Approximation of Profit-and-Loss Distributions

(Management Version)

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1 Introduction

The evaluation of a portfolio and its risk exposure with respect to market movements become difficult as soon as contingent claims are involved. In case only the performance of a portfolio has to be determined, practitioners use the mark-to-market pricing each trading day and observe the value changes of the underlying portfolio ex post. A price series becomes available which reveals not only the performance but also the *risk-return pattern* of the various financial activities untertaken by the portfolio manager. Practitioners prefer this approach due to its simplicity, there is no need for information ex ante, neither for determining the value functions of the financial instruments nor for assessing the probability distribution of the risk factors. However, it is this lack of information, that makes it difficult to rebalance the portfolio for achieving an improved risk-return pattern in an efficient way.

The risk-return pattern

Practitioners often identify the *risk-return* pattern of financial instruments through the average return and the volatility of the return. In this work, we will characterize the risk-return pattern of a portfolio with the so called *profit-and-loss distribution*, associated with a specified planning horizon. Clearly, the profit-and-loss distribution is determined by the value functions of the instruments and the probability distribution of the risk factors. The latter represents the dynamics of the risk factors up to the end of the prespecified holding period. Herein, the notions *risk-return pattern* and *profit-and-loss distribution* are used synonymously.

Requesting an appealing approximation of the profit-and-loss distribution requires to determine the *dynamics of the risk factors* and *the value functions* of the corresponding contingent claims held in the portfolio.

The *dynamics of risk factors* are commonly modelled via normal or lognormal distributions. However, it has been observed empirically that market movements are distributed non-normally, respectively non-lognormally. For taking this into consideration one may think of employing series of high frequency data of the risk factors, which help assess more general distributions for the market movements.

The *value function* of contingent claims are often given implicitly through partial differential equations, which have to be solved. The most common approach still used for valuing derivatives is based on the Black-Scholes model, which allows for solving the underlying partial differential equation analytically. In recent years limited applicability of the Black-Scholes model has been seen by both scientists and practitioners due to the fact that the volatility is supposed to be deterministic and known beforehand; in addition, empirically observed characteristics of certain risk factors, like the mean reverting property of interest rates, cannot be incorporated adequately. In finance, various extensions of the Black-Scholes approach and new valuation models have been developed which take into account various dynamic features for pricing contingent claims in a more realistic manner. Therefore these models build the basis for determining appealing approximations for the value functions, and, finally, for the desired risk-return pattern.

Practical viewpoints

The practice of risk management faces analytical and organizational problems which make the pointwise evaluation of the profit-and-loss distribution onerous.

Analytical problems: The progress of financial engineering has created increasingly sophisticated financial instruments, for which there is no analytically closed-form solution to their value functions. Instead, these value functions are given implicitly, so that even pointwise evaluations of the profit-and-loss distribution become difficult.

Organization problems: Risk management units of large financial companies have to manage a considerable quantity of data. The portfolios of such companies likely contain thousands of financial instruments which depend on hundreds of risk factors. The delocalization of the trading units of a worldwide institution causes the portfolio to be traded continuously, which results in permanent shifts of the portfolio structure. This induces permanent changes of the underlying risk factors and dynamic changes of the risk-return pattern.

Assessing the risk-return pattern of a portfolio provides the portfolio manager with information on the frequency and amount of both, potential loss and potential profit. In case of linear or quadratic value functions and normally distributed risk factors the quantiles of the profit and loss distribution are available pointwise, i.e. with respect to prespecified levels. The challenge lies in the numerical evaluation of quantiles which becomes onerous in case the nonlinear value function of a portfolio is given implicitly and the risk factors are distributed non-normally. In practice, quantiles which represent a loss are also called *value-at- risk*. Herein, *value-at-risk* and *quantiles of a profit-and-loss distribution* are used synonymously.

Coherent risk-measures

Any number which represents the potential loss of a portfolio in an adequate predefined sense may be accepted as a *risk measure*. *Coherent risk measures* pay attention only to those market movements that cause a loss to the portfolio manager. Those market movements which provide profits are not taken into account. Coherent risk measures are not based on market expectations of individual portfolio managers. Instead, a distinguished set of risk factor distributions characterizes the coherent risk measure and allow for identifying the potential loss. In this sense coherent risk measures help regulators assess the capital requirement with respect to a distinguished set of market dynamics for controlling the possibility of bankruptcy.

One common coherent risk measure is the *maximum loss* derived with respect to a predefined feasible region of market movements. The distinguished set of probability measures, which characterizes this measure consists of the one-point distributions at each point of a confidence region and its convexification. The challenge for evaluating the maximum loss lies in the minimization of a nonconvex, high-dimensional value function.

