Approximation of P&L Distributions

(A Numerical Approach for Evaluating VaR based on Extremal Measures)

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Abstract: Value functions (risk profiles) of financial instruments and the real distributions of risk factors are not available in analytically closed forms. These components have to be approximated. In this work, a new approach for risk measurement is introduced. The underlying methodology is based on the utilization of extremal measures for approximating the P&L distribution. A special class of *extremal measures* is employed which exploits the monotonicity of price sensitivities entailed by convexity. Clearly, in case the value functions have monotonous derivatives, the payoff-functions are convex or concave depending on whether a position is held short or long. The incorporated extremal measures provide approximations for both risk factor distribution and risk profiles, and allow for deriving an adequate approximation of the P&L distributions, in particual for appealing VaR-estimates. The basics of this approach are presented and first numerical results are tested against the currently apllied VaR-approaches and the simulation benchmarks established earlier in Allen [1].

1. Introduction

Market risks are managed at several organizational levels of a bank. Fundamental risk policies are formulated by top management reflecting a bank's "risk appetite" or the maximum amount of money that can be tolerated to be lost in a certain time horizon given current business operations and opportunities and actual risk exposure. Top management is supported by personnel (in controlling and support sections) that translate risk policies into operational strategies and standards, do research in risk measurement methodology, collect risk data, support and control trading units, and report risk information. The most important means of risk control is setting limits (e.g., on nominal positions, delta, gamma, theta sensitivities with respect to risk factors, risk classes, and time buckets) for portfolios (books). The way these books are partitioned largely determines the organizational structure of trading and risk control functions within a bank. A fundamental distinction among interest rates, currencies, equities, and commodities is common.

After consolidating and mapping the open positions, VaR concepts are applied, strongly based on the assumption that the primary risk factors are normally distributed and risk profiles are linear. In case of highly non-linear risk profiles, for which separability is lost, and in case primary risk factors are distributed non-normally, the VaR-concepts currently used in practice yield misleading results (see Allen et al. [2]). Up to now, viable alternative approaches do not seem to be available.

For risk reporting to higher organizational levels, VaR-values and sensitivities are commonly presented in one or more of the following ways:

- VaR (which is called DEaR [daily earnings at risk] when calculated on a daily basis) for a given confidence level is plotted over a certain time horizon and compared to empirical (historical) P&L data (with or without trading activity). This way, the validity of the VaR model can be assessed over time (J.P. Morgan & Co [10]).
- Sensitivities are listed and summed up for a given risk factor across portfolios or for a given portfolio across different risk factors, taking into account the variance-covariance matrix.
- Risk profiles are shown for different volatilities and different degrees of aggregation with respect to primary risk factor prices.

Several other techniques for assessing capital at risk, like historical and Monte Carlo simulation, stress testing (maximum loss optimization, factor push method), and neural network hybrid systems are considered in the Swiss banking industry. The following issues were raised in discussions with risk managers and controllers:

- Within the VaR-concept, which of the assumptions on risk factor and P&L distributions are valid and to what degree? How critical is invalidity? Does VaR overestimate risk? Are assumptions on risk profiles, like separability and additivity, justified? What are the adverse consequences, if they are not? How can risk aggregation techniques be improved?
- Are there viable risk concepts other than VaR? How do measurement accuracies and potential scopes of application compare among the different risk concepts? Do combinations and restrictions of application of risk concepts make sense?
- Easily and intuitively understandable, comprehensive and real-time risk assessment tools need to be developed and improved for both trading and top-management.

An open question to academians has been why the well-known downside risk approaches (e.g., shortfall risk, lower partial moments) seemed to be ignored in the context of risk controlling. All these issues and open questions are potential research areas for RiskLab beyond what has been accomplished in the first phase of the project.

In the first phase of RiskLab, the main purpose was to investigate the measurement of market risk in practice. Practitioners measure market risk using various approximation techniques and making assumptions on both statistical properties of market moves and the sensitivity of prices (changes in profit and loss) towards given market moves. Under the guidance of M. Allen (SBC), risk profiles for a large FX-portfolio were selected and presented in normalized polynomial form (see Allen et al. [2]). The profit and loss (P&L) distributions of those risk profiles and associated delta-, delta-gamma-, and piecewise-linear approximations were derived through Monte-Carlo simulation based on normally distributed, multi-dimensional (correlated and uncorrelated) risk factors. Furthermore, P&L distributions for linear, piecewise-linear, and quadratic risk profiles are constructed via analytical methods.

The intention of [2] was to establish benchmarks for evaluating the goodness of those approaches and allowing for the assessment of distribution tails. This has been achieved by analyzing the risk behavior of financial instruments assuming complete knowledge of the functional relations (including their derivatives) between risk factor moves and prices. Currently in practice, these relations may not be obtainable, because of the fact that approximations are based on price information calculated for only few discrete market moves. In this sense, [2] may be seen as an initial step towards investigating the relation between effort required to collect market data and compute prices and the level of attained precision in risk assessment.

