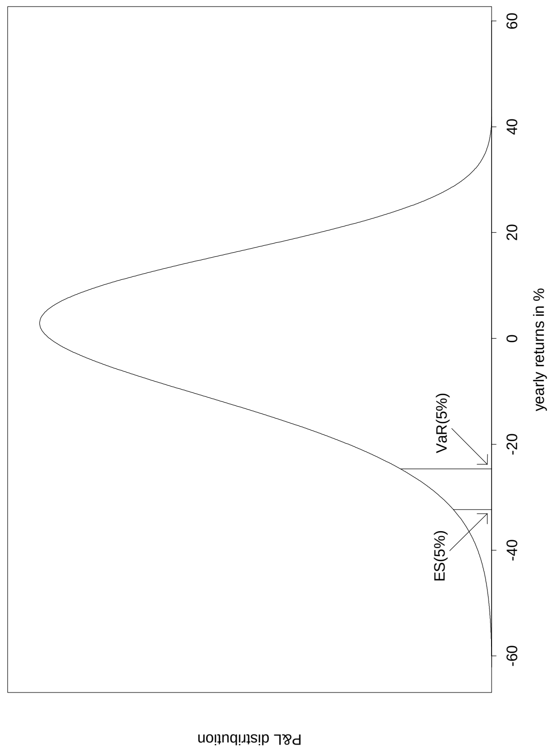
$$ES_\alpha(R) = -\mathbb{E}[R \,|\, R < -VaR_\alpha(R)], \text{ where } R \in \mathbb{L}^1(\Omega).$$

Definition 2 *Given $\alpha \in]0,1[$, the value-at-risk VaR_α at level α of the returns R with distribution $\mathbb{P}$, is*

$$VaR_\alpha(R) = -\inf\{x \in \mathbb{R} \mid \mathbb{P}[R \le x] \ge \alpha\},$$

i.e. VaR is the negative of the α-quantile of R.

The expected shortfall is a *coherent* risk measure in the sense of Artzner, Delbaen, Eber and Heath.
In general, value-at-risk is not !



Key questions

1. Which frequency do we use to fit models?
   - Are long datasets stationary?
   - What are the statistical restrictions? (lack of yearly returns)
   - How can we keep as much information as possible?
2. Do the properties of financial data change when we choose another time horizon?
3. What is the reliability of the time aggregation rule of each model if there is any?
4. How can we compare different time horizons and models?

II Model Description

- Random Walk
- GARCH(1,1)
- Heavy-tailed distribution

Random Walk

Assumption 1: expected log returns are equal to zero

$$\mathbb{E}_t[r_{t+1}] = 0.$$

Assumption 2: Normally distributed, independent log returns with standard deviation σ in each period $[t, t+1]$.

$$\text{Ass. 1 \& 2} \;\Rightarrow\; r_{t+1} \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2).$$

→ The logarithmic asset price follows a random walk with zero drift.

Time aggregation:

$$r_{h,t} := \sum_{i=0}^{h-1} r_{t-i} \overset{\text{iid}}{\sim} \mathcal{N}(0, h\sigma^2).$$

GARCH(1,1)

Let $(X_t, t \in \mathbb{N})$ be a strictly stationary time series representing observations of centered log returns on a financial asset price.

A GARCH(1,1) model for X is defined by

$$X_t = \sigma_t \epsilon_t \quad \text{for } t \in \mathbb{N},$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$
$$\epsilon_t \text{ iid}, \; \mathbb{E}[\epsilon_t] = 0, \; \mathbb{E}[\epsilon_t^2] = 1.$$

Stationarity conditions:
$0 < \alpha_0 < \infty$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and $\alpha_1 + \beta_1 < 1$.

Fit the GARCH(1,1) process by pseudo-maximum-likelihood estimation to obtain the value of the parameters of the conditional volatility.

