Modelling Dependence with Copulas and Applications to Risk Management

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Copula ideas provide

- a better understanding of dependence,
- a basis for flexible techniques for simulating dependent random vectors,
- scale-invariant measures of association similar to but less problematic than linear correlation,
- a basis for constructing multivariate distributions fitting the observed data,
- a way to study the effect of different dependence structures for functions of dependent random variables, e.g. upper and lower bounds.

Example of bounds for linear correlation

For $\sigma > 0$ let

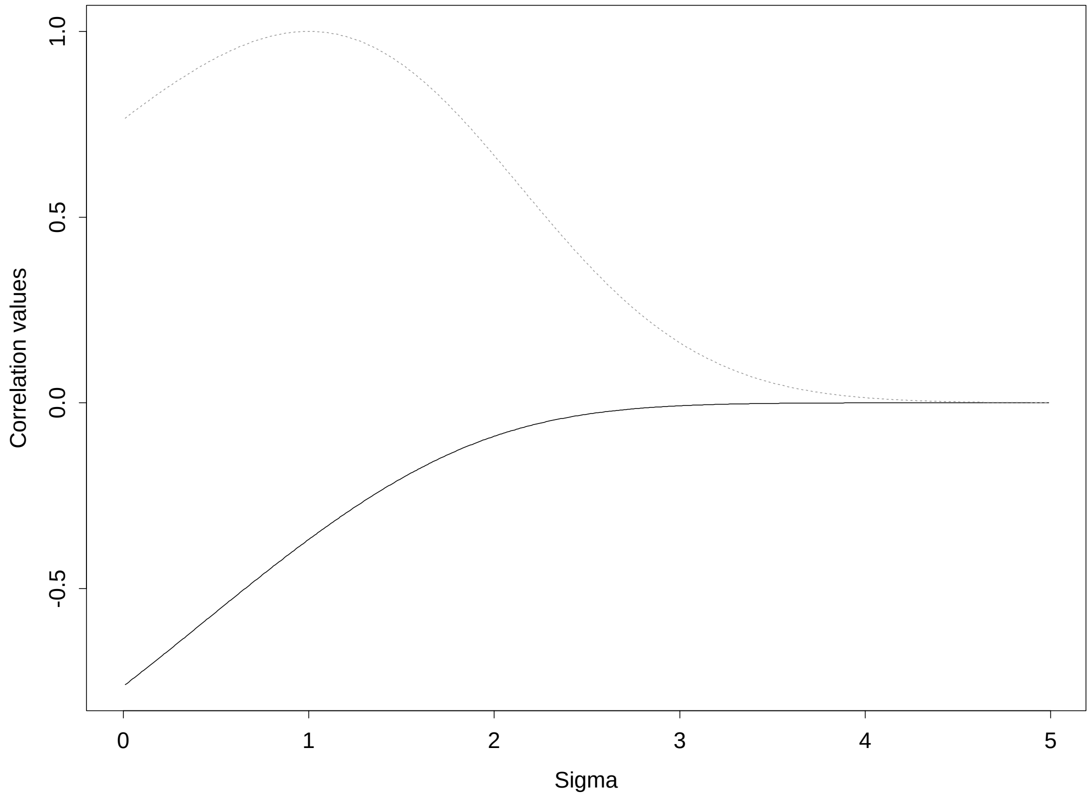
$$X \sim \text{Lognormal}(0, 1),$$
$$Y \sim \text{Lognormal}(0, \sigma^2).$$

Then the minimal obtainable correlation between X and Y (obtained when X and Y are *countermonotonic*) is

$$\rho_l^{\text{min}}(X, Y) = \frac{e^{-\sigma} - 1}{\sqrt{e-1}\sqrt{e^{\sigma^2} - 1}},$$

and the maximal obtainable correlation (obtained when X and Y are *comonotonic*) is

$$\rho_l^{\text{max}}(X, Y) = \frac{e^{\sigma} - 1}{\sqrt{e-1}\sqrt{e^{\sigma^2} - 1}}.$$



The upper bound $\rho_l^{\max}(X, Y)$ and lower bound $\rho_l^{\min}(X, Y)$ for $\sigma \in [0, 5]$.
Note: This holds regardless of the dependence between X and Y.
Note: For $\sigma = 4$, $\rho_l(X, Y) = 0.01372$ means that X and Y are perfectly positively dependent ($Y = T(X)$, T increasing)!

Drawbacks of linear correlation

- Linear correlation is not defined if the variance of X or Y is infinite.

- Linear correlation can easily be misinterpreted.

- Linear correlation is not invariant under non-linear strictly increasing transformations $T : \mathbb{R} \longrightarrow \mathbb{R}$, i.e.,

$$\rho_l\left(T(X), T(Y)\right) \neq \rho_l\left(X, Y\right).$$

- Given margins F and G for X and Y, all linear correlations between -1 and 1 can in general not be obtained by a suitable choice of the joint distribution.

Naive approach using linear correlation

Consider a portfolio of n "risks" $X_1, \ldots, X_n$. Suppose that we want to examine the distribution of some function $f(X_1, \ldots, X_n)$ representing the risk of or the future value of a contract written on the portfolio.

1. Estimate marginal distributions $F_1, \ldots, F_n$.
2. Estimate pairwise linear correlations $\rho_l(X_i, X_j)$ for $i, j \in \{1, \ldots, n\}$ with $i \neq j$.
3. Use this information in some Monte Carlo simulation procedure to generate dependent data.

Questions:

- Is there a multivariate distribution with this linear correlation matrix?

- How do we in general find an appropriate simulation procedure?

Copulas

Definition

A copula, $C : [0,1]^n \mapsto [0,1]$, is a multivariate distribution function whose margins are uniformly distributed on $[0,1]$.

