Modelling Dependent Credit Risks

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Part 1

Copula Ideas

I Dependence concepts, copula families

II Example - risky dependence

Part 2

Common Poisson Shock Models

III Model description

IV Example - large loan portfolios

Drawbacks of linear correlation

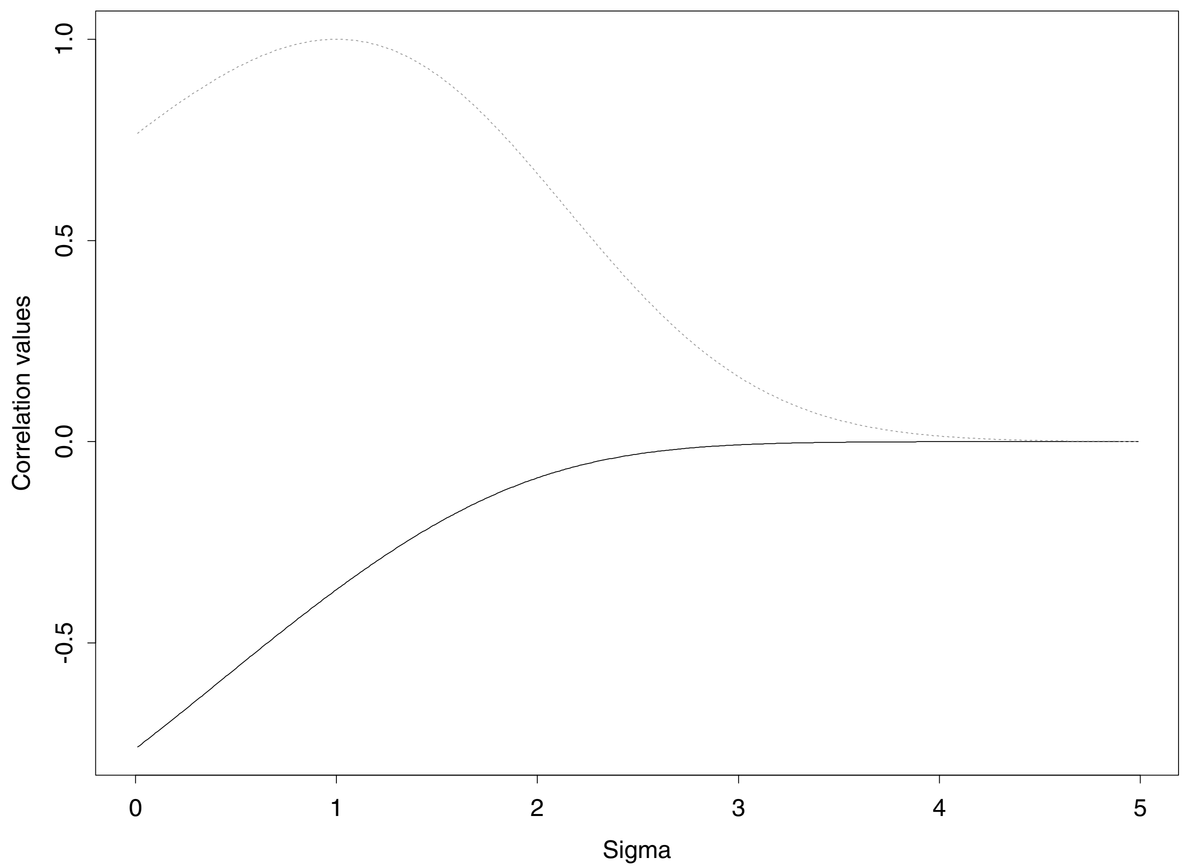
- Linear correlation of random variables X, Y is not defined if the variance of X or Y is infinite.

- Linear correlation can easily be misinterpreted.

- Linear correlation is *not* invariant under non-linear strictly increasing transformations $T : \mathbb{R} \longrightarrow \mathbb{R}$, i.e.,

$$\rho\,(T(X), T(Y)) \neq \rho\,(X, Y)\,.$$

- Given distribution functions F and G for X and Y, all linear correlations between -1 and 1 can in general not be obtained by a suitable choice of the joint distribution.



Upper and lower bounds for $\rho(X, Y)$, where $X \sim \text{Lognormal}(0, 1)$ and $Y \sim \text{Lognormal}(0, \sigma^2)$.

Note:

For $\sigma = 4$, $\rho(X, Y) = 0.01372$ in fact means that $Y = T(X)$, with T strictly increasing!

Copulas

Definition

A copula, $C : [0,1]^n \mapsto [0,1]$, is a joint distribution function (d.f.) of n random variables uniformly distributed on $[0,1]$.

- Let H be an n-dimensional d.f. with continuous margins $F_1, \ldots, F_n$. Then

  $$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

  Conversely, if C is a copula and $F_1, \ldots, F_n$ are d.fs, then H is an n-dimensional d.f. with margins $F_1, \ldots, F_n$.

  Hence the copula of $(X_1, \ldots, X_n) \sim H$ is the d.f. of $(F_1(X_1), \ldots, F_n(X_n))$.

- If $F_1, \ldots, F_n$ are strictly increasing d.f.s, then for every $\mathbf{u} = (u_1, \ldots, u_n)$ in $[0,1]^n$,

  $$C(\mathbf{u}) = H(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)).$$

Special copulas

$$
\begin{aligned}
M^n(\mathbf{u}) &= \min(u_1, u_2, \dots, u_n) \\
W^n(\mathbf{u}) &= \max(u_1 + u_2 + \cdots + u_n - n + 1, 0) \\
\Pi^n(\mathbf{u}) &= u_1 u_2 \dots u_n
\end{aligned}
$$

Note: M^n and Π^n are copulas for all $n \geq 2$ but W^n is a copula only for $n = 2$.