Time aggregation: GARCH coefficients function

Assume: Centered 1-day log returns X_t follow a GARCH(1,1) process with a normally distributed innovation.

$$X_t = \sigma_t \epsilon_t,$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$
$$\epsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1).$$

Drost-Nijman:

$$X_{h,t} := \sum_{i=0}^{h-1} X_{t-i} \quad \text{is weak GARCH(1,1):}$$

$$X_{h,t} = \sigma_{h,t} \epsilon_{h,t},$$
$$\sigma_{h,t}^2 = \alpha_{h,0} + \alpha_{h,1} X_{h,t-1}^2 + \beta_{h,1} \sigma_{h,t-1}^2,$$
$$\epsilon_{h,t} \overset{\text{iid}}{\sim} \mathcal{N}(0,1),$$
$$\alpha_{h,1} \to 0, \beta_{h,1} \to 0 \quad \text{as } h \to \infty.$$

Heavy-tailed distributions

We consider $(r_t, t \in \mathbb{N})$ to be independent and identically distributed (i.i.d.), representing observations of the log returns on a financial asset price.
We assume

$$\mathbb{P}[r_1 < -x] = Cx^{-\alpha} L(x) \quad \text{as } x \to \infty, \tag{1}$$

where C, $\alpha \in \mathbb{R}^+$ and L is a slowly varying function, i.e.

$$\forall t > 0 : \lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1.$$

Distributions satisfying (1) are called *heavy-tailed distributions* since the kth moment is infinite for $k > \alpha$. (1) is also a characterisation of the maximum domain of attraction of the Fréchet distribution.

Time aggregation

Feller's theorem (1971)

Theorem: *Assuming that* $(r_t, t \in \mathbb{N})$ *have heavy-tailed distributions leads to*

$$\mathbb{P}\left[\sum_{t=1}^{h} r_t < -x\right] = hCL(x)x^{-\alpha}[1 + o(1)], \text{ for } x \to \infty,$$

where the scale factor C *is as in* (1).

$\sum_{t=1}^{h} r_t$ corresponds to the h-day log returns. When applicable, this theorem supplements the central limit theorem by providing information concerning the tails.

Dacorogna, Müller, Pictet and de Vries show the following **theorem:**

Suppose r has a finite variance (i.e. $\alpha > 2$). At a constant risk level p, increasing the time horizon h increases the VaR and the expected shortfall numbers for the heavy tailed model by a factor $h^{\frac{1}{\alpha}}$.

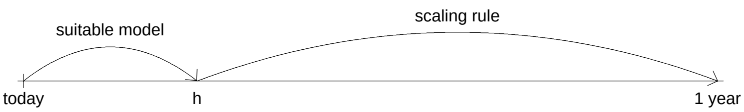
III Backtesting Idea

- Model comparison
- Backtesting description

Model Comparison

No one of the proposed models obviously outperforms the others. Each of them has its deficiencies.

All models only perform well for relatively short time horizons. We have to fix a horizon $h <$ 1 year, for which we can use our models. For the gap between h and 1 year, we will have to use a scaling rule.



Backtesting: Test Description

Idea: compare forecasted expected shortfall $\widehat{\mathrm{ES}}_{t,\alpha}$ with empirical estimation of expected shortfall.

Measure 1: Evaluate values below the negative of the estimated value-at-risk $\widehat{\mathrm{VaR}}_{t,\alpha}$.

We build the difference between the real (i.e. observed) one-year returns R_{t+1} and the negative of the estimation $\widehat{\mathrm{ES}}_{t,\alpha}$.
We calculate the conditional average of these differences, conditioned on $\{R_{t+1} < -\widehat{\mathrm{VaR}}_{t,\alpha}\}$,

$$V_1^{\mathrm{ES}} = \frac{\sum_{t=t_0}^{t_1} \left(R_{t+1} - (-\widehat{\mathrm{ES}}_{t,\alpha})\right) 1_{\{R_{t+1} < -\widehat{\mathrm{VaR}}_{t,\alpha}\}}}{\sum_{t=t_0}^{t_1} 1_{\{R_{t+1} < -\widehat{\mathrm{VaR}}_{t,\alpha}\}}}.$$

A good estimation for expected shortfall will lead to a low absolute value of V_1^{ES}.

Measure 2: Evaluate values below the "1 in $1/\alpha$ event" (for $\alpha = 1\%$: *one in hundred* event).