Sklar's theorem

Let H be an n-dimensional distribution function with margins $F_1, \ldots, F_n$. Then there exists an n-copula C such that for all $x_1, \ldots, x_n$ in $\overline{\mathbb{R}}^n$,

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

Conversely, if C is an n-copula and $F_1, \ldots, F_n$ are distribution functions, then the function H defined above is an n-dimensional distribution function with margins $F_1, \ldots, F_n$.

Hence the copula of $(X_1, \ldots, X_n) \sim H$ is the distribution function of $(F_1(X_1), \ldots, F_n(X_n))$.

If $F_1, \dots, F_n$ are strictly increasing distribution functions (d.f.s), then for every $\mathbf{u} = (u_1, \dots, u_n)$ in $[0, 1]^n$,

$$C(\mathbf{u}) = H(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)).$$

From the multivariate standard normal distribution $\mathcal{N}_n(\mathbf{0}, \rho_l)$ we get the normal or Gaussian n-copula

$$C_{\rho_l}^{\text{Ga}}(\mathbf{u}) = \Phi_{\rho_l}^n(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)),$$

where $\Phi_{\rho_l}^n$ is the d.f. of $\mathcal{N}_n(\mathbf{0}, \rho_l)$, ρ_l is a linear correlation matrix and Φ is the d.f. of $\mathcal{N}(0, 1)$.

The multivariate normal distribution $\mathcal{N}_n(\mu, \Sigma)$ gives the same copula expression, with ρ_l corresponding to Σ.

Further examples of copulas

$$
\begin{aligned}
M^n(\mathbf{u}) &= \min(u_1, u_2, \ldots, u_n) \\
W^n(\mathbf{u}) &= \max(u_1 + u_2 + \cdots + u_n - n + 1, 0) \\
\Pi^n(\mathbf{u}) &= u_1 u_2 \ldots u_n
\end{aligned}
$$

Note: M^n and Π^n are copulas for all $n \geq 2$ but W^n is a copula only for $n = 2$.

Definition

1. X, Y comonotonic
$\iff (X, Y)$ has copula M^2
$\iff (X, Y) =_d (\alpha(Z), \beta(Z))$, α, β increasing and Z is some real valued r.v.

2. X, Y countermonotonic
$\iff (X, Y)$ has copula W^2
$\iff (X, Y) =_d (\alpha(Z), \beta(Z))$, α inc., β dec. and Z is some real valued r.v.

3. $X_1, \ldots, X_n$ independent
$\iff (X_1, \ldots, X_n)$ has copula Π^n.

Properties of copulas

Bounds

For every $\mathbf{u} \in [0,1]^n$ we have

$$W^n(\mathbf{u}) \leq C(\mathbf{u}) \leq M^n(\mathbf{u}).$$

These bounds are the best possible.

Concordance ordering

If C_1 and C_2 are copulas, we say that C_1 is smaller than C_2 and write $C_1 \prec C_2$ if

$$C_1(u,v) \leq C_2(u,v)$$

for all u, v in [0,1].

Copulas and monotone transformations

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are strictly increasing, then $\alpha_1(X_1), \alpha_2(X_2), \ldots, \alpha_n(X_n)$ have the same copula as $X_1, X_2, \ldots, X_n$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be strictly monotone and let $\alpha_1(X_1), \alpha_2(X_2), \ldots, \alpha_n(X_n)$ have copula $C_{\alpha_1(X_1),\alpha_2(X_2),\ldots,\alpha_n(X_n)}$.
Suppose α_1 is strictly decreasing. Then

$$\begin{aligned} &C_{\alpha_1(X_1),\alpha_2(X_2),\ldots,\alpha_n(X_n)}(u_1, u_2, \ldots, u_n) \\ &\quad = C_{\alpha_2(X_2),\ldots,\alpha_n(X_n)}(u_2, \ldots, u_n) \\ &\quad\ - C_{X_1,\alpha_2(X_2),\ldots,\alpha_n(X_n)}(1 - u_1, u_2, \ldots, u_n). \end{aligned}$$

If α and β are strictly decreasing:

$$\begin{aligned} &C_{\alpha(X),\beta(Y)}(u, v) \\ &\quad = v - C_{X,\beta(Y)}(1 - u, v) \\ &\quad = v - \Big(1 - u - C_{X,Y}(1 - u, 1 - v)\Big) \\ &\quad = u + v - 1 + C_{X,Y}(1 - u, 1 - v) \end{aligned}$$

Here $C_{\alpha(X),\beta(Y)}$ is the survival copula, $\widehat{C}$, of X and Y, i.e.,

$$\begin{aligned} \overline{H}(x, y) &= \mathbb{P}[X > x, Y > y] \\ &= \widehat{C}(\overline{F}(x), \overline{G}(y)). \end{aligned}$$

Kendall's tau and Spearman's rho

Let (x, y) and (x', y') be two observations from a random vector (X, Y) of continuous random variables. We say that (x, y) and (x', y') are concordant if $(x-x')(y-y') > 0$, and discordant if $(x - x')(y - y') < 0$.