Definition

1. $X_1, \dots, X_n$ comonotonic
   $\iff (X_1, \dots, X_n)$ has copula M^n
   $\iff (X_1, \dots, X_n) =_d (\alpha_1(Z), \dots, \alpha_n(Z))$,
   $\alpha_1, \dots, \alpha_n$ increasing and Z is some real valued random variable.

2. X, Y countermonotonic
   $\iff (X, Y)$ has copula W^2
   $\iff (X, Y) =_d (\alpha(Z), \beta(Z))$, α inc., β dec. and Z is some real valued r.v.

3. $X_1, \dots, X_n$ independent
   $\iff (X_1, \dots, X_n)$ has copula Π^n.

Properties of copulas

Bounds
For every $\mathbf{u} \in [0,1]^n$ we have

$$W^n(\mathbf{u}) \leq C(\mathbf{u}) \leq M^n(\mathbf{u}).$$

These bounds are the best possible.

Concordance ordering
If C_1 and C_2 are copulas, we say that C_2 is *more concordant* than C_1 ($C_1 \prec_c C_2$) if

$$C_1(\mathbf{u}) \leq C_2(\mathbf{u}) \text{ and } \overline{C}_1(\mathbf{u}) \leq \overline{C}_2(\mathbf{u})$$

for all $\mathbf{u} \in [0,1]^n$. $\overline{C}_1$ and $\overline{C}_2$ are joint survival functions.

Strictly increasing transformations
If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are strictly increasing, then $\alpha_1(X_1), \alpha_2(X_2), \ldots, \alpha_n(X_n)$ have the same copula as $X_1, X_2, \ldots, X_n$.

Rank correlations

Let (X,Y) be a random vector with continuous margins F and G and copula C.

- *Kendall's tau* of (X,Y) is given by

$$\begin{aligned}\tau(X,Y) &:= \mathbb{P}[(X-X')(Y-Y')>0] \\ &\quad - \mathbb{P}[(X-X')(Y-Y')<0] \\ &= 4\iint_{[0,1]^2} C(u,v)\,\mathrm{d}C(u,v)-1,\end{aligned}$$

  where (X,Y) and (X',Y') are independent copies.

- *Spearman's rho* of (X,Y) is given by

$$\begin{aligned}\rho_S(X,Y) &= \rho(F(X),G(Y)) \\ &= 12\iint_{[0,1]^2} uv\,\mathrm{d}C(u,v)-3.\end{aligned}$$

Kendall's tau and Spearman's rho are called rank correlations.

Properties of rank correlation

Let X and Y be continuous random variables with copula C, and let δ denote Kendall's tau or Spearman's rho. The following properties are not shared by linear correlation.

- If T is strictly monotone, then
  $\delta(T(X), Y) = \delta(X, Y)$, T increasing,
  $\delta(T(X), Y) = -\delta(X, Y)$, T decreasing.

- $\delta(X, Y) = 1 \quad \Longleftrightarrow \quad C = M^2$

- $\delta(X, Y) = -1 \quad \Longleftrightarrow \quad C = W^2$

- $\delta(X, Y)$ depends only on the copula of (X, Y).

Given a rank correlation matrix there is always a multivariate distribution with this rank correlation matrix, regardless of the choice of margins.
This is *not* true for linear correlation.

Tail dependence

Let (X, Y) be a random vector with continuous margins F and G and copula C.

- The coefficient of upper tail dependence of (X, Y) is

$$\begin{aligned}\lambda_U &:= \lim_{u \nearrow 1} \mathbb{P}[Y > G^{-1}(u) | X > F^{-1}(u)] \\ &= \lim_{u \nearrow 1} \overline{C}(u, u)/(1 - u)\end{aligned}$$

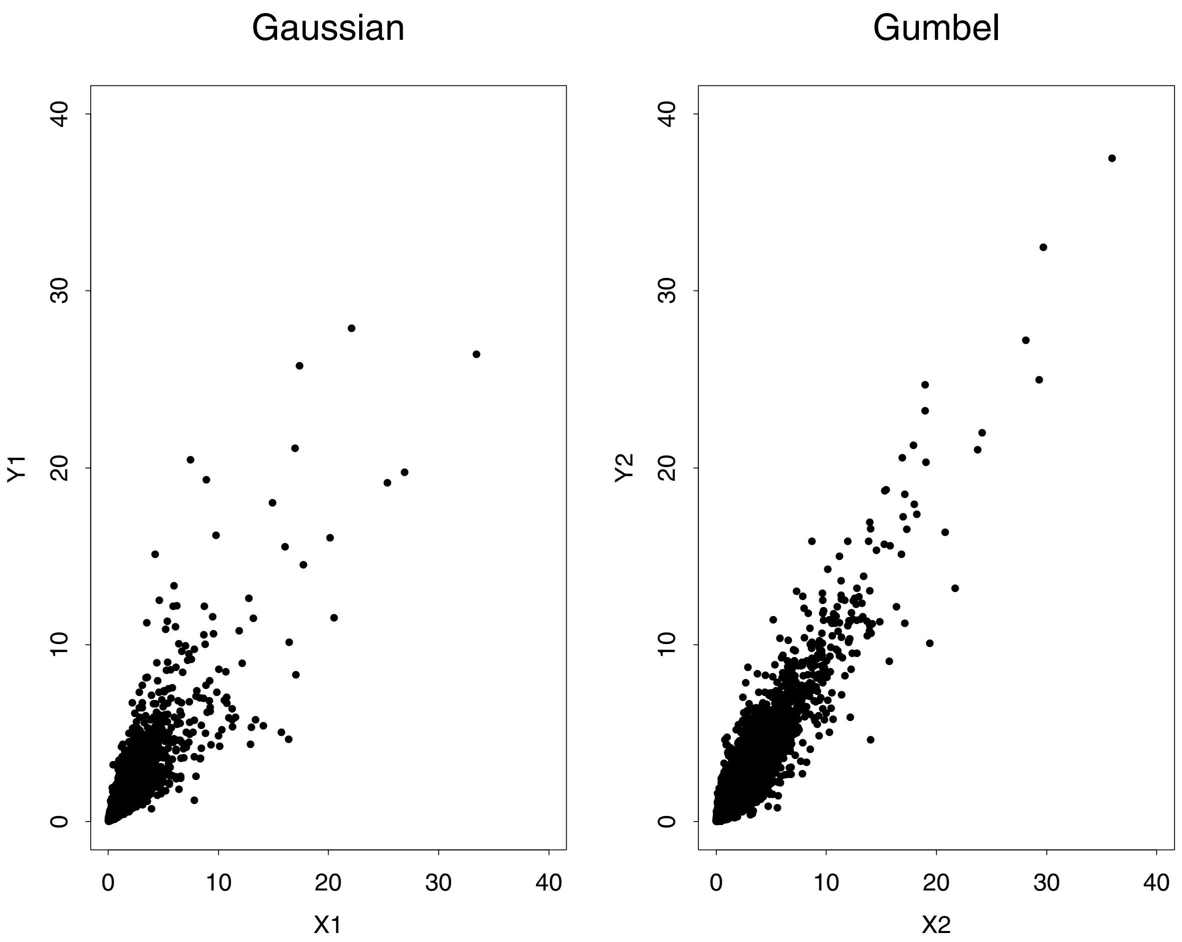
  provided that the limit $\lambda_U \in [0, 1]$ exists. If $\lambda_U > 0$, then (X, Y) has upper tail dependence.

