Pricing and hedging of derivatives in illiquid markets

Pierre Patie

RiskLab

ETH Zürich

Joint work with R. Frey (Swiss Banking Institute)

Frankfurt Math Finance Colloquium

email: patie@math.ethz.ch
homepage: http://www.math.ethz.ch/~patie
RiskLab homepage: http://www.risklab.ch

Pricing and hedging of derivatives in illiquid markets

Outline

The Model (1)

The Market

- ▷ Riskless money market account B with price normalized to $B_t \equiv 1$. Market for B perfectly liquid.
- ▷ Risky asset with price process S. Market for S can be illiquid.

Asset Price Dynamics Let $(S_t, t \geq 0)$, defined on some filtered probability space $(\Omega, (\mathcal{F}_t)_t, P)$, be the solution of the following SDE:

$$dS_t = \sigma S_{t-}\, dW_t + \underbrace{\rho S_{t-}\, d\alpha_t}_{\text{effect of hedging}},$$

where
for $0 \leq t \leq T$, we assume that the large trader holds α_t shares of S,
$\rho \geq 0$ is a liquidity coefficient,
σ is a given reference volatility, $(W_t, t \geq 0)$ is a Brownian motion on $(\Omega, (\mathcal{F}_t)_t), P)$.

The Model (2)

Remarks

- $\rhd$ $\rho = 0 \Rightarrow$ standard Black-Scholes (BS) case where market are assumed to be perfectly liquid (frictionless).

- $\rhd$ ρ large $\Rightarrow$ market illiquid.

- $\rhd$ $\frac{1}{\rho S_t}$ is the market depth at time t.

- $\rhd$ Possible extensions:
  $\rho(.)$ can be function of the asset price or stochastic (as σ).

Example: Stop-Loss Contract

Scenario: large trader holds K shares and protects them by a *stop-loss contract* with trigger $\overline{S}$, i.e., he automatically sells his shares at

$$\tau := \inf\{t > 0, S_t < \overline{S}\}.$$

- ▷ Market perfectly liquid $\Rightarrow$ value of his position always $\geq \overline{V} := K\overline{S}$.
- ▷ What happens in our setup ?

  Strategy equals $\alpha_t := \begin{cases} K \text{ for } t \leq \tau, \\ 0 \text{ for } t \geq \tau. \end{cases}$

  The asset price at τ equals
  $$S_\tau = S_{\tau^-}(1 - \rho K) = \overline{S}(1 - \rho K),$$

  and we have for value of the position at time τ:
  $$V_\tau = KS_\tau = K\overline{S} - \rho\overline{S}K^2 < \overline{V}.$$

  $\Longrightarrow$ Stop-loss yields imperfect protection!

Market Volatility – Feedback-Effects from Hedging

Class of strategies considered: $\alpha_t = \Phi(t, S_t)$ for some smooth function $\Phi : [0, T] \times \mathbb{R}^+ \to \mathbb{R}$ with derivative Φ_S satisfying $\rho S \Phi_S(t, S) < 1$.

Proposition: If large trader uses strategy $\Phi(t, S_t)$ the asset price follows diffusion of the form:

$$dS_t = v(t, S_t) S_t \, dW_t + b(t, S_t) S_t \, dt, \tag{1}$$

where

$$v(t, S) = \frac{\sigma}{1 - \rho S \Phi_S(t, S)},$$

$$b(t, S) = \frac{\rho}{1 - \rho S \Phi_S(t, S)} \left(\Phi_t(t, S) + \frac{\sigma^2 S^2 \Phi_{SS}}{2(1 - \rho S \Phi_S(t, S))^2} \right).$$

Remarks:

- Volatility depends on Φ_S, i.e. on "Gamma".
- Volatility is increased if $\Phi_S > 0$, it decreased if $\Phi_S < 0$.

Hedging of Derivatives – Basic Concepts Used

Hedger uses strategy $(\alpha_t, \beta_t) \Rightarrow$ stock price $S_t(\alpha)$.

Mark to market value: $V_t^M = \alpha_t S_t(\alpha) + \beta_t$.

Value of a self-financing strategy: $V_T^M = V_0^M + \int_0^T \alpha_{s-}\, dS_s(\alpha)$.

Definition: consider a derivative with payoff $h(S_T)$ and a self-financing hedging strategy (α_t, β_t). The tracking error e_T^M of this strategy equals

$$e_T^M = h\left(S_T(\alpha)\right) - \left(V_0^M + \int_0^T \alpha_{s-}\, dS_s(\alpha)\right), \tag{2}$$

e_T^M measures loss (profit) from hedging.

Remark: one can prove that if the large trader uses the Black-Scholes strategy e_T^M is always positive.

Perfect Option Replication

Problem: can we replicate a derivative *perfectly* (i.e. $e_T^M = 0$) if we adapt the strategy?

This is a *fixed-point* problem: volatility structure used in computing the hedge must be the one resulting from hedging activity.

Proposition: suppose that the smooth function $u(t, S)$ solves the parabolic partial differential equation (PDE)

$$\begin{aligned} u_t + \frac{1}{2} S^2 \frac{\sigma^2}{(1 - \rho S u_{SS})^2} u_{SS} &= 0, \\ u(T, S) &= h(S). \end{aligned}$$

Then $\Phi(t, S_t) := u_S(t, S_t)$ is a replicating strategy, $u(t, S_t)$ is the hedge cost.