In [2], the results show that the presented approximations do not allow for the derivation of significant P&L distributions. Those are not capable of capturing true risk properties. Their associated risk measures deviated from the benchmarks by large amounts. The preliminary study [2] indicates that an acceptable level of precision in risk assessment cannot be reached by local information (e.g., 1st/2nd order derivatives) on the price change. The proposition that deviations from benchmark values cancel out when a portfolio is subject to higher-dimensional risk could not be validated.

Comparing errors incurred by risk profile approximation to simulation errors (subject to sample size), the study [2] shows that the latter are neglectible compared to the former. In a separate work, the impact of various random number generators to the risk measures was investigated. Such an analysis was motivated by Mr. Schlegel (UBS) in an early RiskLab meeting. The results illustrate that i) the inaccuracy of VaR is increasing with decreasing α, ii) the inaccuracy of VaR is of order 5% to 10% for sample sizes less than $50'000$, and iii) higher moments, like skewness and kurtosis, are estimated with insufficient accuracy. For details see Frauendorfer [6].

These experiences strongly motivate that current research activities should be focused on deriving better risk profile approximations rather than increasing simulation efforts. The attained results emphasize that more (not just local) information on P&L values and derivatives needs to be utilized in order to get better approximations. This has been the motivation for employing a special class of *extremal probability measures*, which have proven extremely useful in *stochastic optimization*.

In this work, it is discussed how extremal probability measures resulting from barycentric approximation help exploit monotonous (first-order) price sensitives for approximating the P&L distribution of a portfolio. Section 2 states the underlying problem formally. In section 3, monotonicity of first-order price sensitivities is discussed for standard pricing models. Extremal measures are introduced in section 4 and applied for approximating P&L distributions (section 5). First numerical results are outlined in section 6. This work ends with a summary of the achieved results, with conclusions and an outlook to the research activities lying ahead (section 7).

2. Problem Statement

Let $\omega = (\omega_1, \cdots, \omega_M) \in \mathbb{R}^M$ represent stochastic changes of primary *risk factors*. The current value of the risk factors is supposed to be 0. Stochasticity of risk factors is modelled through P_t on $(\mathbb{R}^M, \mathcal{B})$, which represents the induced (M-dimensional) probability measure for the random change ω in the risk factors at the end of the *holding period* t. Typical reporting periods are one-day or two-week periods. In reality, P_t is unknown and may vary with the length of the reporting period. In this work, no additional assumptions are posed on P_t. In practice, however, it is often assumed that P_t is a multivariate normal distribution with mean 0 and a variance-covariance matrix changing with the length of the reporting period. Thus

$$P_t := N(0, \Sigma_t) \quad \text{with} \quad \Sigma_t := t \cdot \Sigma, \tag{1}$$

and the existence of all moments are ensured. It should be stressed that there is empirical evidence that random changes of risk factors show greater density in their tails than a normal distribution (see J.P.Morgan & Co. 1995 [10]), probably without the existence of second and higher order moments.

Let $g_i(\omega, t) : \mathbb{R}^M \times [0, T] \rightarrow \mathbb{R}$ denote *price functions* (*value functions, risk profiles*) of a financial instruments $i = 1, \cdots, I$ at the end of the reporting period (time t), with risk factors changing by ω. These functions can be determined through pricing models, however they cannot be specified analytically. At best, function values, first and second order derivatives (or differences) of these functions can be computed for small, one-dimensional collections of preselected points (risk factor levels). Within the reporting period, the change of the corresponding portfolio value is given by

$$v(\omega, t) = \sum_{i=1}^{I} g_i(\omega, t). \tag{2}$$

Clearly, the value change $v(\cdot, t)$ of the portfolio is stochastic and represents a loss, if negative, and a profit, if positive. The following information is of importance for risk managers and controllers:

1. What is the profit & loss (P&L) distribution of $v(\cdot, t)$?
2. What are the various capital at risk values subject to some predefined confidence level α ?
3. In case of existence, what are the first four moments, skewnesses, kurtoses of $v(\cdot, t)$ over the reporting period $[0, t]$?