- If

$$\lim_{u \searrow 0} C(u, u)/u = \lambda_L > 0$$

  exists, then (X, Y) has lower tail dependence.

- Tail dependence is a *copula* property.



Two bivariate distributions with standard lognormal margins and Kendall's tau 0.7, but different dependence structures. Gumbel copulas have upper tail dependence, but Gaussian copulas have not.

Elliptical copulas

A *spherical distribution* is an extension of the multivariate normal distribution $\mathcal{N}_n(\mathbf{0}, \mathbf{I}_n)$ and an *elliptical distribution* is an extension of $\mathcal{N}_n(\mu, \Sigma)$. Recall that $\mathcal{N}_n(\mu, \Sigma)$ can be defined as the distribution of

$$\mathbf{X} = \mu + A\mathbf{Y},$$

where $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I}_n)$ and $\Sigma = AA^T$.

A random vector $\mathbf{X}$ has a spherical distribution if

$$\mathbf{X} =_d R\mathbf{U}$$

for some positive random variable R independent of the random vector $\mathbf{U}$ uniformly distributed on the unit hypersphere.

A random vector $\mathbf{X}$ has an elliptical distribution with parameters μ and Σ if

$$\mathbf{X} =_d \mu + A\mathbf{Y},$$

where $\mathbf{Y}$ is spherical and $\Sigma = AA^T$.

- Let $\mathbf{Z} = (Z_1, \dots, Z_n) \sim \mathcal{N}_n(\mathbf{0}, \Sigma)$ with non-degenerate margins.

$$(F_1^{-1}(\Phi(Z_1)), \dots, F_n^{-1}(\Phi(Z_n))$$

has the Gaussian copula

$$C_\rho^{\text{Ga}}(\mathbf{u}) = \Phi_\rho^n(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)),$$

where ρ is the linear correlation matrix corresponding to the covariance matrix Σ.

- Let $\mathbf{X} = \frac{\sqrt{\nu}}{\sqrt{S}}\mathbf{Z}$, where $\mathbf{Z}$ and $S \sim \chi_\nu^2$ are independent.