We build the difference between the real (i.e. observed) one-year return R_{t+1} and the negative of the estimation $\widehat{\text{ES}}_{t,\alpha}$:

$$D_t := R_{t+1} - (-\widehat{\text{ES}}_{t,\alpha}).$$

We calculate the conditional average of these differences, conditioned on $\{D_t < D^\alpha\}$,

$$V_2^{\text{ES}} = \frac{\sum_{t=t_0}^{t_1} D_t \, 1_{\{D_t < D^\alpha\}}}{\sum_{t=t_0}^{t_1} 1_{\{D_t < D^\alpha\}}},$$

where D^α denotes the empirical α quantile of $\{D_t\}_{t_0 \le t \le t_1}$.

A good estimation for expected shortfall will lead to a low absolute value of V_2^{ES}.

We evaluate two additional measures that provide information about the goodness of our estimators:

- **Frequency of exceedance**

  Calculate the percentage of times the observed returns fall below the negative VaR estimate:

$$V^{\text{freq}} = \frac{1}{t_1 - t_0 + 1} \sum_{t=t_0}^{t_1} 1_{\{R_{t+1} < -\widehat{\text{VaR}}_{t,\alpha}\}}.$$

- **Relative amount of exceedance**

  Relative size of values exceeding VaR:

$$V^{\text{size}} = \frac{\sum_{t=t_0}^{t_1} \frac{R_{t+1} - (-\widehat{\text{VaR}}_{t,\alpha})}{-\widehat{\text{VaR}}_{t,\alpha}} 1_{\{R_{t+1} < -\widehat{\text{VaR}}_{t,\alpha}\}}}{\sum_{t=t_0}^{t_1} 1_{\{R_{t+1} < -\widehat{\text{VaR}}_{t,\alpha}\}}}.$$

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Normal Distribution

We assume for the h-day log returns:

$$r_h \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_h^2).$$

We estimate the one year expected shortfall at a level p by:

$$\widehat{\text{ES}}_t^p(r) = \sqrt{\frac{261}{h}} \hat{\sigma}_{t,h} \frac{\varphi(x_p)}{p},$$

where:

x_p : $p-$quantile of a standard normal random variable,

φ : density of a standard normal random variable,

$\hat{\sigma}_{t,h}^2 = \frac{1}{N-1} \sum_{i=0}^{N-1} r_{t-i,h}^2,$

the h-day sample variance.

GARCH(1,1)

A GARCH(1,1) model with normally distributed innovation process for the h-day log returns r_h is defined by

$$r_{t,h} = \sigma_{t,h}\,\epsilon_t \quad \text{for } t \in \mathbb{N},$$
$$\sigma^2_{t,h} = \alpha_{0,h} + \alpha_{1,h} r^2_{t-1,h} + \beta_{1,h}\sigma^2_{t-1,h},$$

where $\epsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$.

We estimate the 1 year expected shortfall at a level p in 4 steps:

1. we fit the GARCH(1,1) process with the h-day log returns,
2. we apply the Drost-Nijman rule (scaling rule) to get the parameters of the yearly conditional volatility,
3. we forecast the yearly volatility using the following recursive relation:
$$\hat{\sigma}^2_{t+1,y} = \hat{\alpha}_{0,y} + \hat{\alpha}_{1,y} r^2_{t,y} + \hat{\beta}_{1,y}\hat{\sigma}^2_{t,y},$$
starting with $\hat{\sigma}^2_{1,y} = \frac{261}{N-1}\sum_{i=1}^{N}(r_i - \overline{r})^2$,
4. $\widehat{\text{ES}}^p_t(r) = \hat{\sigma}_{t+1,y}\frac{\varphi(x_p)}{p}$.

Heavy Tailed Distribution

We assume the h-day log returns to be independent, further

$$\mathbb{P}[r_1 < -x] = Cx^{-\alpha}L(x) \quad \text{as } x \to \infty, \qquad (2)$$

where $C,\ \alpha \in \mathbb{R}^+$ and
L is a slowly varying function.

By inverting (2) and using the scaling rule for heavy tailed distributions (Feller's theorem) we can easily derive estimates for the one year expected shortfall at a level p:

$$\widehat{ES}^p_t = \left(\frac{261}{h}\right)^{\frac{1}{\hat{\alpha}_{k,N}}}\left(\frac{\hat{\alpha}_{k,N}}{\hat{\alpha}_{k,N}-1}\right)\hat{x}^p_{k,N}, \qquad \hat{\alpha}_{k,N} > 1,$$

where

$$\hat{x}^p_{k,N} = r_{k,N}\left(\frac{kp}{N}\right)^{-\frac{1}{\hat{\alpha}_{k,N}}}$$

$$\hat{\alpha}^{-1}_{k,N} = \frac{1}{k}\sum_{i=1}^{k}\log\left(\frac{r_{i,N}}{r_{k,N}}\right) \text{ (Hill estimator)},$$

N is the sample size,
$r_{k,N}$ is the kth order statistics.