Let (X', Y') be an independent copy of (X, Y). Then *Kendall's tau* between X and Y is

$$\begin{aligned}\tau(X, Y) \;=\; & \mathbb{P}[(X - X')(Y - Y') > 0] - \\ & \mathbb{P}[(X - X')(Y - Y') < 0]\end{aligned}$$

For a sample of size n from (X, Y), with c concordant pairs and d discordant pairs the sample version of Kendall's tau is given by

$$\frac{c - d}{c + d} = (c - d)/\binom{n}{2}$$

Kendall's tau can be expressed only in terms of the copula C of (X, Y)

$$\tau(X,Y) = \tau(C) = 4\iint_{[0,1]^2} C(u,v)\,\mathrm{d}C(u,v) - 1$$

and this is also true for *Spearman's rho*

$$\begin{aligned}\rho_S(X,Y) &= \rho_S(C) = 12\iint_{[0,1]^2} uv\,\mathrm{d}C(u,v) - 3\\ &= 12\iint_{[0,1]^2} C(u,v)\,\mathrm{d}u\,\mathrm{d}v - 3.\end{aligned}$$

Note that for $(U,V) \sim C$

$$\begin{aligned}\rho_S(C) &= 12\iint_{[0,1]^2} uv\,\mathrm{d}C(u,v) - 3\\ &= 12\mathbb{E}(UV) - 3 = \frac{\mathbb{E}(UV) - 1/4}{1/12}\\ &= \frac{\mathbb{E}(UV) - \mathbb{E}(U)\mathbb{E}(V)}{\sqrt{\mathrm{Var}(U)}\sqrt{\mathrm{Var}(V)}}.\end{aligned}$$

Since $(F(X), G(Y)) \sim C$ we get

$$\rho_S(X,Y) = \rho_l(F(X), G(Y)).$$

Kendall's tau and Spearman's rho are called *rank correlations*.

Properties of rank correlation

Let X and Y be continuous random variables with copula C, and let δ denote Kendall's tau or Spearman's rho. The following properties are not shared by linear correlation.

- If T is strictly monotone, then
  $\delta(T(X),Y)=\delta(X,Y)$, T increasing,
  $\delta(T(X),Y)=-\delta(X,Y)$, T decreasing.

- $\delta(X,Y)=1 \quad \Longleftrightarrow \quad C=M^2$

- $\delta(X,Y)=-1 \quad \Longleftrightarrow \quad C=W^2$

- $\delta(X,Y)$ depends only on the copula of (X,Y).

Given a proper rank correlation matrix there is always a multivariate distribution with this rank correlation matrix, regardless of the choice of margins. This is *not* true for linear correlation.

Tail dependence

Let X and Y be random variables with continuous distribution functions F and G. The coefficient of upper tail dependence of X and Y is

$$\lim_{u \nearrow 1} \mathbb{P}[Y > G^{-1}(u) | X > F^{-1}(u)] = \lambda_U$$

provided that the limit $\lambda_U \in [0,1]$ exists.
If a bivariate copula C is such that

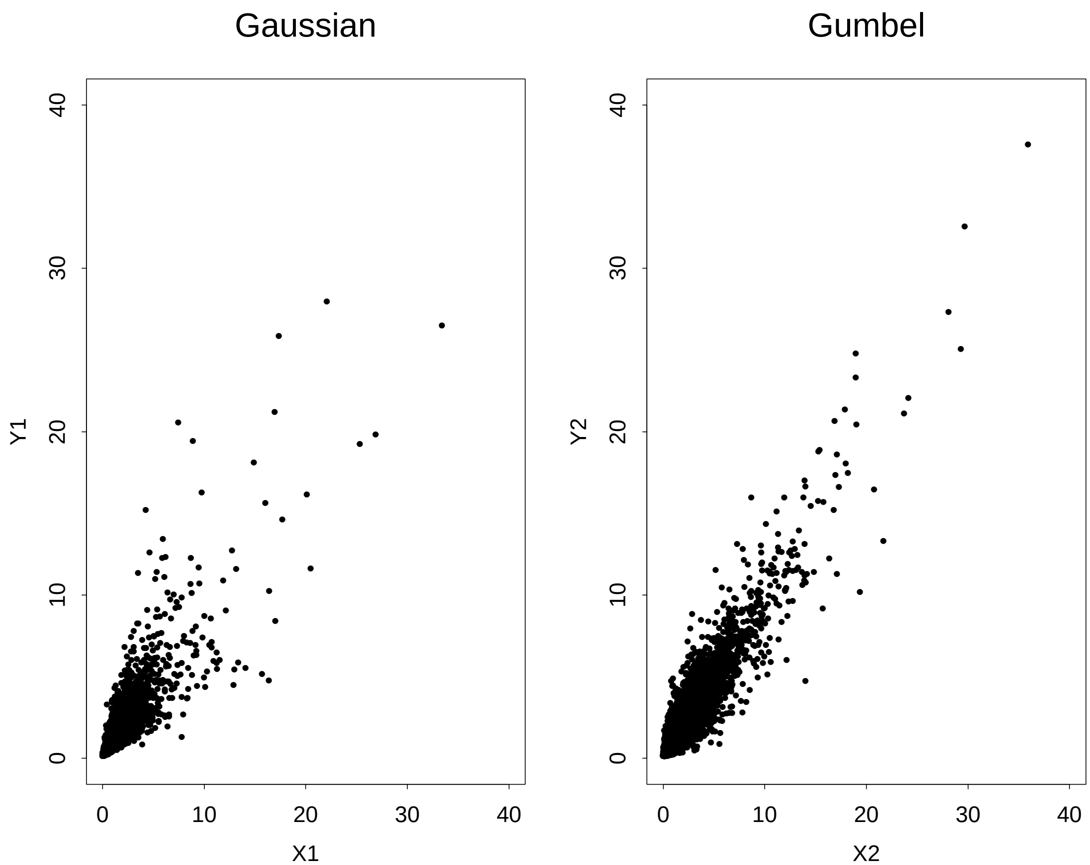
$$\lim_{u \nearrow 1} \overline{C}(u,u)/(1-u) = \lambda_U > 0$$

exists, then C has upper tail dependence.
Recall that $\overline{C}(u,u) = 1 - 2u + C(u,u)$.
If