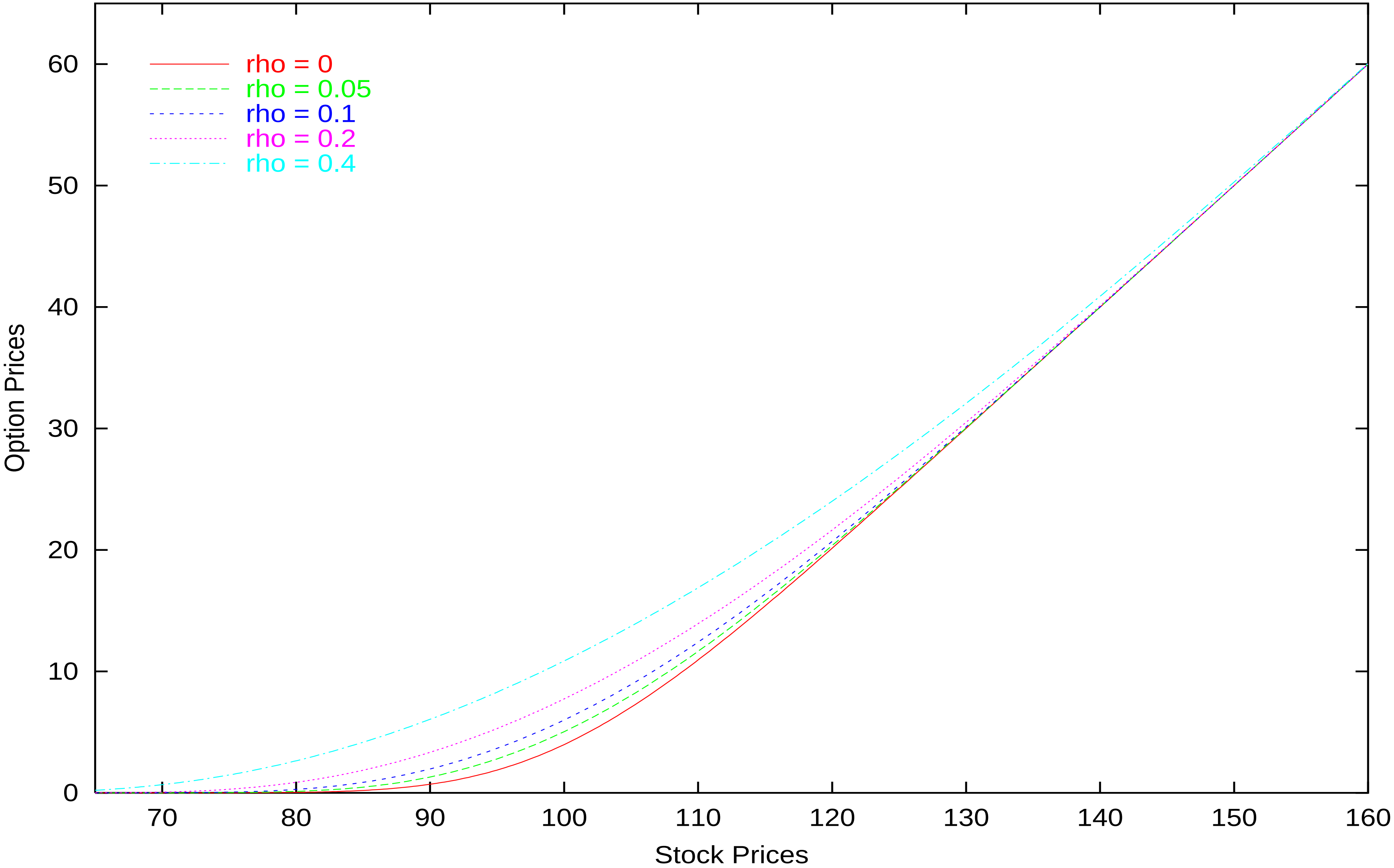
Numerical Solution (1)

- ▷ To avoid problems with the volatility range, we considered the modified operator

$$u_t + \frac{1}{2} S^2 \max\{\delta_0, \frac{\sigma^2}{(1 - \min\{\delta_1, \rho S u_{SS}\})^2}\} u_{SS}.$$

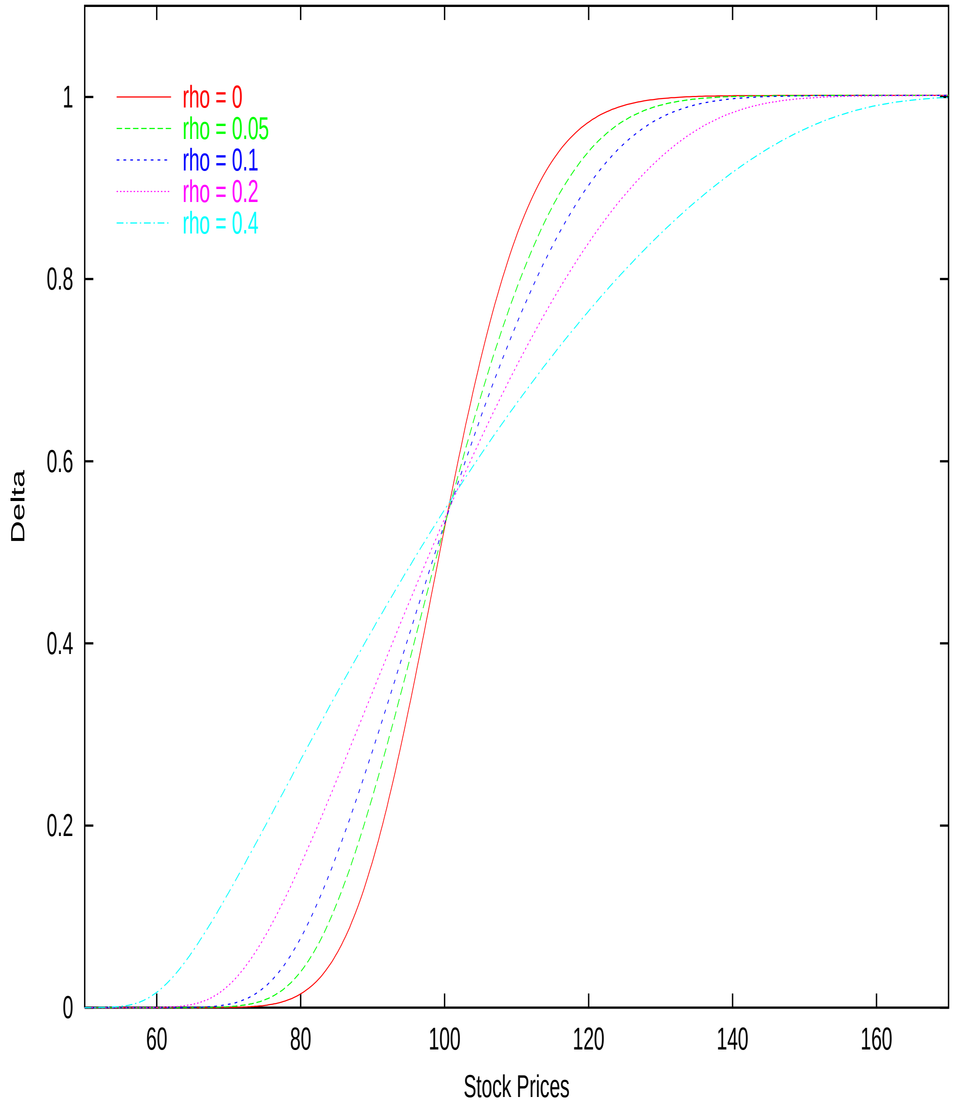
- ▷ To solve the nonlinear PDE we proceed as follows:
  - ≫ time and space discretization by finite differences methods,
  - ≫ implicit scheme for space derivatives approximation, (unconditionally stable scheme)
  - ≫ we solve the resulting nonlinear system by using the Newton method for each time step.