Contents of this paper

We focus on algorithmic procedures for determining the risk-return pattern of a portfolio. As mentioned above, the risk-return pattern characterized by the profit-and-loss distribution is completely determined by the value functions of the underlying instruments and by the risk factor distribution. For the reason of adequate benchmarking, we focus on normal distributed risk factors and on value functions which stem from the Black-Scholes model. It will become clear below for which methodologies these assumptions may be relaxed in what way. This work is seen as a contribution for helping develop and improve risk assessement tools for both trading and management.

2 Problem statement

The profit-and-loss distribution (i.e. the risk-return pattern) is then given by the induced probability measure of the portfolio value. It is stressed that value-at-risk reveals only the frequency (i.e. probability) that portfolio value exceeds a certain limit. There is no information available concerning to what extent the portfolio value does not come up to that limit.

The positions in the portfolio are supposed to remain fixed within a pregiven holding period. Due to its characterization a *coherent risk measure* provides neither information on the frequency (i.e. probability) of the potential loss nor on the frequency and amount of potential profits. Hence, no information can be deduced for the risk-return pattern and its asymmetry, which provides the basis for improving the portfolio management. This is the motivation for focusing on the profit-and-loss distribution. It is stressed that quantiles if accepted as risk measures do not fulfill the subadditivity condition and, hence, are not coherent. Evaluating quantiles requires that a unique probability measure is used for modeling the market movements. This unique probability measure may be represented by the martingale measure or by the individial investor's expectations of future market movements.

For the ease of understanding the profit-and-loss distribution of a warrant, which has been issued recently by a Swiss private bank, is evaluated. This *ROE warrant on ABB* is seen as an alternative to money-market instruments. The warrant expires on June 24, 1998 and has been priced with SFr. 1875.- on June 15, 1997. At this date the price of the ABB stock has been SFr. 2130.-. Depending on the ABB stock price at the expiration date two payments are possible: i) in case the price is greater or equal the cap of SFr. 2100.- then the holder of one warrant obtains the cap of Sfr. 2100.- as payment; ii) in case the ABB stock price closes below that cap SFr. 2100.- then the holder of one warrant receives one ABB stock.

We are interested in the value-at-risk corresponding to level 1%, 3% and 5%, and in the sensitivity of the value-at-risk with respect to the ABB stock price. The time horizon is 3 months. The two-dimensional risk profile of the warrant is illustrated in Figure 1. The current stock price p_0

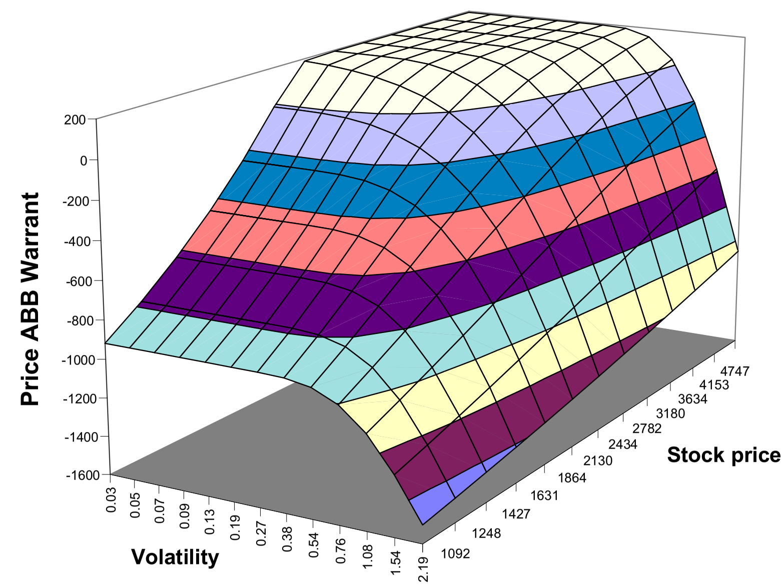

Figure 1: Risk profile of the ROE warrant on ABB

is Sfr. 2130.- with a volatility σ_0 set to 26%. The risk factor changes of price ω_1 and volatility ω_2 are normally distributed $N(\mu, \Sigma)$ with parameters

$$\mu = 0 \in \mathbb{R}^2 \qquad \Sigma^{ABB} := \begin{pmatrix} 0.070756 & 0.037240 \\ 0.037240 & 0.490000 \end{pmatrix}.$$

The associated profit-and-loss distribution is continuous and shown in Figure 2. The value-at-risk numbers of levels $1\%, 3\%$ and 5% are listed in Table 1 and reveal information on the asymmetry of the risk-return pattern.