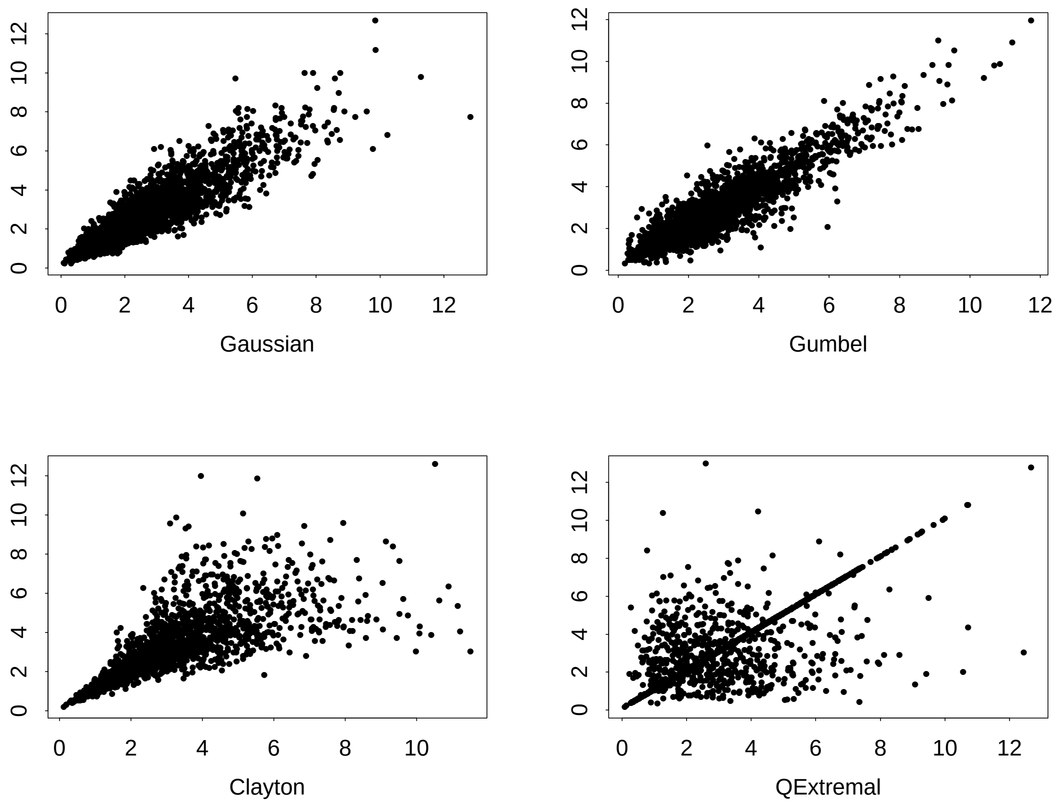
$$\lim_{u \searrow 0} C(u,u)/u = \lambda_L > 0$$

exists, then C has lower tail dependence.

Note that tail dependence is a *copula* property.



Two bivariate distributions with standard lognormal margins and Kendall's tau 0.7, but different dependence structures. Gumbel copulas (defined later) have upper tail dependence, but Gaussian copulas have not.



Four bivariate distributions with Gamma(3, 1) margins and Kendall's tau 0.7, but different dependence structures.

Marshall-Olkin copulas

Consider a two component system where the components are subjects to shocks, which are fatal to one or both components.

Let X_1 and X_2 denote the lifetimes of the components.

Assume that the shocks form three independent Poisson processes with parameters λ_1, λ_2 and $\lambda_{12} \geq 0$, where the index indicate whether the shocks kill only component 1, only component 2 or both. Then the times Z_1, Z_2 and Z_{12} of occurrence of these shocks are independent exponential random variables with these parameters. The joint survival function of X_1 and X_2 is

$$\overline{H}(x_1, x_2) = \widehat{C}(\overline{F}_1(x_1), \overline{F}_2(x_2)).$$

The univariate survival margins

$$\overline{F}_1(x_1) = \exp(-(\lambda_1 + \lambda_{12})x_1),$$

$$\overline{F}_2(x_2) = \exp(-(\lambda_2 + \lambda_{12})x_2)$$

and the survival copula (a *Marshall-Olkin copula*)

$$\begin{aligned}\widehat{C}(u_1, u_2) &= C_{\alpha_1,\alpha_2}(u_1, u_2) \\ &= \min(u_1^{1-\alpha_1} u_2, u_1 u_2^{1-\alpha_2})\end{aligned}$$

with parameters

$$\alpha_1 = \lambda_{12}/(\lambda_1 + \lambda_{12}), \quad \alpha_2 = \lambda_{12}/(\lambda_2 + \lambda_{12}).$$

Spearman's rho, Kendall's tau and the coefficient of upper tail dependence

$$\begin{aligned}\rho_{\alpha_1,\alpha_2} &= 12 \iint_{[0,1]^2} C_{\alpha_1,\alpha_2}(u, v)\, \mathrm{d}u\, \mathrm{d}v - 3 \\ &= \cdots \\ &= \frac{3\alpha_1\alpha_2}{2\alpha_1 + 2\alpha_2 - \alpha_1\alpha_2}, \\ \tau_{\alpha_1,\alpha_2} &= 4 \iint_{[0,1]^2} C_{\alpha_1,\alpha_2}(u, v)\, \mathrm{d}C_{\alpha_1,\alpha_2}(u, v) - 1 \\ &= \cdots \\ &= \frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1\alpha_2}, \\ \lambda_U &= \lim_{u \nearrow 1} \frac{\overline{C}(u, u)}{1 - u} = \min(\alpha_1, \alpha_2).\end{aligned}$$