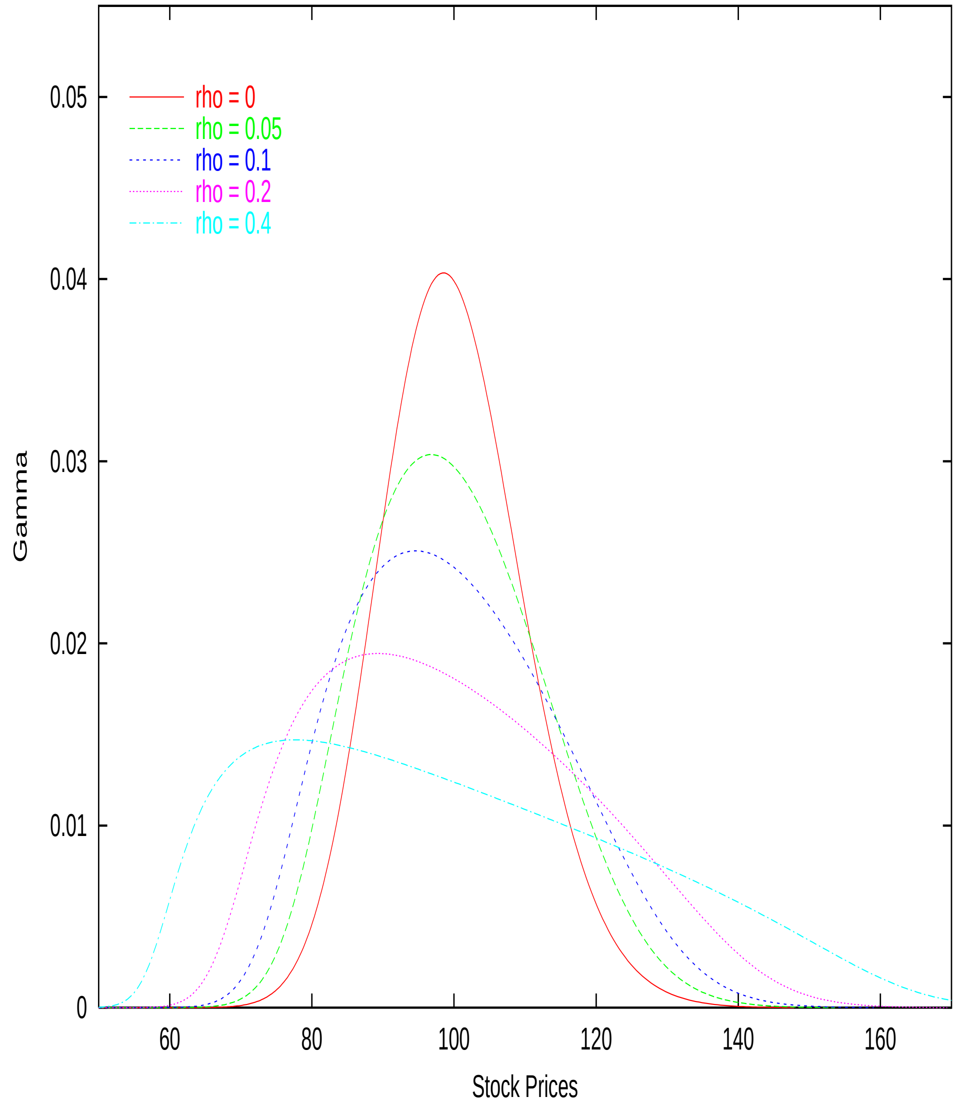
European Call Prices



Hedge cost of European call $u(S, T)$ for various values of ρ
(Strike $=$ 100, $\sigma = 0.4$, $T - t = 0.25$ years).