$$(F_1^{-1}(t_\nu(X_1)), \dots, F_n^{-1}(t_\nu(X_n))$$

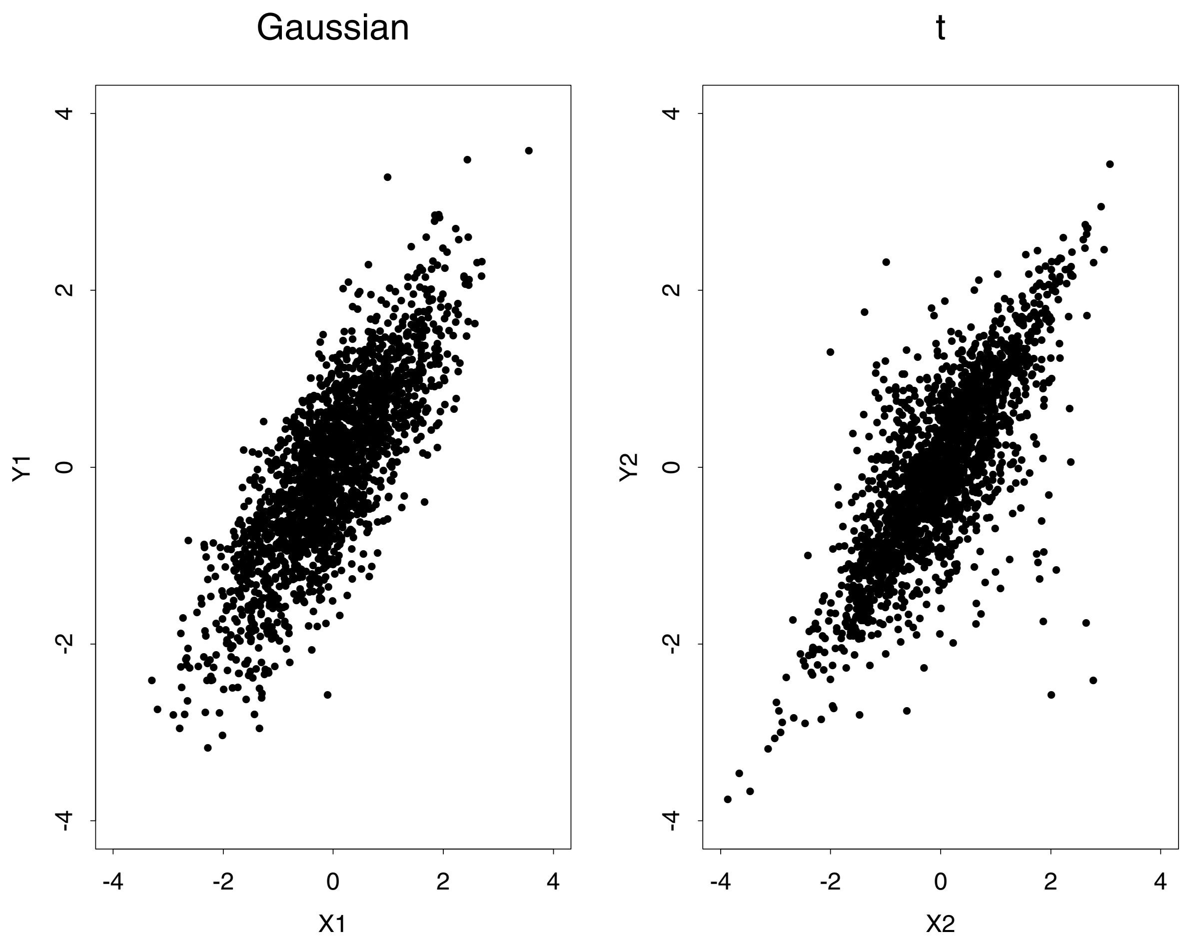
has the t_ν-copula

$$C_{\nu,\rho}^{\text{t}}(\mathbf{u}) = t_{\nu,\rho}^n(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_n)).$$

$C_{\nu,\rho}^{\text{t}}$ has upper and lower tail dependence.

- For all elliptical copulas

$$\tau(i,j) = \frac{2}{\pi} \arcsin \rho(i,j).$$



Two bivariate distributions with standard normal margins and $\tau = 0.6$. Gaussian and t_2-copulas.

Archimedean copulas

Let φ be a continuous strictly decreasing convex function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1)=0$. Then