A natural multivariate extension

Consider n components. Assign shock intensities $\lambda_1, \dots \lambda_l$ to each of the $l = 2^n - 1$ non-empty subsets of components, then $(X_1, \dots, X_n)$ has a survival copula whose bivariate margins for $i, j \in \{1, \dots, n\}$ for $i \neq j$ are Marshall-Olkin copulas C_{α_i, α_j} with

$$\alpha_i = \Big(\sum_{k=1}^{l} a_{ik} a_{jk} \lambda_k\Big) \Big/ \Big(\sum_{k=1}^{l} a_{ik} \lambda_k\Big),$$
$$\alpha_j = \Big(\sum_{k=1}^{l} a_{ik} a_{jk} \lambda_k\Big) \Big/ \Big(\sum_{k=1}^{l} a_{jk} \lambda_k\Big),$$

where $a_{ik} \in \{0, 1\}$ indicates whether a shock of subset k kills component i.

Shock models are used as models in e.g. insurance and credit risk.

Remark: A multivariate extension is an n-copula whose bivariate margins are in the bivariate copula family and whose higher dimensional margins are of the same multivariate form.

Elliptical copulas

A *spherical distribution* is an extension of the multivariate normal distribution $\mathcal{N}_n(\mathbf{0}, \mathbf{I}_n)$ and an *elliptical distribution* is an extension of $\mathcal{N}_n(\mu, \Sigma)$. Recall that $\mathcal{N}_n(\mu, \Sigma)$ can be defined as the distribution of

$$\mathbf{X} = \mu + A\mathbf{Y},$$

where $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I}_n)$ and $\boldsymbol{\Sigma} = AA^T$.

A random vector $\mathbf{X}$ is said to have a spherical distribution if for every orthogonal matrix Γ ($\Gamma^T\Gamma = \Gamma\Gamma^T = \mathbf{I}_n$)

$$\Gamma\mathbf{X} =_d \mathbf{X}.$$

Alternatively, $\mathbf{X}$ has a spherical distribution if

$$\mathbf{X} =_d R\mathbf{U}$$

for some positive random variable R independent of the random vector $\mathbf{U}$ uniformly distributed on the unit hypersphere $S_{n-1} = \{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{z}^T\mathbf{z} = 1\}$.

A random vector $\mathbf{X}$ is said to have an elliptical distribution with parameters μ and Σ if

$$\mathbf{X} =_d \mu + A\mathbf{Y},$$

where $\mathbf{Y}$ has a spherical distribution of dimension $k = \text{rank}(\Sigma)$ and A is an $(n \times k)$-matrix with $AA^T = \Sigma$.

An elliptical distribution is uniquely determined by its mean, covariance matrix and the type of its margins (t_ν, normal, etc.).

Hence the copula of an multivariate elliptical distribution is uniquely determined by its correlation matrix and knowledge of its type.

One example is the Gaussian n-copula

$$C_{\rho_l}^{\text{Ga}}(\mathbf{u}) = \Phi_{\rho_l}^n(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)),$$

where ρ_l is a linear correlation matrix.

Distributions with Gaussian copulas

If $(X_1, \ldots, X_n)$ have a multivariate normal distribution with linear correlation matrix ρ_l, then for $i, j \in \{1, \ldots, n\}$

- Spearman's rho $\rho_S(X_i, X_j) = \frac{6}{\pi} \arcsin \frac{\rho_l(i,j)}{2}$,
- Kendall's tau $\tau(X_i, X_j) = \frac{2}{\pi} \arcsin \rho_l(i, j)$

It follows from transformation invariance for strictly increasing d.f.s $F_1, \ldots, F_n$ that

- the elements of the Kendall's tau and Spearman's rho correlation matrix for $(F_1^{-1}(\Phi(X_1)), \ldots, F_n^{-1}(\Phi(X_n)) \sim H$ is given by the above expressions,

- all proper rank correlations matrices can be obtained for H for all choices of $F_1, \ldots, F_n$.

t_ν-copulas

The multivariate t_ν-copula is given by

$$C^{\mathrm{t}}_{\nu,\rho_l}(\mathbf{u}) = \Theta^n_{\nu,\rho_l}(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_n)),$$

where Θ^n_{ν,ρ_l} is the d.f. of the n-dimensional standard t_ν distribution with linear correlation matrix ρ_l and t_ν^{-1} is the quantile function of the univariate standard t_ν distribution with ν degrees of freedom.

An n-dimensional t_ν distribution can be obtaind from $\mathcal{N}_n(\mathbf{0}, \Sigma)$ and χ^2_ν via

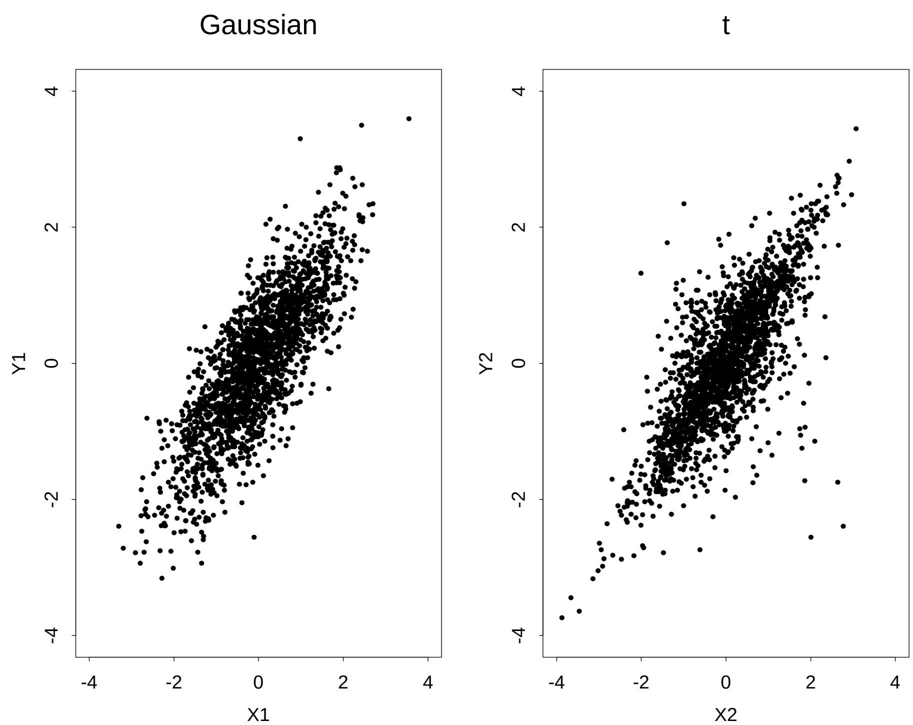
$$\mathbf{X} = \frac{\sqrt{\nu}}{\sqrt{S}}\mathbf{Z} + \mu,$$