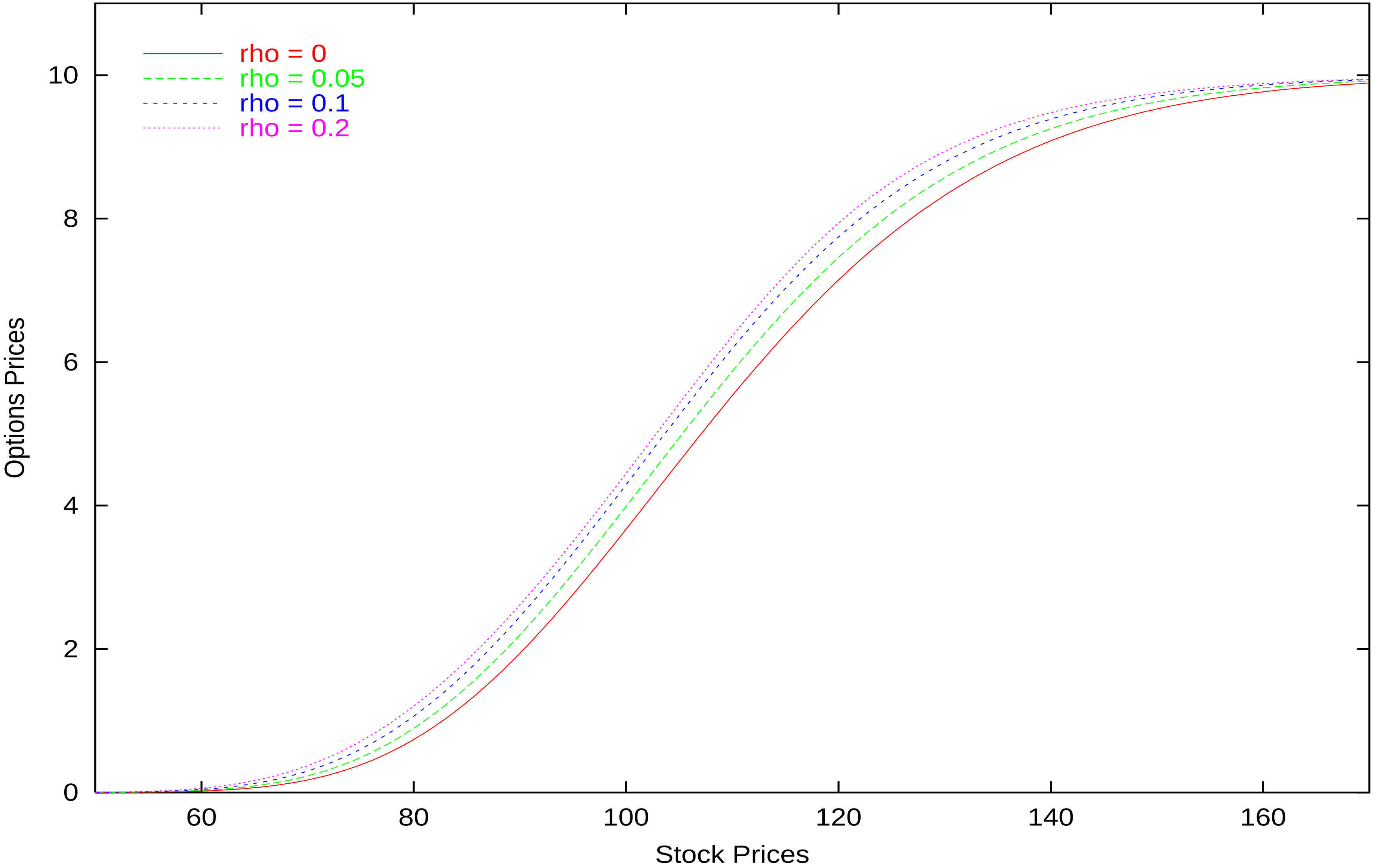
European Call Greeks





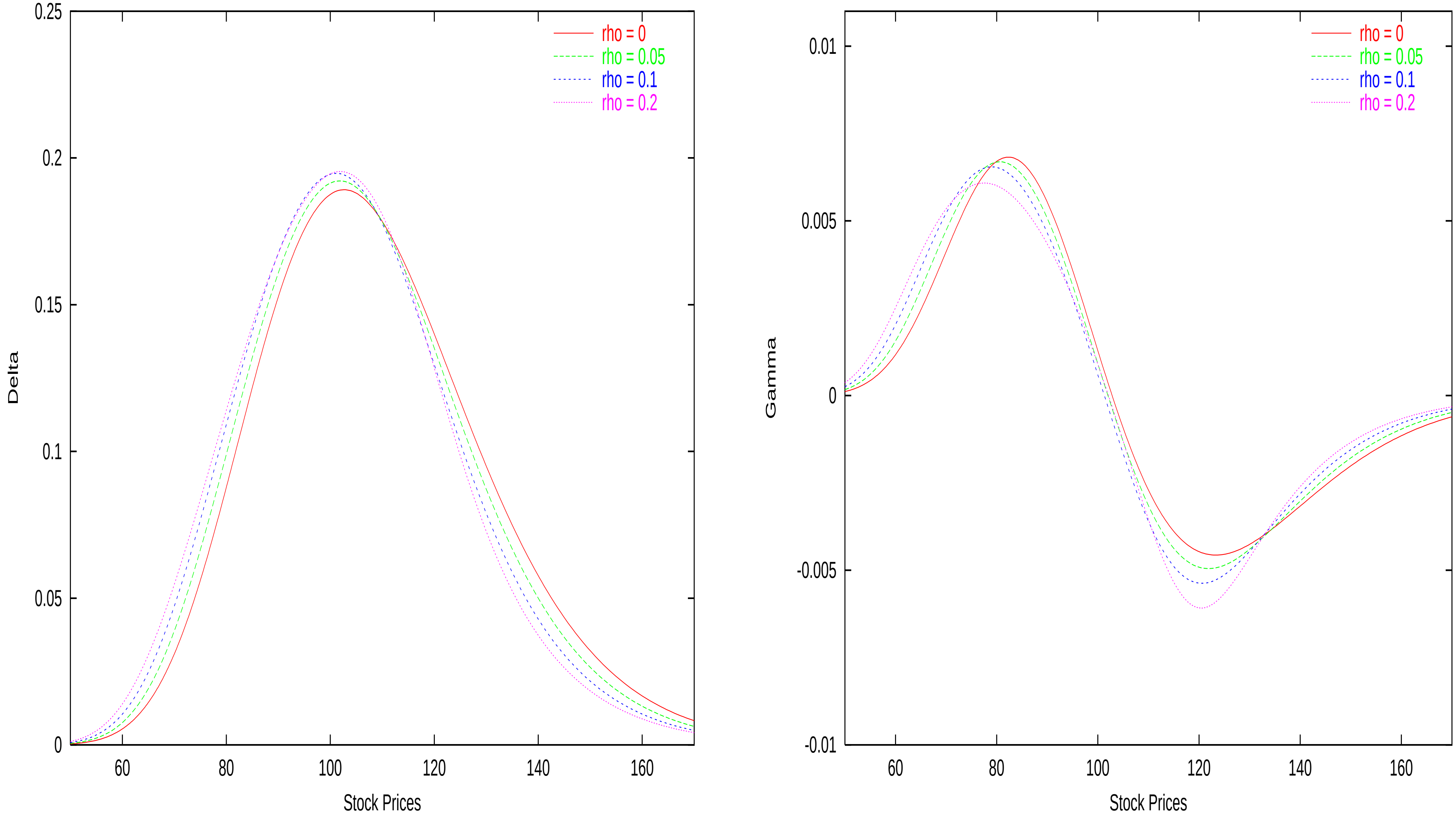
Hedge ratio u_S and Gamma u_{SS} for an European call for various values of ρ (Strike $=100$, $\sigma = 0.4$, $T-t = 0.25$ years).

Call Spread Prices



Hedge cost of Call Spread $u(S,T)$ for various values of ρ
(Strike 1 $=$ 100, Strike 2 $=$ 110, $\sigma = 0.4$, $T - t = 0.25$ years).

Call Spread Greeks



Hedge ratio u_S and Gamma u_{SS} for a Call Spread for various values of ρ
(Strike 1 = 100, Strike 2 = 110, $\sigma = 0.4$, $T - t = 0.25$ years).