α	$v^{+,\alpha}$	$v^{-,\alpha}$
1%	378.123	180.683
3%	300.441	163.559
5%	261.284	151.669

Table 1: VaR for the ROE Warrant on ABB

3 Existing Approaches

We briefly review existing value-at-risk methodologies for measuring the potential loss of a portfolio risk profile with respect to a corresponding domain. Goodness of the value-at-risk

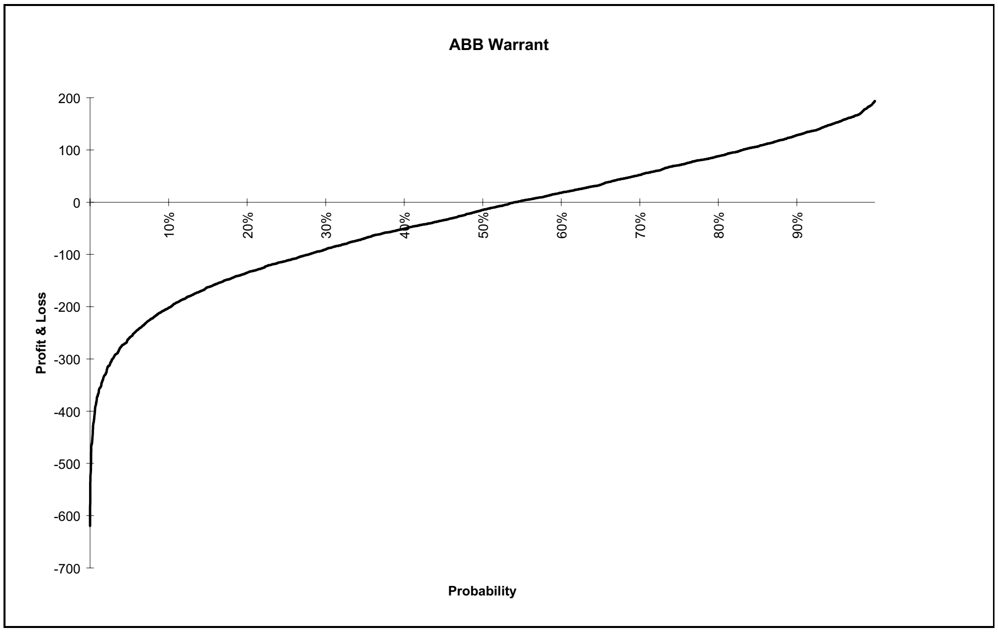

Figure 2: Profit-and-loss distribution of the ROE warrant on ABB

proxies and their associated numerical effort is discussed. It will become clear from the arguments below that each approach offers valuable information on the risk exposure. However, each has to be utilized with care.

Delta approximation of the risk-profile

Linear approximations of the value functions at current price levels are denoted *Delta approximations*. These are widely used in classical risk management, and are known as *duration analysis* in bond management or *Delta hedging* in portfolio management. Substituting the risk profile by linear functions locally helps overcome the difficulty of implicitly given value functions and provide analytical ways for determining the value-at-risk in case the risk factors are normally distributed. Clearly, their goodness depends on the degree of nonlinearity of the risk profile. It should be stressed that the goodness of the linear approximation decreases with increasing holding period if options are included.

The value of a Delta-hedged portfolio remains unchanged for small changes in the risk factor, the value-at-risk is close to 0. However, it might have severe impact on the portfolio value if risk factor changes leave some neighbourhood of 0.

Applying the Delta approximation to the ROE warrant yields the value-at-risk numbers listed in Table 2. These results illustrate that the accuracy is insufficient and the severe asymmetric shape of the true profit-and-loss distribution is not mapped adequately.

α	$v^{+,\alpha}$	$v_{\Delta}^{+,\alpha}$	$v^{-,\alpha}$	$v_{\Delta}^{-,\alpha}$
1%	378.123	289.425	180.683	315.600
3%	300.441	230.891	163.559	251.365
5%	261.284	199.966	151.669	212.613

Table 2: Delta approximation of VaR for the ROE warrant

Delta-Gamma approximation of the risk profile

To incorporate the nonlinearity of a risk profile second order information of the risk profile is used. Substituting the risk profile locally by a quadratic function helps overcome the difficulty of implicitly given value functions and provides additional information on the curvature of the risk profile. Normally distributed risk factors allow for deriving the value-at-risk numbers analytically, which serve as proxies for the corresponding value-at-risk numbers of true risk profile. In literature, this approach is known as the *Delta-Gamma approximation*. Obviously, the approximate portfolio value is no longer distributed normally. However, the corresponding distribution function is representable as a combination of non-central χ^2-distributions, whose corresponding quantiles are given in analytical form.

JP Morgan has analysed the goodness of its own Delta-Gamma approximation for the pricing of call and put options. The widely used Black and Scholes formula serves as a benchmark. The results show that the relative error is dependent on the relation of the spot and strike price and on the time to maturity. The error increases when the option approaches expiration at-the-money. An obvious explanation is offered by the nondifferentiability of the risk profile at the strike price when the option expires.