where $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, \Sigma)$ is independent of $S \sim \chi^2_\nu$. Then $\mathbf{X}$ will have covariance matrix Σ (for $\nu > 2$) and expectation μ (for $\nu > 1$).

t_ν-copulas have upper (and equal lower) tail dependence:

$$\lambda_U = 2\,\bar{t}_{\nu+1}\Big(\sqrt{\nu+1}\sqrt{1-\rho_l}\,\big/\,\sqrt{1+\rho_l}\Big),$$

$\bar{t}_{\nu+1}(x) = 1 - t_{\nu+1}(x)$ and ρ_l is the linear correlation coefficient for a bivariate t_ν distribution.



Two bivariate distributions with standard normal margins and linear correlation 0.8. Gaussian and t_2-copulas.

Simulation using $\mathbf{t}_\nu$-copulas

Suppose we want to simulate random vectors from a distribution, H, with continuous margins $F_1, \ldots, F_n$, a t_ν-copula and a given Kendall's tau rank correlation matrix τ.

- Set $\rho_l(i,j) = \sin(\pi\tau(i,j)/2)$.

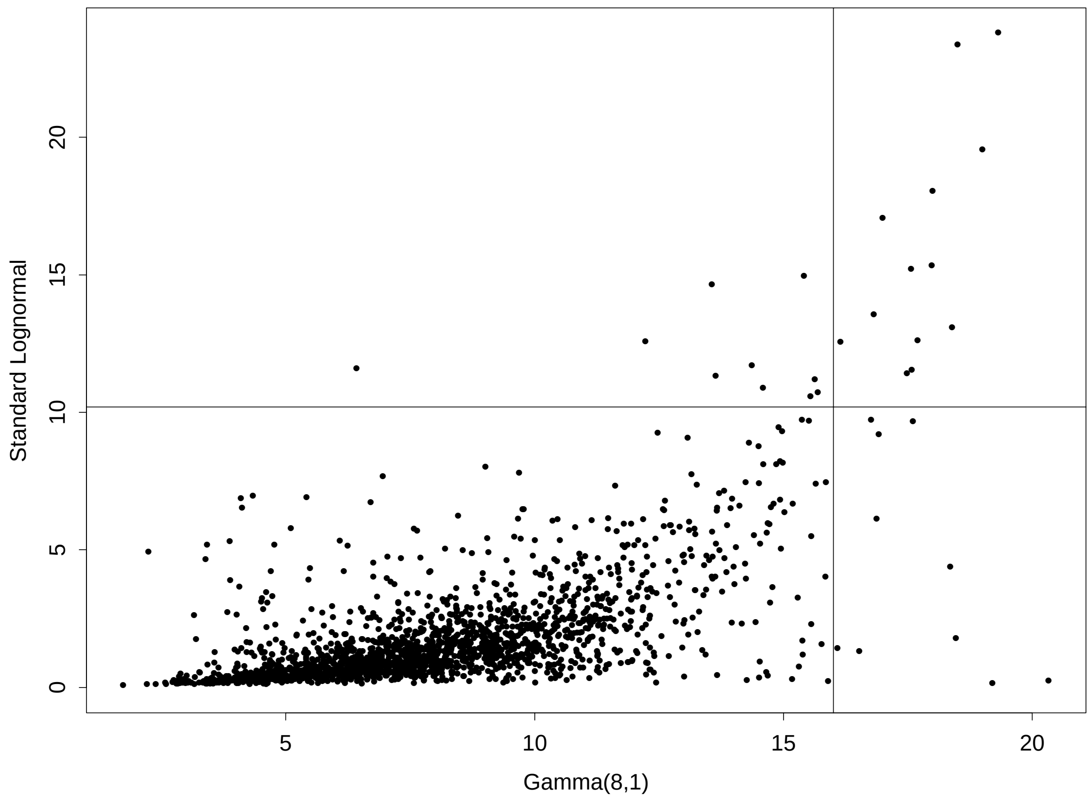
- Set $\mathbf{X} = \frac{\sqrt{\nu}}{\sqrt{S}}\mathbf{Z}$ where $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, \rho_l)$ and $S \sim \chi^2_\nu$. $\mathbf{X}$ has a multivariate standard t_ν distribution.

- $(t_\nu(X_1), \ldots, t_\nu(X_n)) \sim C^{\mathrm{t}}_{\nu,\rho_l}$

- $(F_1^{-1}(t_\nu(X_1)), \ldots, F_n^{-1}(t_\nu(X_n))) \sim H$

Hence simulating from H is easy.