$$\varphi^{[-1]}(\varphi(u)+\varphi(v)), \quad u,v\in[0,1],$$

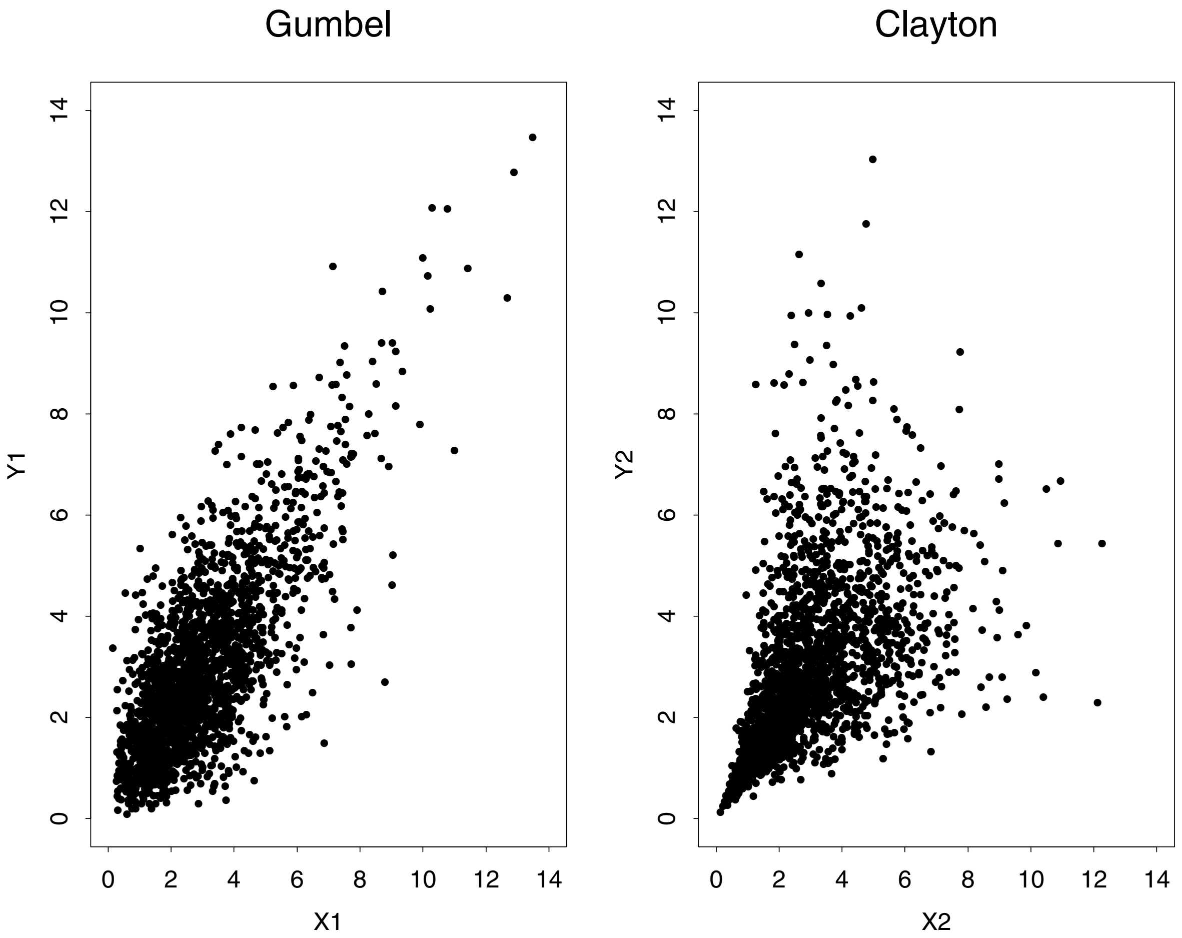
is a copula with generator φ, where

$$\varphi^{[-1]}(t)=\begin{cases}\varphi^{-1}(t), & t\leq\varphi(0),\\ 0, & t\geq\varphi(0).\end{cases}$$

If $\varphi(0)=\infty$, then $\varphi^{[-1]}=\varphi^{-1}$.

- Gumbel copula:
  Take $\varphi(t)=(-\ln t)^{\theta}$ with $\theta\in[1,\infty)$,
  $C_\theta(u,v)=\exp(-[(-\ln u)^\theta+(-\ln v)^\theta]^{1/\theta})$

- Clayton copula:
  Take $\varphi(t)=(t^{-\theta}-1)/\theta$ with $\theta\in[-1,\infty)\setminus\{0\}$,
  $C_\theta(u,v)=\max([u^{-\theta}+v^{-\theta}-1]^{-1/\theta},0)$



Two bivariate distributions with Gamma$(3,1)$ margins and $\tau = 0.5$.
Gumbel copulas have upper tail dependence,
Clayton copulas have lower tail dependence.

Risky dependence

- Risks $X_1, \ldots, X_n$.

- Thresholds $k_1, \ldots, k_n$.

- Default occurs for company i if $X_i > k_i$.

What is the probability of at least l defaults?

Assume historical data are available allowing estimation of
- marginal distributions,
- pairwise *rank* correlations.

$N = \left|\left\{i \in \{1, \ldots, n\} \;\middle|\; X_i > k_i\right\}\right|$ is the number of defaults.

The probability of all companies defaulting is given by

$$\begin{aligned}\mathbb{P}(N = n) &= \overline{H}(k_1, \ldots, k_n) \\ &= \widehat{C}(\overline{F}_1(k_1), \ldots, \overline{F}_n(k_n)),\end{aligned}$$

where $\widehat{C}$ is the *survival copula* of $(X_1, \ldots, X_n)$. We can evaluate $\mathbb{P}(N \geq l)$ for various copulas.

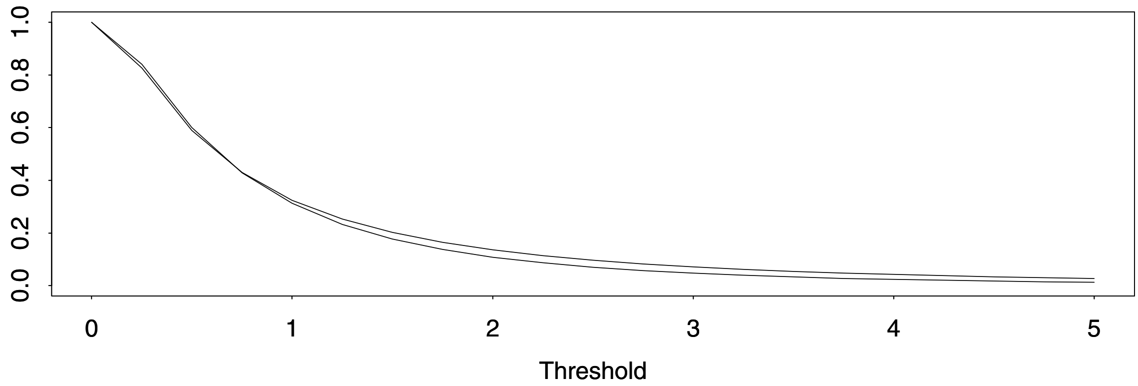
Illustration: $l = n = 3$; $X_i \sim$ Lognormal$(0, 1)$; $k_i = k$ for all i; $\tau(X_i, X_j) = 0.5$ for all $i \neq j$.

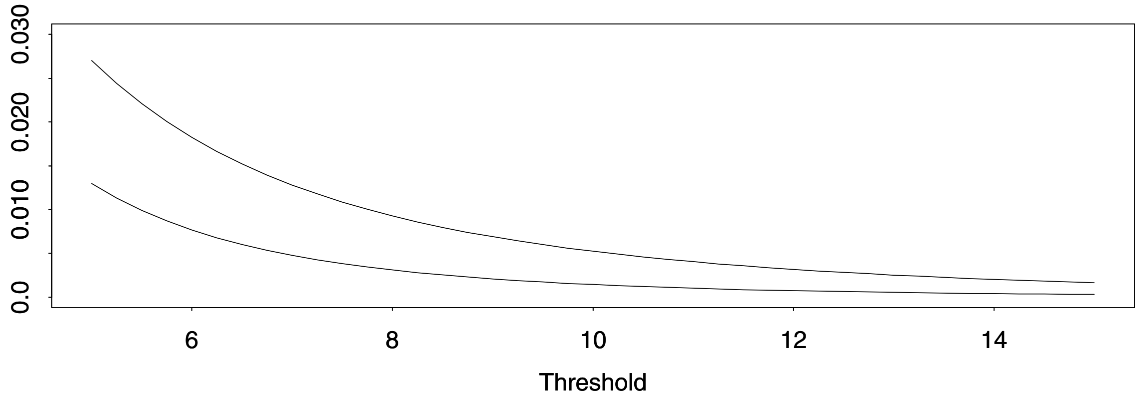
We compare trivariate Gaussian and Gumbel copulas and use the relations

$$\rho = \sin(\pi\tau/2) \quad \text{and} \quad \theta = \frac{1}{1-\tau}$$

to parametrisize the respective copulas so that they have a common Kendall's tau rank correlation matrix.