Applying the Delta-Gamma approximation to the ROE warrant yields the value-at-risk numbers listed in 3. These results illustrate sufficient accuracy and an adequate mapping of the asymmetric profit-and-loss distribution.

α	$v^{+,\alpha}$	$v_{\Delta-\Gamma}^{+,\alpha}$	$v^{-,\alpha}$	$v_{\Delta-\Gamma}^{-,\alpha}$
1%	378.123	369.296	180.683	195.977
3%	300.441	293.088	163.559	166.341
5%	261.284	253.719	151.669	150.037

Table 3: Delta-Gamma approximation of the VaR for the ROE warrant

Monte Carlo simulation

The risk factor movements are drawn randomly with a certain sample size. The underlying portfolio is priced for each of the samples. This way, the *Monte Carlo simulation* yields an empirical distribution and, hence, an approximation of the real profit-and-loss distribution. The received quantiles are proxies.

The applicability of the Monte-Carlo simulation is limited due to the fact that the mapping of both the probability measure and the risk profile is adequate only for a large sample size. Monte-Carlo simulation or modern versions, like the quasi-random Monte Carlo simulations is therefore used in practice with care. It should be noted, that the goodness of random number generators impacts the goodness of the value-at-risk numbers. Numerical tests with using various random number generators have indicated that the variability of the quantiles of the empirical distribution taken with respect to various generators is between $1-5\%$. Therefore the variability of the value-at-risk numbers with respect to different generators may be accepted as negligible, at least at this stage.

Historical simulation

In the *historical simulation* the portfolio is evaluated with respect to risk factor movements of the past. This yields an empirical distribution of the portfolio changes which serves as approximation. Evaluating the skewness and kurtosis of the historical data illustrates whether the normal distribution of the risk factors is valid. This allows conclusions on the goodness of the Delta approximation and the Delta-Gamma approximation at least *ex-post*.

Again, the applicability of the historical simulation is limited by its size. The length of the past period considered is a trade-off between the sample size and the representativeness of the data. It is capturing possible fat tails but also outliers of the distribution. One has to be aware that the past observations map the future risk factor dynamics. Hence, the goodness of the so derived value-at-risk numbers depends on how accurate the future risk factor movements obey the past movements probabilistically.

In practice, the daily returns are often used as an empirical distribution although the underlying portfolios are modified in the daily business. In this case, the empirical distribution reveals neither information on the riskiness or on the risk-return pattern of the current portfolios, nor can this be utilized for improving the risk-return pattern and, hence, for improving the performance of a portfolio manager. The daily returns do reveal information on the risk attitude of the portfolio manager if its volatility is benchmarked to that of indices.

4 Sensitivity of the Value-at-Risk

The quantiles of the profit-and-loss distribution depend on the probability measure of a risk factor space and on the risk profile of the underlying portfolio. The quantiles of the profit-and-loss distribution represent the value-at-risk (VaR) with respect to a predefined level α. Whether this value-at-risk number reflects the real risk exposure of the portfolio in the prespecified sense depends on how good the probability measure maps the future risk factor movements up to the end of the holding period, and on how good the risk profiles maps the market value of the portfolio. How to model the risk factor dynamics and the pricing mechanism is of interest for scientists and practitioners. To clarify the contribution of this work, the issue is the evaluation of the quantiles of the profit-and-loss distribuition given the probability measure of the risk factros and the risk profile. Due to the complexity of the problem the quantiles are to be determined not analytically but numerically with some level of inaccuracy. An efficient algorithmic procedure should behave reasonably fast and accurate. The sensitivity of the quantiles with respect to the parameters of market dynamics is one indicator for the model risk. Key parameters for modelling the dynamics are the volatility and the correlation structure of the risk factors. Accurate estimates of this parameters are of capital importance for the goodness of value-at-risk numbers.

Various test environments are specified which help illustrate the sensitivity of value-at-risk with respect to volatility and correlation in the subsequent section. These environments are characterized by the probability measure of the risk factors and an underlying risk profile which represents the payoff structure of any underlying portfolio.

The aim is to illustrate the sensitivity of the value-at-risk with respect to changes in the market parameter and with respect to various levels. This is done for the above outlined environments independent of the methodology. As mentioned above, the sensitivity of the quantiles with respect to the key parameters, volatility and correlation structure, is one indicator for the model risk one is exposed to with using the value-at-risk approach. This will motivate to pay attention to the slope-curvature relation $\frac{\Gamma}{|\Delta|}$ along the risk factor components.

Sensitivity of the value-at-risk with respect to the level α: The value-at-risk corresponding to a concave risk profile is more sensitive with respect to the level than the value-at-risk to a convex risk profile. In case of a positive curvature the relative change is less severe. This has not been expected from the delta, but could have been from the curvature-slope relation $\frac{\Gamma}{|\Delta|}$. A delta-hedged portfolio with negative curvature has a large negative curvature-slope relation which indicates that the value-at-risk number is very sensitive with respect to changes in the level α. The value-at-risk numbers of a portfolio with almost no curvature behave like $1,64 : 1,88 : 2,33$ when levels $\alpha = 1\%, 3\%$ and 5% are chosen. The value-at-risk numbers of a portfolio with positive curvature behave even below the relation $1,64 : 1,88 : 2,33$.