Note: $C^{\mathrm{t}}_{\nu,\rho_l} \longrightarrow C^{\mathrm{Ga}}_{\rho_l}$ as $\nu \longrightarrow \infty$



A sample from (X, Y), where $X \sim$ Gamma$(8, 1)$, $Y \sim$ Lognormal$(0, 1)$ and (X, Y) has the copula $C^{\mathrm{t}}_{2,0.7}$. The linear correlation for the data is 0.6, Spearman's rho is 0.66 and Kendall's tau is 0.49. The upper right rectangle shows simultaneous exceedences of the respective 99% quantiles.

Archimedean copulas

Let φ be a continuous, strictly decreasing convex function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1)=0$. Then

$$\varphi^{[-1]}(\varphi(u)+\varphi(v)), \quad u,v\in[0,1],$$

is a copula with generator φ, where

$$\varphi^{[-1]}(t)=\begin{cases}\varphi^{-1}(t), & t\le\varphi(0),\\ 0, & t\ge\varphi(0).\end{cases}$$

If $\varphi(0)=\infty$, then $\varphi^{[-1]}=\varphi^{-1}$.

- Gumbel copula:
  Take $\varphi(t)=(-\ln t)^\theta$ with $\theta\in[1,\infty)$,
  $C_\theta(u,v)=\exp(-[(-\ln u)^\theta+(-\ln v)^\theta]^{1/\theta})$

- Clayton copula:
  Take $\varphi(t)=(t^{-\theta}-1)/\theta$ with $\theta\in[-1,\infty)\setminus\{0\}$,
  $C_\theta(u,v)=\max([u^{-\theta}+v^{-\theta}-1]^{-1/\theta},0)$

One possible trivariate extension:

$$\varphi_1^{-1}(\varphi_1 \circ \varphi_2^{-1}(\varphi_2(u_1) + \varphi_2(u_2)) + \varphi_1(u_3))$$

Given that a few additional conditions are satisfied this is a copula and the bivariate (1,2)-margin is an Archimedean copula with generator φ_2, and the bivariate (1,3)- and (2,3)-margins are Archimedean copulas with generator φ_1.

This extension generalizes to any dimension.

Problem: Only $n-1$ of $n(n-1)/2$ bivariate margins are different.

Advantages: Simulating from this class of n-copulas is easy, a great variety of copulas belong to this class and many of them possess nice mathematical and statistical properties (multivariate extreme value copulas etc.).

An insurance example

Consider a portfolio of n risks $X_1, \ldots, X_n$. Let the risks represent potential losses in dependent lines of business for an insurance company and let $k_1, \ldots, k_n$ be some thresholds/retensions. Suppose the insurer seeks reinsurance for the situation that l of n losses exceed their retensions. In this case these losses will be paid in full by the reinsurer.

Assume historical data are available allowing estimation of

- marginal distributions
- pairwise $rank$ correlations

$N = |\Big\{i \in \{1, \ldots, n\} | X_i > k_i\Big\}|$ is the number of losses exceeding their retensions.

$L_l = \mathbf{1}_{\{N \geq l\}} \sum_{i=1}^{n} \Big(X_i \mathbf{1}_{\{X_i > k_i\}}\Big)$ is the loss to the reinsurer.

The probability that all losses exceed their retensions is given by

$$\begin{aligned}\mathbb{P}\{N = n\} &= \overline{H}(k_1, \ldots, k_n) \\ &= \widehat{C}(\overline{F}_1(k_1), \ldots, \overline{F}_n(k_n))\end{aligned}$$

We can evaluate/estimate $\mathbb{P}\{N \geq l\}$ and $\mathbb{E}[L_l]$ for various copulas.

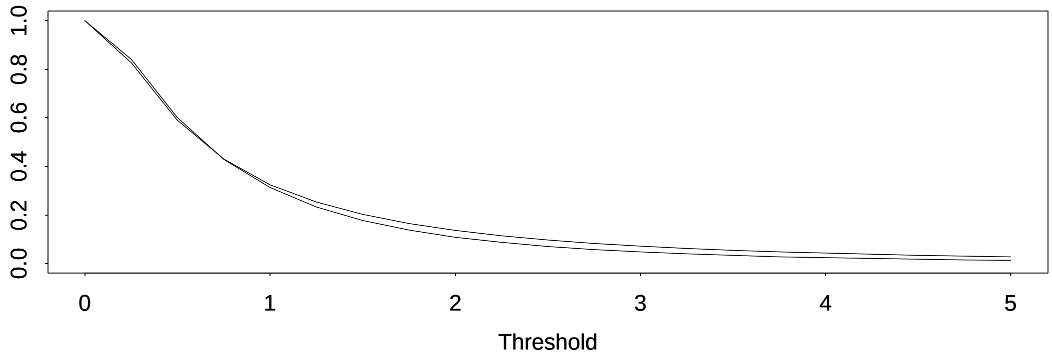
Illustration: $l = n$; $X_i \sim$ Lognormal$(0, 1)$ and $k_i = k$ for all i; $\tau(X_i, X_j) = 0.5$ for all $i \neq j$.