Probability of joint default





Note that for $k = 5 \approx \text{VaR}_{0.95}(X_i)$

$$\frac{\mathbb{P}^{\text{Gumbel}}\{N = 3\}}{\mathbb{P}^{\text{Gaussian}}\{N = 3\}} \approx 2$$

and that for $k = 10 \approx \text{VaR}_{0.99}(X_i)$

$$\frac{\mathbb{P}^{\text{Gumbel}}\{N = 3\}}{\mathbb{P}^{\text{Gaussian}}\{N = 3\}} \approx 4.$$

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- Independent Poisson shock processes

$$\{N^{(e)}(t), t \geq 0\}, \quad e = 1, \ldots, m,$$

with intensities $\lambda^{(e)} > 0$.
$N^{(e)}(t)$ is the number of shocks of type e during the time period $[0, t]$.

- At the rth occurrence of shock e the indicators $I_{j,r}^{(e)} \sim Be(p_j^{(e)})$ indicates whether a loss of type $j \in \{1, \ldots, n\}$ occurs.

$$\mathbf{I}_r^{(e)} = (I_{1,r}^{(e)}, \ldots, I_{n,r}^{(e)})'$$

for $r = 1, \ldots, N^{(e)}(t)$ are considered to be independent and identically distributed with a multivariate Bernoulli distribution.

For higher order marginal probabilities of $\mathbf{I}_r^{(e)}$ we use the notation

$$P(I_j^{(e)} = i_j, I_k^{(e)} = i_k) = p_{j,k}^{(e)}(i_j, i_k).$$

The loss counters

The processes

$$\{N_j(t), t \geq 0\}, \quad j = 1, \ldots, n,$$

counts losses of type j occurring in $[0, t]$. The counting processes for shock events and losses are thus linked by

$$N_j(t) = \sum_{e=1}^{m} \sum_{r=1}^{N^{(e)}(t)} I_{j,r}^{(e)}.$$

The process

$$\{(N_1(t), \ldots, N_n(t)), t \geq 0\}$$

is multivariate Poisson with

$$E(N_j(t)) = t \sum_{e=1}^{m} \lambda^{(e)} p_j^{(e)}.$$

The covariance structure is given by

$$\text{cov}\left(N_j(t), N_k(t)\right) = t \sum_{e=1}^{m} \lambda^{(e)} p_{j,k}^{(e)}(1,1).$$

The total number of losses

The total number of losses $N(t) = \sum_{j=1}^{n} N_j(t)$ is not Poisson but rather *compound Poisson*.

Moments

The moments of $N(t)$ are given recursively by

$$E\left(N(t)^p\right) = t \sum_{k=0}^{p-1} \binom{p-1}{k} \varphi(k+1) E\left(N(t)^{p-k-1}\right)$$

where

$$E(N(t)) = t \sum_{j=1}^{n} \sum_{e=1}^{m} \lambda^{(e)} p_j^{(e)}$$

and

$$\varphi(k) = \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \sum_{e=1}^{m} \lambda^{(e)} p_{j_1,\ldots,j_k}^{(e)}(1,\ldots,1).$$

Probability distribution

The probabilities $P(N(t) \leq k)$ can be calculated using Panjer recursion.

Times to first losses

Let $T_j = \text{inf}\{t : N_j(t) > 0\}$. $(T_1, \ldots, T_n)$ has a multivariate exponential distribution whose survival copula is a Marshall-Olkin copula.

$$P(T_i > t_i, T_j > t_j) = C_{\alpha_i,\alpha_j}(\overline{F}_i(t_i), \overline{F}_j(t_j))$$

where

$$\overline{F}_i(t_i) = \exp\left\{-t \sum_{e=1}^{m} \lambda^{(e)} p_i^{(e)}\right\}$$

and

$$C_{\alpha_i,\alpha_j}(u,v) = \min(u^{1-\alpha_i}v, uv^{1-\alpha_j})$$

with

$$\alpha_i = \frac{\sum_{e=1}^{m} \lambda^{(e)} p_{i,j}^{(e)}(1,1)}{\sum_{e=1}^{m} \lambda^{(e)} p_i^{(e)}}, \alpha_j = \frac{\sum_{e=1}^{m} \lambda^{(e)} p_{i,j}^{(e)}(1,1)}{\sum_{e=1}^{m} \lambda^{(e)} p_j^{(e)}}.$$

Linear and Kendall's tau correlation structure:

$$\rho(T_i, T_j) = \tau(T_i, T_j) = \frac{\alpha_i\alpha_j}{\alpha_i + \alpha_j - \alpha_i\alpha_j}.$$

Large loan portfolios

Consider a loan portfolio consisting of $n = 100000$ obligors, divided into sectors and rating categories. We will consider

- a global shock process which could be thought of as recessions in the world economy;
- sector shock processes which represent adverse economic conditions for specific industries or geographical sectors;
- idiosyncratic "bad management" shock processes for each obligor.

Simplifying notation

- Denote the conditional default probability for an l-rated k-sector obligor by s_l^k given a k-sector shock and g_l^k given a global shock.