Sensitivity of the value-at-risk with respect to market volatility σ: In case the volatility increases by 30%, the value-at-risk may change by about 70% for the concave risk profile, it likely changes by about 20% for the convex risk profile. These sensitivity results behave in line with the underlying curvature-slope relations.

One may state that the value-at-risk is less sensitive with respect to the volatility of the risk factors when the risk profile is convex. This indicates less model risk when the risk exposure of a portfolio is measured. To the contrary, negative curvature in the loss region of the risk profile should be carefully analyzed. This documents that insufficient estimates of the volatility can have significant impact on the identified value-at-risk.

Further, $16-$ and $4-$dimensional linear-quadratic risk profiles have been considered with the intention to examine the sensitivity of the value-at-risk with respect to the curvature of a quadratic risk profile and the correlation structure of the risk factors. The risk factors are supposed to be normally distributed. Evaluations have been performed for different degrees of correlation and for different Hessian. In particular, we have considered the uncorrelated case, a medium correlated case, and a highly correlated case. In addition, the impact of high and low curvature relative to the slope is investigated.

The correlation structure of the risk factors plays a significant role for the value-at-risk of both curvature-slope relations in the multidimensional negative definite case. The greater the correlation and the curvature, the higher the value-at-risk. Underestimating the market volatility by 25% yields an underestimation of the value-at-risk by about 28% for a risk profile with a small negative curvature-slope relation. Overestimating the volatility by 30% yields an overestimation

G	exchange rate						
volatility	$p_0 - 3\sigma_p$	$p_0 - 2\sigma_p$	$p_0 - \sigma_p$	p_0	$p_0 + 1\sigma_p$	$p_0 + 2\sigma_p$	$p_0 + 3\sigma_p$
$v_0 - 2\sigma_v$	$g_{1,1}$	$g_{1,2}$	$g_{1,3}$	$g_{1,4}$	$g_{1,5}$	$g_{1,6}$	$g_{1,7}$
$v_0 - 1\sigma_v$	$g_{2,1}$	$g_{2,2}$	$g_{2,3}$	$g_{2,4}$	$g_{2,5}$	$g_{2,6}$	$g_{2,7}$
v_0	$g_{3,1}$	$g_{3,2}$	$g_{3,3}$	$g_{3,4}$	$g_{3,5}$	$g_{3,6}$	$g_{3,7}$
$v_0 + 1\sigma_v$	$g_{4,1}$	$g_{4,2}$	$g_{4,3}$	$g_{4,4}$	$g_{4,5}$	$g_{4,6}$	$g_{4,7}$
$v_0 + 2\sigma_v$	$g_{5,1}$	$g_{5,2}$	$g_{5,3}$	$g_{5,4}$	$g_{5,5}$	$g_{5,6}$	$g_{5,7}$

Table 4: Lattice representation of risk profile

of the value-at-risk by about 37% for a risk profile with a small negative curvature-slope relation. For risk profiles with larger curvature the value-at-risk is more sensitive with respect to the market volatility. It results an underestimation of the VaR by about 40% and an overestimation of VaR by about 60% in the multidimensional case. The degree of correlation and the dimension of the risk factor space have less impact on the sensitivity of the value-at-risk with respect to market volatility. It is the curvature of the risk profile which counts for the sensitivity.

These results confirm that negative curvature and estimating volatility play a significant role for the stability of value-at-risk estimates.

A Swiss bank has motivated the following lattice representation. Applied to a FX-portfolio with K foreign currencies, this results in the evaluation of K risk matrices, where each *risk matrix* (see Table 4) consists of the value change of the underlying portfolio with respect to one pair of risk factors; in case of a FX portfolio the risk factors cross-rate and its volatility are used. Of course, the lattice representation is also applicable to fixed-income or equity portfolios.

The entries $g_{i,j}$ of the matrix G represent the value change of the underlying portfolio with respect to multiple changes in the price $\pm k\sigma_p$ and in the volatility $\pm k\sigma_v$. 7 cross-rate movements and 5 volatility movements are considered here. The current price and volatility is given by p_0 and v_0. It is noted that by construction $g_{34} = 0$ in the above example. Hence, the value of the portfolio is known for finitely many points. For determining the value change with respect to different factor movements one has to apply adequate inter- or extrapolation, which provides an approximation of the real risk profile.