Numerical Solution (2)

First order approximation (Papanicolaou and Sircar (1999))

First, we denote by L_{BS} the Black-Scholes operator:

$$L_{BS}C := C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} + r(SC_S - C).$$

For small ρ ($\rho << 1$), we construct a regular perturbation series:

$$C(S,t,\rho) = C^{BS}(S,t) + \rho\overline{C}(S,t) + \mathcal{O}(\rho^2),$$

where

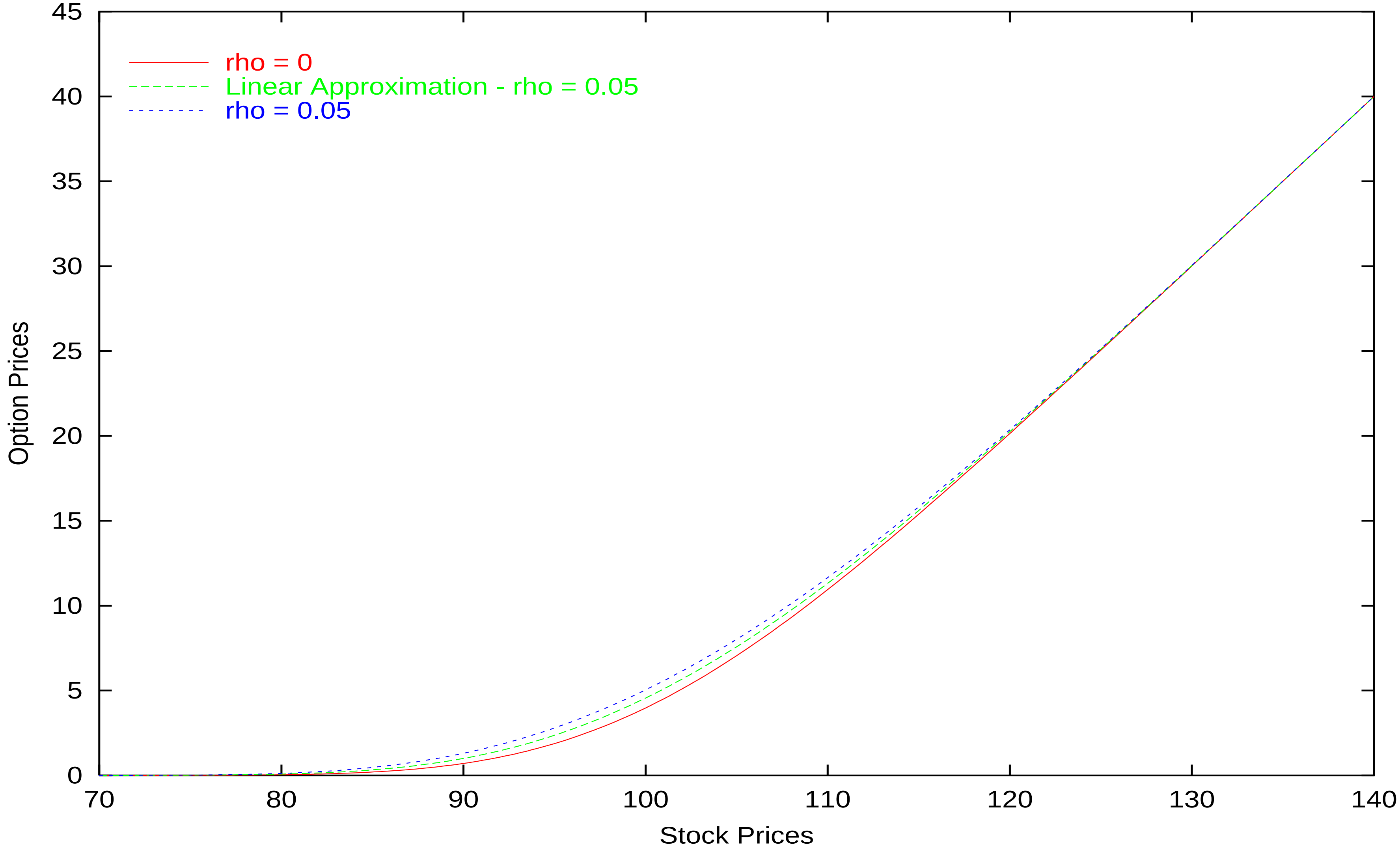
$$L_{BS}C^{BS} = 0,$$

and

$$L_{BS}\overline{C} = -\sigma^2 S^3 (C^{BS}_{SS})^2.$$

Therefore we can approximate the solution of the non-linear PDE by computing successively solution of two Black-Scholes linear PDE. We compared prices obtained with the direct solver and the approximation for European call options.

European Call Prices - Linear Approximation



Hedge cost of European call $u(S,T)$ for various values of ρ
(Strike = 100, $\sigma = 0.4$, $T - t = 0.25$ years).

Hedge Simulation – Tracking Error Computation (1)

In order to check the robustness of our model and to compare different hedging strategies we carry out some hedge simulation. First, we use the stochastic differential equation (SDE) (1) satisfied by the stock price process under feedback:

$$dS_t \;=\; v(t, S_t)S_t\, dW_t + b(t, S_t)S_t\, dt.$$

Then we use Euler-Maruyama scheme to solve it numerically. We discretisize the time interval $[0, T]$ with a fixed step-size ($\Delta_t = \frac{T}{n}$) and for $k = 0, \dots, n-1$,

$$\begin{cases} \tilde{S_0} = S_0, \\ S^i_{(k+1)\Delta_t} = S^i_{k\Delta_t} + v(k\Delta_t, S^i_{k\Delta_t})\left(W^i_{(k+1)\Delta_t} - W^i_{k\Delta_t}\right) + b(k\Delta_t, S^i_{k\Delta_t})\Delta_t, \end{cases}$$

where

$\left(W_{(k+1)\Delta_t} - W_{k\Delta_t}\right)_{(0\le k\le n-1)}$ denote independent $N(0, \Delta_t)$-distributed Gaussian random variables.

Hedge Simulation – Tracking Error Computation (2)

Then, for each simulated path i, $1 \leq i \leq N$, we approximate the tracking error (1) as follows:

$$e_T^i \approx h(S_T^i) - \left(V_0 + \sum_{k=0}^{n-1} \Phi\left(k\Delta_t, S_{k\Delta_t}^i\right)\left(S_{(k+1)\Delta_t}^i - S_{k\Delta_t}^i\right)\right),$$