- For the jth obligor belonging to sector $k = k(j)$ and rating class $l = l(j)$

$$\lambda_j = \lambda_{\text{idio}}^{l(j)} + s_{l(j)}^{k(j)} \lambda_{\text{sector}}^{k(j)} + g_{l(j)}^{k(j)} \lambda_{\text{global}}.$$

- Companies with the same rating l have the same overall default intensity λ_{total}^l where

$$\lambda_j = \lambda_{\text{total}}^{l(j)}.$$

- Let $n_{l,k}$ denote the number of companies in rating class l and sector k.

Parameter assumptions

- $t = 1$ year;
- two rating classes, four sectors;
- $\lambda^1_{\text{total}} = 0.005$;
- $\lambda^2_{\text{total}} = 0.02$;
- The sizes $n_{l,k}$ of the 8 homogeneous groups are

$$\begin{array}{ll} n_{1,1} = 10000, & n_{1,2} = 20000, \\ n_{1,3} = 15000, & n_{1,4} = 5000, \\ n_{2,1} = 10000, & n_{2,2} = 25000, \\ n_{2,3} = 10000, & n_{2,4} = 5000. \end{array}$$

- The default indicators for a group of companies are conditionally independent given the occurrence of a global or sector shock.

Case 1

For both rating classes we attribute

- 40% of defaults to idiosyncratic shocks:

$$\lambda^l_{\text{idio}}/\lambda^l_{\text{total}} = 0.4.$$

- To sector specific causes:
  20% of defaults of sector 1 obligors,
  50% of defaults of sector 2 obligors,
  10% of defaults of sector 3 obligors,
  40% of defaults of sector 4 obligors:

$$s^k_l \lambda^k_{\text{sector}}/\lambda^l_{\text{total}} = 0.2, 0.5, 0.1, 0.4.$$

Moreover we believe that the frequencies of sector and global shocks are in the ratio

$$\lambda^1_{\text{sector}} : \lambda^2_{\text{sector}} : \lambda^3_{\text{sector}} : \lambda^4_{\text{sector}} : \lambda_{\text{global}} = 1 : 5 : 2 : 4 : 1.$$

The model is specified up to a factor $f > 0$:

Shock intensities:

$$(\lambda^1_{\text{sector}}, \lambda^2_{\text{sector}}, \lambda^3_{\text{sector}}, \lambda^4_{\text{sector}}, \lambda_{\text{global}}) \\ = f(0.2, 1.0, 0.4, 0.8, 0.2)$$

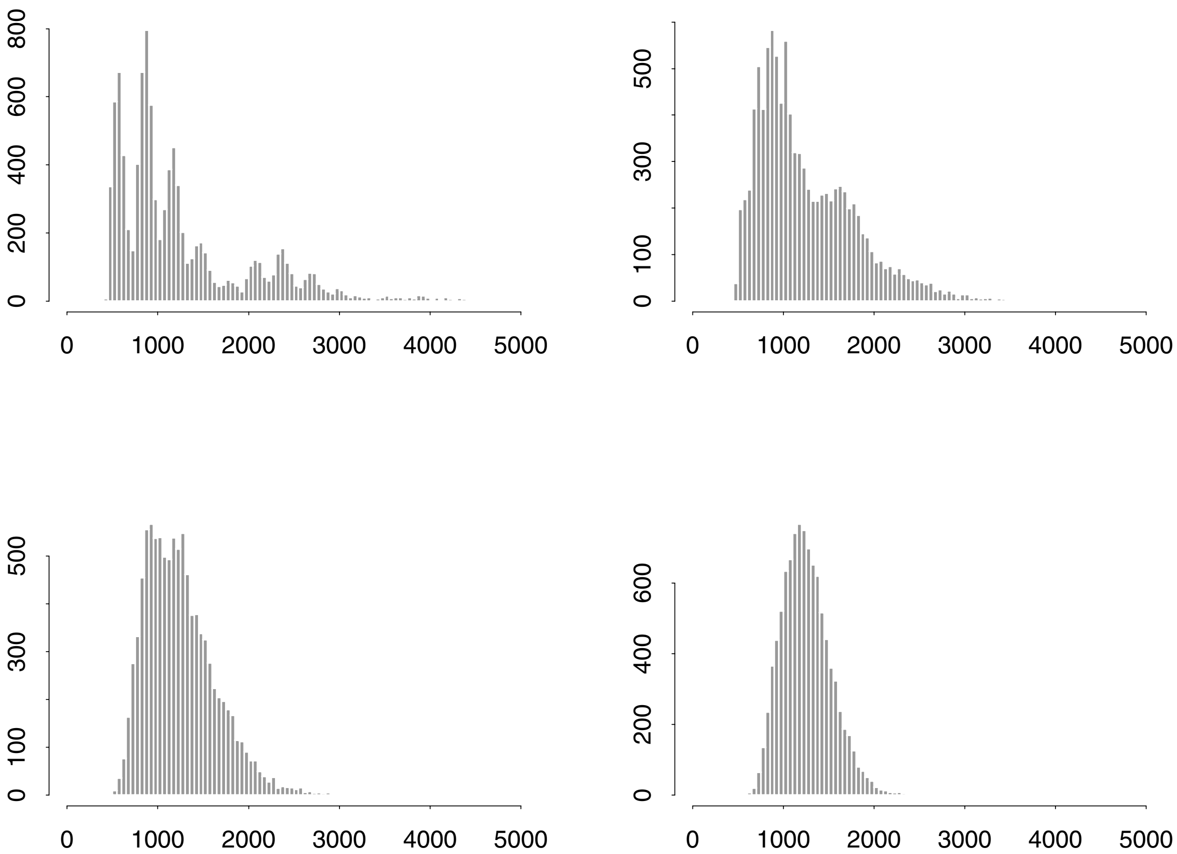
Univariate conditional default probabilities given sector shocks:

$$(s^1_1, s^2_1, s^3_1, s^4_1, s^1_2, s^2_2, s^3_2, s^4_2) \\ = \frac{1}{f}(0.5, 0.25, 0.125, 0.25, 2, 1, 0.5, 1)10^{-2}$$

Univariate conditional default probabilities given global shocks:

$$(g^1_1, g^2_1, g^3_1, g^4_1, g^1_2, g^2_2, g^3_2, g^4_2) \\ = \frac{1}{f}(1, 0.25, 1.25, 0.5, 4, 1, 5, 2)10^{-2}.$$

- How important is the choice of f?