Observe that the nonseparability of risk profiles with respect to the prices and with respect to the volatilities is lost when the lattice representation is used in the above way. Only the nonseparability of price and volatility of one underlying currency is taken into account. For measuring that impact we have taken a FX-portfolio with 8 major currencies whose 16-dimensional risk profile has been approximated by a linear-quadratic function.

A 16-dimensional risk profile of a FX-portfolio has been represented by finitely many points through 8 risk matrices with the components cross-rate and volatility for each of the 8 currencies. These matrices have been used for investigating the impact of separability.

For determining the value change with respect to various factor movements we have applied bilinear and quadratic interpolation, both of which provide an approximation of the real risk profile. As mentioned, the nonseparability of risk profiles with respect to the pairs of risk factors, i.e. with respect to cross-rates and volatilities, is lost. Only the nonseparability of price and volatility of each underlying currency is taken into account.

The numerical results show that working with lattice representations may result in a severe over- or underestimation of the value-at-risk. Surprisingly, the way the lattice points are interpolated has less impact. Bilinear and quadratic interpolation yield similiar value-at-risk numbers.

Finally, a FX portfolio PF_1^V has been constructed which covers 8 exchange rates and which consists of 200 different call and put options. The portfolio PF_1^V contains 40% of instruments with maturity 7 days, 40% with maturity 3 months, 10% with maturity 6 months and 10% with maturity one year. 40% of the derivatives are at-the-money, 20% are each mid in-the-money or out-of-the-money and 10% are each deep in- or out-of-the-money.

Subsets of the above portfolio are defined according to the following rules: Portfolio PF_2^V contains in-the-money instruments; portfolio PF_3^V out-of-the-money instruments; portfolio PF_4^V in-the-money calls and out-of-the-money puts; portfolio PF_5^V contains in-the-money puts and out-of-the-money calls. The maturity structures of all these subportfolios are the same.

The Black-Scholes approach is used for valuing the portfolio yielding the corresponding five risk profiles. The numerical results demonstrate that the sensitivity of the value-at-risk numbers does not depend on the weights of out-of-the-money, in-the-money and at-the-money options within the portfolio.

5 The Barycentric Approximation

The complexity of interaction between time and uncertainty makes practical decision and planning problems to utmost difficult applications of probability and optimization theory. The *Barycentric Approximation* represents a methodology which has been developed for analysing interaction effects between decision making and uncertainty within *stochastic programming* (a field of activity within *mathematical programming*).

Contrary to stochastic control problems, stochastic programs are solved once per period, taking into account periodically updated forecasts of the involved stochastic processes with respect to the future periods. It is today's optimal policy, which is of importance, adopted with respect to the current stochastic dynamics of prices, returns, cash-flows, and also with respect to the optimal policies in future periods, which in turn are adopted with respect to new information on stochastic dynamics. It is this dynamic planning mechanism that characterizes stochastic programming and has received increasing attention in finance in the U.S. and in Great Britain.

The above mentioned dynamic planning mechanism is solved when integration and optimization of value functions has been performed with a prescribed level of accuracy. Barycentric approximation helps overcome the difficulties in the multidimensional integration and optimization of recursively given value functions by sophisticated discretization of the discrete-time stochastic processes. In theory, the convergence of the approximate solutions and the corresponding values are enforced by the weak convergence of the discrete measures. In practice, its application within stochastic programming has provided promising results when the decision space is high-dimensional and the probability space is low-dimensional. This has motivated the application of the barycentric approximation methodology for evaluating profit-and-loss distributions numerically. Former research achtivities by the authors have focused on exploiting structural properties of the value functions. Although, convergence of the quantiles is ensured by the weak convergence of the discrete probability measures, numerical results have indicated that the approximation of the quantiles of the associated profit-and-loss distribution are less practical when the portfolio ascertains a reasonable complexity. Even when the level α is kept fixed, the corresponding quantile is approximated with insufficient accuracy. This reveals that *weak convergence* appears to be not strong enough for evaluating quantiles numerically in a satisfactory way.

The above mentioned experience has motivated the authors to focus on information offered by the barycentric approximation but still unused in the first project phase. It has been realized that the component which represents the dual to the derived discrete probability measure helps evaluate the quantiles in a better way. It is the piecewise linearization of the risk profile over a simplicial partition of the risk factor space, which provides an appealing approximation of the profit-and-loss distribution. The generalized barycenters of the subsimplices may be viewed as distinguished market scenarios subject to which the portfolio has to be evaluated. It is stressed that the methodology poses no assumption on the risk factor distribution and is applicable for general multivariate or empirical distributions in case the variance-covariance matrix exists.

It is observed that the value-at-risk numbers and the accuracy of the barycentric approximation increases with the degree of correlation at a fixed level α. It is known from the above that the sensitivity of the VaR with respect to the market volatility increases with decreasing curvature-slope relation. Having in mind the characteristic features of the barycentric approximation it is not surprising that the accuracy of the barycentric approximation decreases with the curvature-slope relation. Also, the accuracy of the barycentric approximation is insensitive with respect to the level α for fixed variance-covariance matrix.