where

$h(S_T^i)$ is the payoff of the derivative at maturity ($h(S) = (S-K)^+$),

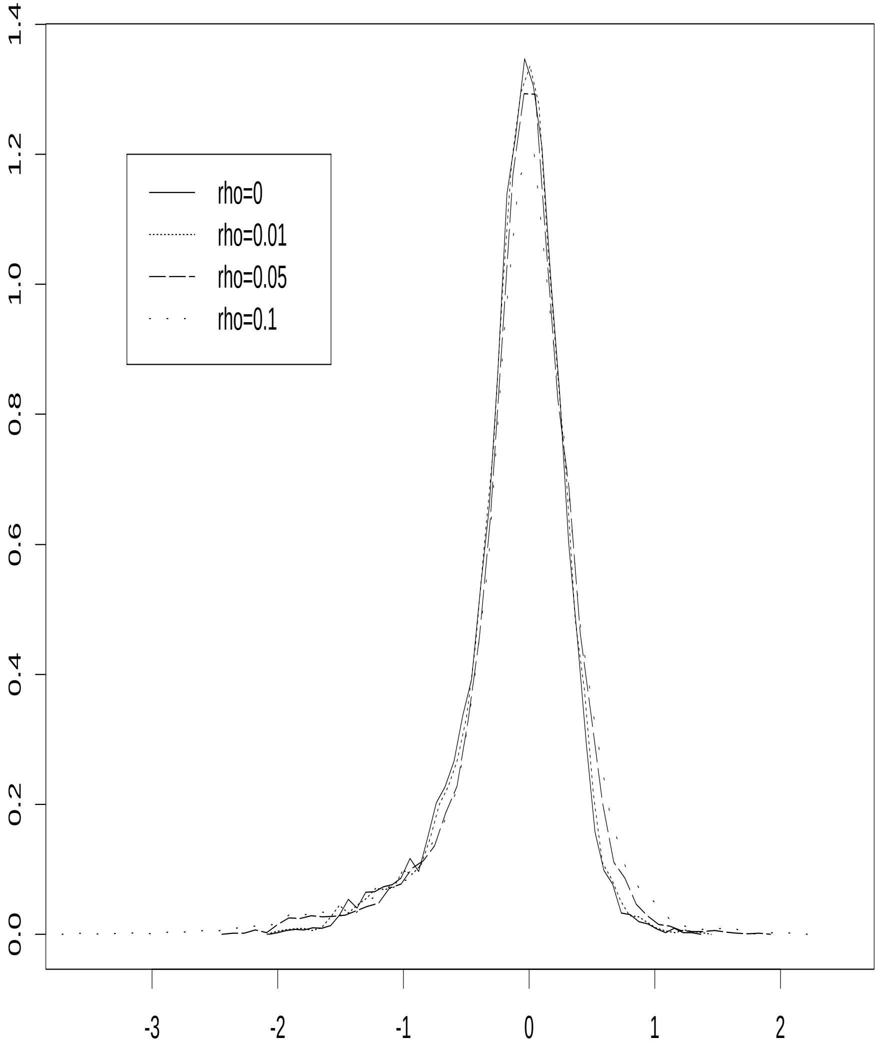
V_0 is the initial value of the hedge-portfolio,

$\Phi(k\Delta_t, S_{k\Delta_t}^i)$ is the hedging strategy value.

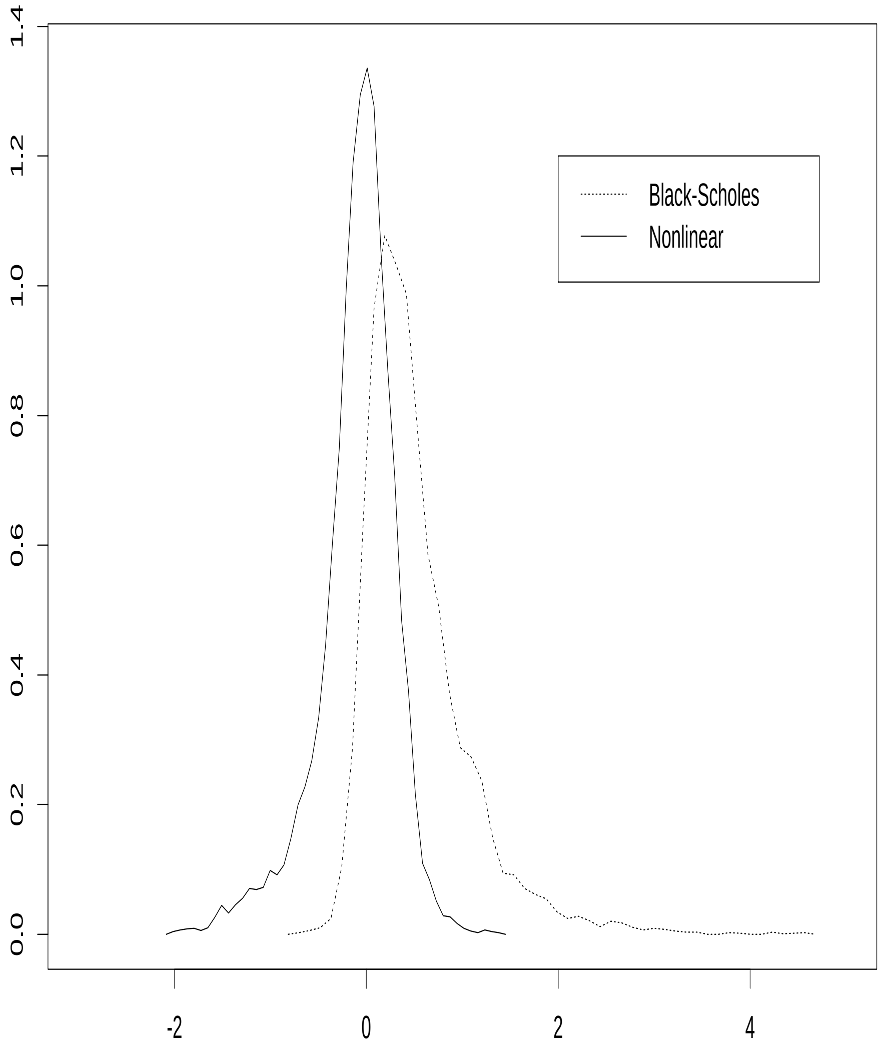
We define the tracking error average by

$$\overline{e}_T = \frac{1}{N}\sum_{i=1}^{N} e_T^i.$$

Tracking Error Density



Tracking error density in an illiquid market using the nonlinear strategy for various values of ρ ($N = 5000, n = 240, T = 0.5$ years).



Tracking error density in an illiquid market using various strategies ($\rho = 0.02, N = 5000, n = 240, T = 0.5$ years).

Properties of the Tracking Error Distribution (1)

ρ	0	0.01	0.02	0.05
$\overline{e}_T^M$	− 0.08	− 0.08	− 0.08	− 0.07
$VaR_{0.99}\left(e_T^M\right)$	0.67	0.7	0.73	0.83
$ES_{0.99}\left(e_T^M\right)$	0.84	0.89	0.93	1.07

Properties of the tracking error distribution for the nonlinear hedging strategy used to replicate an European call option for different values of ρ ($T = 0.5$, $K = 100$, $S_0 = 100$, 5000 simulations with $n = 240$ (number of trades)).