Backtesting in Action

Problem: Not enough yearly data for estimating model parameters *and* proceeding the backtesting!

Solution: We use 22 stock samples (German stocks), each containing 23.5 years of data, we carry out the backtesting on each sample independently, then we aggregate the results.

For each sample we proceed as follows:

1. for each model we estimate the yearly forecasted expected shortfall $\widehat{\text{ES}}^p_t$ on a window of size N (e.g. N=2000 daily data). We also use non-overlapping data for lower frequency,
2. we compare the estimates with the following returns R_{t+1} using different measures,
3. we move the window by one, then we repeat steps 1 and 2 up to the end of the whole dataset.

Backtesting Measures

Measure 1: Evaluate values below the negative of the estimated value-at-risk $\widehat{\text{VaR}}^p_t$:

$$V_1^{\text{ES}} = \frac{\sum_{t=t_0}^{t_1}\left(R_{t+1}-(-\widehat{\text{ES}}^p_t)\right)1_{\{R_{t+1}<-\widehat{\text{VaR}}^p_t\}}}{\sum_{t=t_0}^{t_1}1_{\{R_{t+1}<-\widehat{\text{VaR}}^p_t\}}}.$$

Measure 2: Evaluate values below the "1 in $1/\alpha$ event" (for $\alpha = 1\%$: *one in hundred* event):

$$D_t := R_{t+1} - (-\widehat{\text{ES}}^p_t),$$
$$V_2^{\text{ES}} = \frac{\sum_{t=t_0}^{t_1} D_t\,1_{\{D_t<D^\alpha\}}}{\sum_{t=t_0}^{t_1}1_{\{D_t<D^\alpha\}}},$$

where D^α is the α quantile of $\{D_t\}_{t_0\le t\le t_1}$.

Frequency of exceedance:

$$V^{\text{freq}} = \left(\frac{1}{t_1-t_0+1}\sum_{t=t_0}^{t_1}1_{\{R_{t+1}<-\widehat{\text{VaR}}^p_t\}}\right).$$

Relative amount of exceedance:

$$V^{\text{size}} = \frac{\sum_{t=t_0}^{t_1}\frac{R_{t+1}-(-\widehat{\text{VaR}}^p_t)}{-\widehat{\text{VaR}}^p_t}1_{\{R_{t+1}<-\widehat{\text{VaR}}^p_t\}}}{\sum_{t=t_0}^{t_1}1_{\{R_{t+1}<-\widehat{\text{VaR}}^p_t\}}}.$$

The 5% One Year Expected Shortfall

Model	Freq	ES		VaR	
		V_1^{ES}	V_2^{ES}	V^{freq}	V^{size}
Expected		0%	0%	5%	
Normal	1	6.4	10.9	0.5	1.1
	5	4.3	7.2	3.2	9.1
	22	3.7	6.8	4.1	11.
	65	1.9	2.2	5.4	30.7
GARCH	1	−2.7	−7.4	8.6	−42.9
	5	−2.5	−6.8	8.2	−40.9
	22	−2.7	−6.5	7.7	−41.2
HT	1	1.6	5.3	7.8	40.5

Backtesting Results
22 backtesting samples of length 3884 each.

Freq	N	N GARCH
1	2000	3070
5	400	614
22	90	139
65	30	47
261	7	11

Number of data used
with respect to the frequency (in days).

Conclusions

- The Normal approach gives good results but only for low frequency (i.e. more than 3 months). This may come from the fact that low frequency data get closer to the normal distribution. Also the square root of time rule does not suit for high frequency data. It leads to overestimation of the risk.
- Heavy tailed distributions perform well for high frequency data (daily data).
- GARCH underestimates the risk: problem of stationarity, or – as for the Normal approach – the scaling rule does not perform well for high frequency data (in the opposite way).

  Next steps:

  – investigate the Normal inverse Gaussian distribution,

  – consider the problem of risk aggregation.

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