As learned from the low-dimensional case already, both risk factor space and risk profile have to be taken into account to fulfill the needs for applicability in the high-dimensional case. The accuracy of the barycentric approximation correlates with the sensitivity of the VaR due to its characteristic features and has proven to be competitive with the Delta-Gamma approximation.

The barycentric approximation has been applied to a ROE warrant on the ABB stock and has been benchmarked by the Delta-Gamma approximation for various risk profiles. The numerical results have illustrated the risk-return pattern of the ROE warrant and the various risk profiles. The asymmetry of the risk-return pattern reveals the risk attitude of the investors which proclaim those risk profiles.

The value-at-risk associated with ROE warrant on the ABB stock is listed with respect to various stock prices in Tables 5 and 6. $v^{+,\alpha}_{B.A.}$ and $v^{-,\alpha}_{B.A.}$ are obtained by applying the barycentric approximations for $J = 100$ refinements, $v^{+,\alpha}_{\Delta-\Gamma}$ and $v^{-,\alpha}_{\Delta-\Gamma}$ correspond to the Delta-Gamma approximation. The accuracy of the Delta-Gamma approximation is within the range $[-4.52\%, 8.47\%]$ for levels $\alpha = 1\%, 3\%, 5\%$, that of the barycentric approximation is within $[-2.95\%, 3.60\%]$ (see Table 7). It becomes clear how the asymmetry of the profit-and-loss distribution changes with respect to different prices of the underlying ABB stock.

$\alpha = 1\%$	long			short		
Underlying	$v^{+,\alpha}_{B.A.}$	$v^{+,\alpha}$	Error	$v^{-,\alpha}_{B.A.}$	$v^{-,\alpha}$	Error
1800	402.275	407.056	-1.17%	295.152	291.349	1.31%
1900	397.223	403.917	-1.66%	262.858	257.826	1.95%
2000	388.211	396.749	-2.15%	229.661	223.997	2.53%
2100	374.691	382.426	-2.02%	196.281	190.697	2.93%
2130	370.074	378.123	-2.13%	185.945	180.683	2.91%
2200	360.637	371.608	-2.95%	163.709	158.717	3.15%
$\alpha = 3\%$						
1800	329.088	330.313	-0.37%	255.870	248.882	2.81%
1900	324.292	325.225	-0.29%	231.170	224.033	3.19%
2000	316.261	318.154	-0.59%	205.086	197.950	3.60%
2100	303.502	304.709	-0.40%	176.377	171.595	2.79%
2130	298.672	300.441	-0.59%	168.904	163.559	3.27%
2200	285.573	290.241	-1.61%	149.869	145.687	2.87%
$\alpha = 5\%$						
1800	293.890	293.811	-0.03%	226.890	221.746	2.32%
1900	287.328	288.333	-0.35%	207.383	201.917	2.71%
2000	277.895	279.054	-0.42%	184.834	179.951	2.71%
2100	264.019	265.605	-0.60%	161.336	158.470	1.81%
2130	259.508	261.284	-0.68%	154.558	151.669	1.90%
2200	249.626	250.410	-0.31%	138.152	136.166	1.46%

Table 5: Barycentric approximation for the ROE warrant

$\alpha = 1\%$	long			short		
Underlying	$v_{\Delta-\Gamma}^{+,\alpha}$	$v^{+,\alpha}$	Error	$v_{\Delta-\Gamma}^{-,\alpha}$	$v^{-,\alpha}$	Error
1800	418.560	407.056	2.83%	302.531	291.349	3.84%
1900	409.234	403.917	1.32%	272.693	257.826	5.77%
2000	389.240	396.749	-1.89%	239.327	223.997	6.84%
2100	377.615	382.426	-1.26%	205.527	190.697	7.78%
2130	369.296	378.123	-2.33%	195.977	180.683	8.46%
2200	354.800	371.608	-4.52%	172.167	158.717	8.47%
$\alpha = 3\%$						
1800	334.473	330.313	1.26%	247.144	248.882	-0.70%
1900	325.085	325.225	-0.04%	224.860	224.033	0.37%
2000	312.928	318.154	-1.64%	199.744	197.950	0.91%
2100	296.848	304.709	-2.58%	174.399	171.595	1.63%
2130	293.088	300.441	-2.45%	166.341	163.559	1.70%
2200	283.806	290.241	-2.22%	149.365	145.687	2.52%
$\alpha = 5\%$						
1800	295.990	293.811	0.74%	217.352	221.746	-1.98%
1900	285.413	288.333	-1.01%	198.393	201.917	-1.75%
2000	271.974	279.054	-2.54%	177.008	179.951	-1.64%
2100	258.168	265.605	-2.80%	156.616	158.470	-1.17%
2130	253.719	261.284	-2.90%	150.037	151.669	-1.08%
2200	241.574	250.410	-3.53%	134.090	136.166	-1.52%