Properties of the Tracking Error Distribution (2)

ρ	0	0.01	0.02	0.05
$\overline{e}_T^M$	$-$ 0.08	0.24	0.51	2.15
$VaR_{0.99}\left(e_T^M\right)$	0.67	1.44	2.37	26.06
$ES_{0.99}\left(e_T^M\right)$	0.84	1.7	2.88	40.9

ρ	0	0.01	0.02	0.05
$\overline{e}_T^M$	$-$0.08	0.04	0.12	1.15
$VaR_{0.99}\left(e_T^M\right)$	0.67	1.24	1.98	25.06
$ES_{0.99}\left(\overline{e}_T^M\right)$	0.84	0.85	2.49	39.9

Properties of the tracking error distribution for the Black-Scholes strategy, starting respectively with the hedge-cost given by the Black-Sholes model and the nonlinear PDE, used to replicate an European call option for different values of ρ ($T = 0.5$, $K = 100$, $S_0 = 100$, 2500 simulations with $n = 240$ (number of trades)).

A General Criterion for Derivative Prices (1)

Suppose that at time $t = 0$, the large trader has sold a portfolio of m derivatives contract with the same maturity T, and with terminal payoff $H := \sum_{i=1}^{m} n_i H_i$. From its replicating trading strategy α^H, we have:

$$H = H_0 + \int_0^T \alpha_s^H \, dS_s(\rho, \alpha^H). \tag{3}$$

Definition: Suppose that the large trader uses a trading strategy α_s^H with (3) and that the stock price $(S_t(\rho, \alpha^H))_t$ is arbitrage-free for a small investors. Denote by $\mathcal{M}^e$ the set of equivalent martingale measures for the process $(S_t(\rho, \alpha^H))_t$. Then a vector $\mathbf{H}_0 = (H_{1,0}, \ldots, H_{m,0})'$ is a fair price system for the derivatives at $t = 0$, if there is some $Q \in \mathcal{M}^e$ such that:

(i) $H_{i,0} = E^Q(H_i \,\big|\, \mathcal{F}_0)$ for all $i = 1, \ldots, m$,

(ii) $M_t := \int_0^t \alpha_s^H dS_s(\alpha)$ is a Q-martingale.

A General Criterion for Derivative Prices (2)

We give conditions under which a vector $\mathbf{H}_0$ of fair prices is *uniquely* determined.

Proposition: Assume that the semimartingale $S(\rho, \alpha^H)$ admits a nonempty set $\mathcal{M}^e$ of equivalent martingale measures and that we have, for all $1 \leq i \leq m$, the representation:

$$H_i = H_{0,i} + \int_0^T \alpha_{i,s}\, dS_s(\rho, \alpha^H), \tag{4}$$

for adapted trading strategies $\left(\alpha_{i,t}\right)_t$. Suppose moreover that $M_{i,t} := \int_0^t \alpha_{i,s} dS_s(\rho, \alpha^H)$, $1 \leq i \leq m$ and $M_t := \int_0^t \alpha_s^H dS_s(\rho, \alpha^H)$ are Q-martingales for all $Q \in \mathcal{M}^e$.
Then $\mathbf{H}_0 := \left(H_{0,1}, \ldots, H_{0,m}\right)$ is the only fair price system for the derivatives.

Application to Terminal Value Claims

Suppose that H_i is given by $h_i(S_T)$ for a smooth function $h_i : \mathbb{R}^+ \to \mathbb{R}$, and define the function h by $h(x) := \sum_{i=1}^m n_i h_i(x)$. We define u as solution to the nonlinear PDE:

$$u_t + \frac{1}{2}x^2\sigma^2 \frac{1}{(1-\rho\lambda(x)xu_{xx})^2} u_{xx} = 0, \quad u(T,x) = h(x); \tag{5}$$

then a hedging strategy for the claim with payoff $h(S_T)$ is given by $\alpha_t^h := u_x(t, S_t)$. We introduce the function

$$\sigma_u : [0,T] \times \mathbb{R}^+ \to \mathbb{R}; \quad (t,x) \mapsto \sigma_u(t,x) := \frac{\sigma}{1-\lambda(x)xu_{xx}(t,x)}. \tag{6}$$

Then the price for the claims with payoff h_i (from the viewpoint of the small investors) is given by the solution u^i of the PDE:

$$(u_i)_t + \frac{1}{2}x^2\sigma_u^2(u_i)_{xx} \ = \ 0, \quad u_i(T,x) = h_i(x). \tag{7}$$

Granted some regularity on the derivatives u_x and u_x^i, the fair price of the claim with payoff $h_i(S_T)$ is given by $u_i(t_0, S_{t_0}(\rho, \alpha^h))$.

Market Liquidity and Smile Patterns of Implied Volatility

Hypothesis: part of smile/skew pattern of implied volatility can be explained by lack of market liquidity.

Trader's view: "smile/skew due to additional selling pressure in a falling market."

Scientific studies: Grossmann-Zhou (1995), Platen-Schweizer (1998), in a related model.

Our idea: Smile and skew are caused by *fluctuations in liquidity.*
In particular: liquidity drops, i.e., ρ increases, if stock price drops a lot relatively to current asset-price level; in line with "market psychology".

Liquidity Profiles

We model market-liquidity profile by the following 2-parameter function (liquidity profile):