Density estimates (histograms) plotted by row for $f = 1, 2, 4, 8$.

Case 2

Suppose we are able to quantify the probabilities with which sector or global shocks cause the default of individual obligors. Set

$$(s_1^1, s_1^2, s_1^3, s_1^4, s_2^1, s_2^2, s_2^3, s_2^4)$$
$$= (0.25, 0.08, 0.05, 0.1, 1, 0.3, 0.25, 0.25)10^{-2}$$

and

$$(g_1^1, g_1^2, g_1^3, g_1^4, g_2^1, g_2^2, g_2^3, g_2^4)$$
$$= (0.25, 0.1, 0.4, 0.1, 1, 0.5, 1.5, 1)10^{-2}.$$

We vary the shock intensities

$$(\lambda_{\text{idio}}^1, \lambda_{\text{idio}}^2, \lambda_{\text{sector}}^1, \lambda_{\text{sector}}^2, \lambda_{\text{sector}}^3, \lambda_{\text{sector}}^4, \lambda_{\text{global}})$$

progressively in the following way

$$(0.005, 0.02, 0.0, 0.0, 0.0, 0.0, 0.0) \rightarrow$$
$$(0.004, 0.016, 0.2, 1.0, 0.4, 0.8, 0.2) \rightarrow$$
$$(0.002, 0.008, 0.6, 3.0, 1.2, 2.4, 0.6) \rightarrow$$
$$(0.0, 0.0, 1.0, 5.0, 2.0, 4.0, 1.0).$$

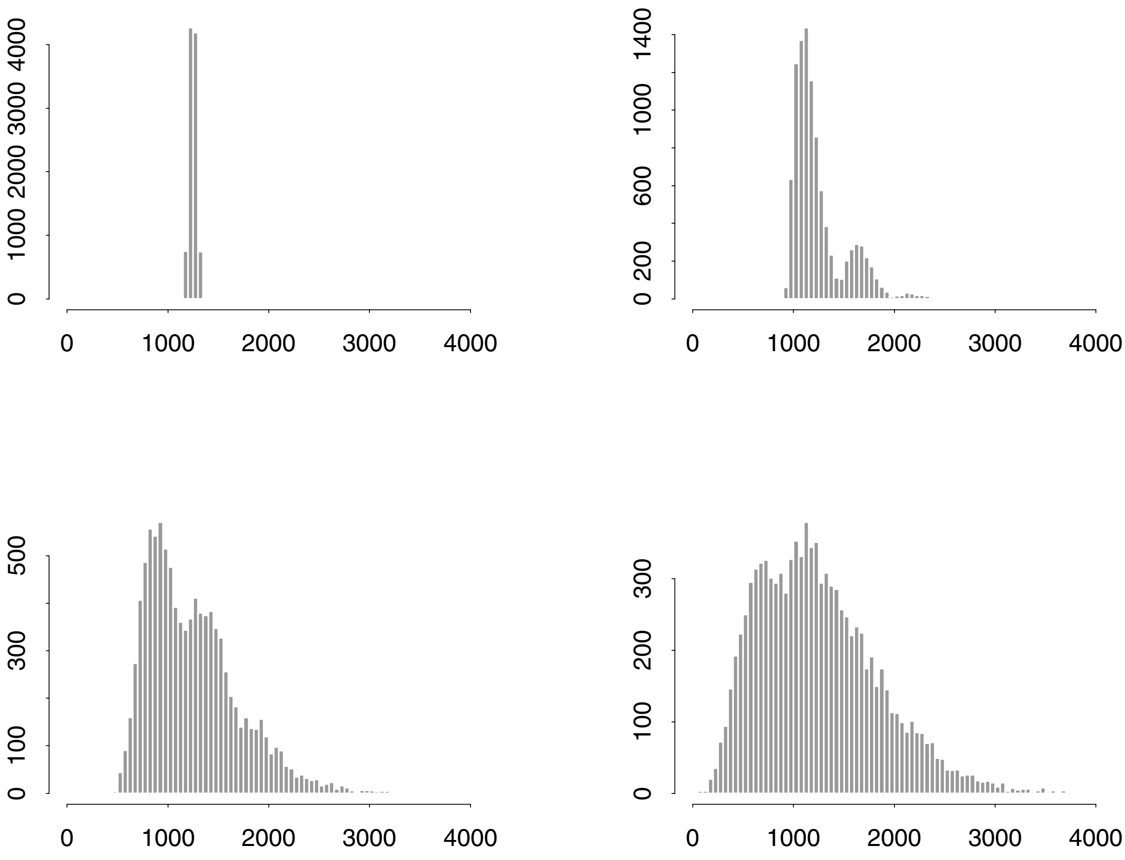
Hence

- we start with the special case of no common shocks and a situation where the individual default counters $N_j(1)$ are independent Poisson and the total number of defaults $N(1)$ is Poisson.

- In the second model we still attribute 80% of the default intensities λ_j to idiosyncratic shocks, but we now have 20% in common shocks.

- In the third model we have 40% in idiosyncratic and 60% in common shocks.

- In the final model we have only common shocks.



The effect of increasing the intensities of common shocks while decreasing the intensities of idiosyncratic shocks when the univariate conditional default probabilities are held fixed.