Table 6: Delta-Gamma approximation for the ROE warrant

Error	$\alpha = 1\%$		$\alpha = 3\%$		$\alpha = 5\%$	
Underlying	$v_{B.A.}^{+,\alpha}$	$v_{\Delta-\Gamma}^{+,\alpha}$	$v_{B.A.}^{+,\alpha}$	$v_{\Delta-\Gamma}^{+,\alpha}$	$v_{B.A.}^{+,\alpha}$	$v_{\Delta-\Gamma}^{+,\alpha}$
1800	-1.17%	2.83%	-0.37%	1.26%	-0.03%	0.74%
1900	-1.66%	1.32%	-0.29%	-0.04%	-0.35%	-1.01%
2000	-2.15%	-1.89%	-0.59%	-1.64%	-0.42%	-2.54%
2100	-2.02%	-1.26%	-0.40%	-2.58%	-0.60%	-2.80%
2130	-2.13%	-2.33%	-0.59%	-2.45%	-0.68%	-2.90%
2200	-2.95%	-4.52%	-1.61%	-2.22%	-0.31%	-3.53%
Underlying	$v_{B.A.}^{-,\alpha}$	$v_{\Delta-\Gamma}^{-,\alpha}$	$v_{B.A.}^{-,\alpha}$	$v_{\Delta-\Gamma}^{-,\alpha}$	$v_{B.A.}^{-,\alpha}$	$v_{\Delta-\Gamma}^{-,\alpha}$
1800	1.31%	3.84%	2.81%	-0.70%	2.32%	-1.98%
1900	1.95%	5.77%	3.19%	0.37%	2.71%	-1.75%
2000	2.53%	6.84%	3.60%	0.91%	2.71%	-1.64%
2100	2.93%	7.78%	2.79%	1.63%	1.81%	-1.17%
2130	2.91%	8.46%	3.27%	1.70%	1.90%	-1.08%
2200	3.15%	8.47%	2.87%	2.52%	1.46%	-1.52%

Table 7: Error of the barycentric and Delta-Gamma approximations for the ROE warrant

6 Conclusions and Outlook

We have started with assessing the sensitivity of the value-at-risk with respect to market volatility. The curvature-slope relation of a risk profile evaluated at the current market situation reveals information on that sensitivity, and, hence, on the model risk, one is exposed to when the parameter volatility is over- or underestimated.

Based on the experience that weak convergence of the probability measures is not strong enough to receive adequate approximates of the value-at-risk with a reasonable numerical effort, we have focused on the dual view of the barycentric approximation. It is the piecewise linearization of the risk profile over a simplicial partition of the risk factor space, which provides an appealing approximation of the profit-and-loss distribution. The generalized barycenters of the subsimplices may be viewed as distinguished market scenarios subject to which the portfolio has to be evaluated. The approach is also applicable to nonnormally distributed risk factors for which the variance-covariance matrix exists. As learned from the low-dimsional case already, both risk factor space and risk profile have to be taken into account to fulfill the needs for applicability in the high-dimensional case. The accuracy of the barycentric approximation correlates with the sensitivity of the VaR due to its characteristic features and has proven to be competitive with the Delta-Gamma approximation.

On the whole, it is recognized that the barycentric approximation is competitive with the Delta-Gamma approximation. The accuracy of both approximations is insensitive with respect to the level α. The asymmetry of the profit-and-loss distribution is realized from the value-at-risk proxies at various levels. Having in mind that the barycentric approximation is applicable for general multivariate distributions, for which the variance covariance matrix exists, makes this methodology a promising tool for developing and improving risk assessment systems for both trading and management.

This work is seen as one step towards controling and managing market risk with the value-at-risk approach. The key for being efficient lies in an adequate mapping of the risk-return pattern that corresponds to the underlying portfolio. Based on the current developments and the achieved experiences the focus of future research activities is therefore posed on various issues. The way the refinement process of the simplicial partition is designed is still judged as rather crude by the authors. The information on the variability of the slope and the curvature, which is available locally at the barycenters and the vertices of the subsimplices, reveals the goodness of the piecewise linearization. This information is still unused although it appears to be of major importance to the authors, not only for assessing the model risk but also for improving the convergence behaviour of the approximate risk-return pattern.

As mentioned, the barycenters represent market scenarios subject to which the portfolio has to be analyzed. These portfolio values and their sensitivities provide the basis for evaluating and implementing optimized hedging activities. This requires that the value-at-risk approach becomes embedded into a stochastic optimization problem. The challenge of these future activities lies in determining the dynamic of its risk-return pattern and how this can be incorporated in an active portfolio management.

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