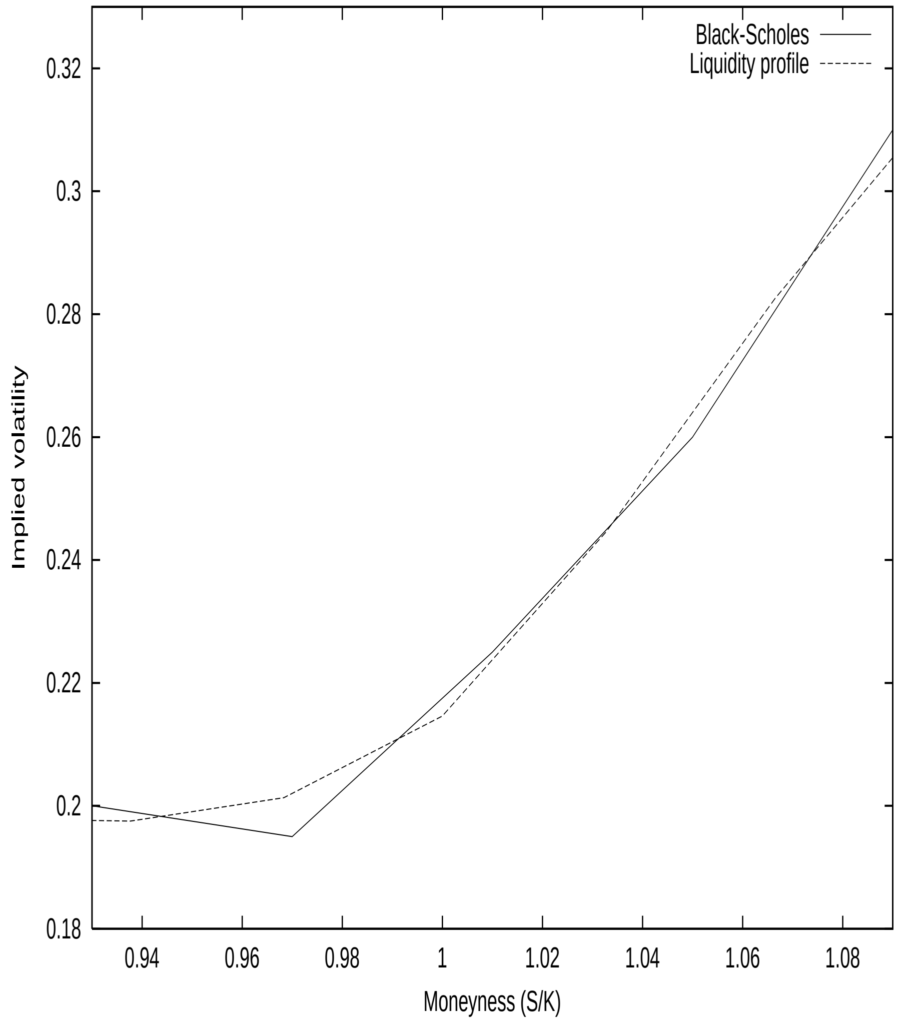
$$\lambda(S) = 1 + (S - S_0)^2 (a_1 \mathbf{1}_{\{S \le S_0\}} + a_2 \mathbf{1}_{\{S > S_0\}}).$$

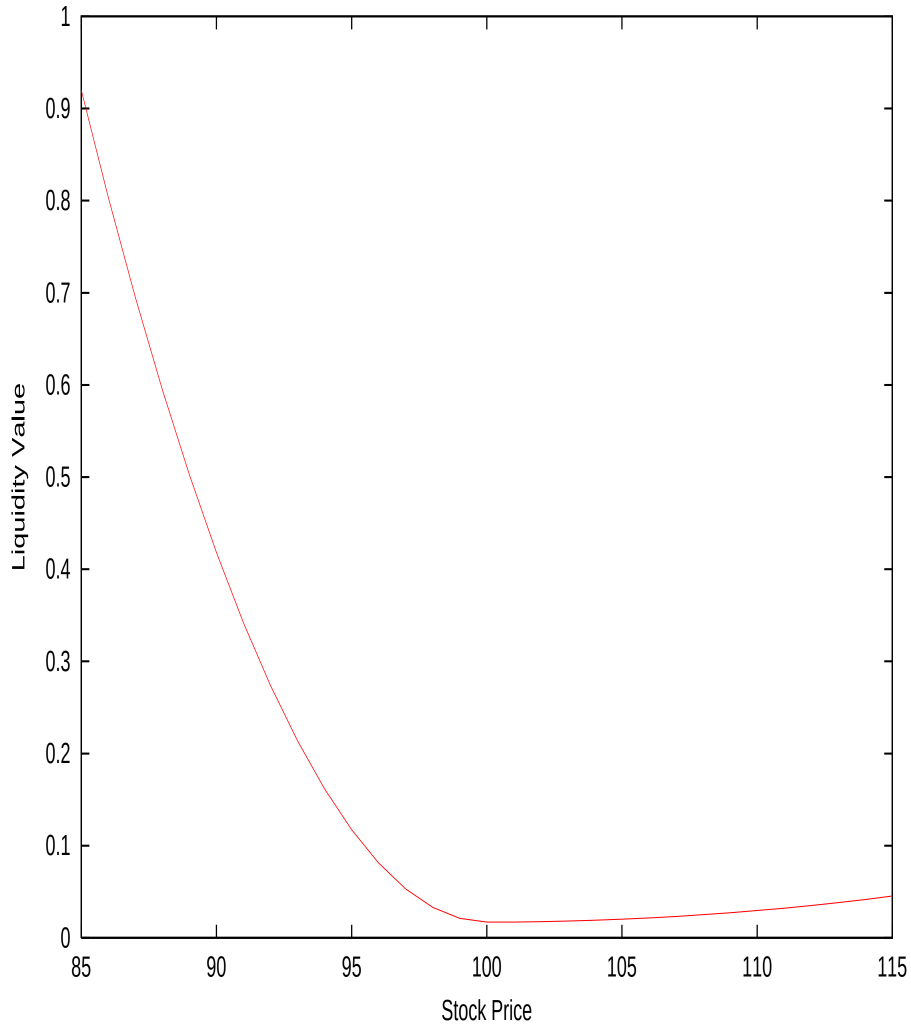
Determination of parameters: given prices C_i for traded options with strikes K_i and maturity T, $1 \le i \le N$. Denote by $C(S_0, K_i, T; \rho, a_1, a_2)$ prices from our nonlinear PDE-model using liquidity profile $\lambda(.)$.
Parameters ρ^*, a_1^* and a_2^* are estimated (numerically) as

$$(\rho^*, a_1^*, a_2^*) = \arg \min_{\rho, a_1, a_2} \sum_{i=1}^{N} \left(C(S_0, K_i, T; \rho, a_1, a_2) - C_i \right)^2 ,$$

i.e., by minimizing the squared distance of the option prices from our model and the observed option prices.

Smile Pattern and Liquidity Profiles





left: implied volatilities on the S&P 500 (July 1990-December 1990) estimated from the Black-Scholes model and from the nonlinear model. For the last model we used as liquidity profile parameters $\rho = 0.017$, $a_1 = 0.236$, $a_2 = 0.0074$ (see *right* graph)
(T-t $= 0.25$, $\sigma = 17.47\%$, $S_0 = 100$)

Outlook and Conclusion

- ▷ More computations for the implied liquidity, (estimating ρ from observed option prices).

- ▷ Stochastic liquidity.

- ▷ Properties of the solution of the PDE.