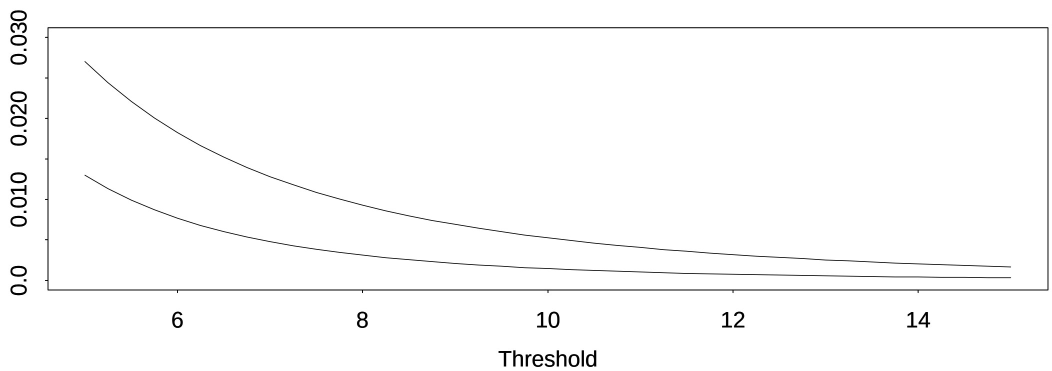
We compare trivariate Gaussian and Gumbel copulas and use the relations

$\rho_l = \sin(\pi\tau/2)$ and $\theta = \frac{1}{1-\tau}$

to parametrizise the respective copulas so that they have a common Kendall's tau rank correlation matrix.

Probability of payout





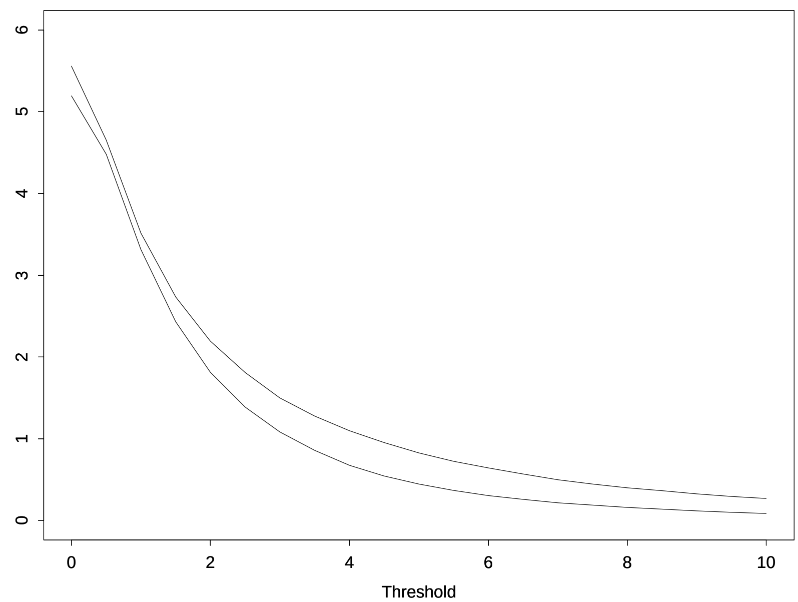
Note that for $k = 5 \approx \text{VaR}_{0.95}(X_i)$

$$\frac{\mathbb{P}^{\text{Gumbel}}\{N = 3\}}{\mathbb{P}^{\text{Gaussian}}\{N = 3\}} \approx 2$$

and that for $k = 10 \approx \text{VaR}_{0.99}(X_i)$

$$\frac{\mathbb{P}^{\text{Gumbel}}\{N = 3\}}{\mathbb{P}^{\text{Gaussian}}\{N = 3\}} \approx 4.$$

Expected loss to the reinsurer



Simulation results for $\mathbb{E}^{\text{Gumbel}}[L_n]$, (upper curve) and $\mathbb{E}^{\text{Gaussian}}[L_n]$ (lower curve).

If the dependence structure is given by a copula with upper tail dependence the expected loss to the reinsurer is much bigger than for the Gaussian case (for high retensions).

- $\lambda^{Ga} = 0$
- $\lambda^{t_2} = 2\overline{t_3}\Big(\sqrt{2+1}\sqrt{1-\rho_l}\;\Big/\;\sqrt{1+\rho_l}\Big) \approx 0.52$
- $\lambda^{Gu} = 2 - 2^{1/\theta} = 2 - 2^{1-0.5} \approx 0.59$

Splus Simulations

```
> copvals<-mvsncopula(2000,F,rcorm)
> rnvals<-simulRn(copvals,c("Tdistr","Gdistr",
"Lognormal","Exponential"),c(2,3,0,2),4)
>
> cor(copvals)
          [,1]      [,2]      [,3]      [,4]
[1,] 1.0000000 0.7124862 0.7004455 0.7127428
[2,] 0.7124862 1.0000000 0.7173173 0.7037622
[3,] 0.7004455 0.7173173 1.0000000 0.7061682
[4,] 0.7127428 0.7037622 0.7061682 1.0000000
>
> cor(rnvals)
          [,1]      [,2]      [,3]      [,4]
[1,] 1.0000000 0.5321945 0.5242888 0.4891848
[2,] 0.5321945 1.0000000 0.6367366 0.6996728
[3,] 0.5242888 0.6367366 1.0000000 0.6189779
[4,] 0.4891848 0.6996728 0.6189779 1.0000000
```

We see that linear correlation can change a lot when changing the margins.

```
> cor(rank(copvals[,1]),rank(copvals[,2]))
[1] 0.7125685
> cor(rank(copvals[,1]),rank(copvals[,3]))
[1] 0.7009264
> cor(rank(copvals[,1]),rank(copvals[,4]))
[1] 0.7128279
>
> cor(rank(rnvals[,1]),rank(rnvals[,2]))
[1] 0.7125685
> cor(rank(rnvals[,1]),rank(rnvals[,3]))
[1] 0.7009264
> cor(rank(rnvals[,1]),rank(rnvals[,4]))
[1] 0.7128279
```

Spearman's rho (and Kendall's tau) is not changed when changing the margins.
"rnvals" is a sample from a 4-variate distribution with a Gaussian copula and t_2, Gamma$(3,1)$, Lognormal$(0,1)$ and Exp(2) margins and all pairwise Spearman's rank correlations 0.7.