The *profit and loss distribution* F_t is given through

$$F_t(\hat{v}) = P_t(v(\omega, t) \leq \hat{v}). \tag{3}$$

In case of existence, the first moment of $v(\cdot, t)$ is denoted

$$\bar{v}(t) = \int v(\omega, t) dP_t(\omega), \tag{4}$$

and the moments of higher order($r \geq 2$) with respect to the mean $\bar{v}(t)$ are denoted

$$\bar{v}_r(t) := \int \; [v(\omega, t) \Leftrightarrow \bar{v}(t)]^r dP_t(\omega). \tag{5}$$

Clearly, the second moment ($r = 2$) is the variance of $v(\cdot, t)$. From the moments $\bar{v}_2(t), \bar{v}_3(t), \bar{v}_4(t)$, the skewness $s(t)$ and kurtosis $k(t)$ of the P&L distribution are given by:

$$s(t) := \frac{\bar{v}_3(t)}{\sqrt{\bar{v}_2(t)^3}}, \qquad k(t) := \frac{\bar{v}_4(t)}{\bar{v}_2(t)^2}. \tag{6}$$

Even if the risk factors are normally distributed, the *profit and loss distribution* of an underlying portfolio has to be approximated with sophisticated techniques. The skewness of the normal distribution is 0, and its kurtosis 3. So, in case skewness and/or kurtosis of the P&L distribution differ from 0 and 3 significantly, it's not adequate to approximate the P&L distribution by a Normal distribution.

In the literature (see e.g. Allen [1], Wilson [13], Beckström and Campbell [3]), a technique which estimates potential changes in portfolio values based on statistical confidence intervals of risk factor changes is called VaR. VaR-estimates measure the current portfolio's risk under the assumption that the composition of a given portfolio remains fixed over the holding period $[0, t]$. In other words, VaR-estimates may be interpreted as the amount of capital currently at risk. Two major types of VaR-estimates $v_\alpha(t)$ exist:

The VaR-estimate of the first type $v_\alpha^I(t)$ is defined as

$$F_t(v_\alpha^I(t)) = \alpha. \tag{7}$$

Practically, if $v_\alpha^I(t) \leq 0$ for $\alpha = 0.05$, one may state that in 5 out of 100 holding periods $[0, t]$ the portfolio decreases by more than $|v_\alpha^I(t)|$. Analogously, if $v_\alpha^I(t) > 0$ for $\alpha = 0.9$, one may state that in 10 out of 100 reporting periods $[0, t]$, the portfolio increases by more than $v_\alpha^I(t)$. It is stressed that $v_\alpha^I(t)$ always exist, even if the moments of ω do not.

The VaR-estimate of the second type $v_\alpha^{II}(t)$ is defined according to

$$v_\alpha^{II}(t) := \min\{v(\omega, t) | \omega \in \hat{\Omega}, P_t(\hat{\Omega}) \leq \alpha\}; \tag{8}$$

it represents the maximum loss with respect to a compact confidence region $\hat{\Omega}$ of level α. Clearly, the associated risk factor events $\omega_\alpha^{II}(t)$, defined by

$$\omega_\alpha^{II}(t) \in \ \text{argmin}\{v(\omega, t) | \omega \in \hat{\Omega}, P_t(\hat{\Omega}) \leq \alpha\}, \tag{9}$$

exist if continuity of the value function and compactness of the confidence region hold. These events represent worst case scenarios (see Wilson [13]).

VaR-estimates of the first type commonly refer to the distribution tails of losses (given small α) and are investigated in this work. Likewise, VaR-estimates of the second type are restricted to worst case scenarios subject to confidence regions with a preselected confidence level α, being investigated in Studer [12].

It is common for risk managers that the risk factor distribution is modelled as a multivariate normal distribution with mean 0 and an estimate of the associated variance-covariance matrix Σ_t. Furthermore in practice, the price functions of the financial instruments are evaluated at one-dimensional collections of few predefined risk factor values, separately in their components ω_m. This yields the associated portfolio price change $v_m(\omega_m, t)$ with respect to the risk components $\omega_m,\ m = 1, \cdots, M$. From the viewpoint of risk controllers it becomes a necessity to rely on the relation

$$v(\omega, t) \approx \sum_{m=1}^{M} v_m(\omega_m, t). \tag{10}$$

Relying on these simplifications, one can even expect to get closed form expressions for the VaR-estimates of the first type. For example, let risk factors be normally distributed and let the price functions $v_m(\omega_m, t)$ be replaced by their first-order approximations $\Delta_m \cdot \omega_m$ for $m = 1, \cdots, M$. As the associated P&L distribution of the entire portfolio value is a normal distribution with mean 0 and standard deviation

$$\sigma_v := \sqrt{\Delta' \cdot \Sigma \cdot \Delta},$$

the VaR-estimate of first type is given analytically according to

$$v_\alpha^I(t) := a(\alpha) \cdot \sigma_v \cdot \sqrt{t}, \tag{11}$$

where, for example, $a(0.975) = 2$ or $a(0.998) = 3$. Clearly, the goodness of this VaR-value depends on the linearity assumption of the risk profile (hence, on the validity of 10) and on the assumption of normally distributed risk factors. It should be stressed that the linearity assumption of the risk profiles may be accepted only for very short holding periods $[0, t]$, where the assumption of normally distributed risk factors may be accepted for holding periods of sufficient length. According to the study in [2], closed form expression (11) is NOT applicable.

As mentioned above, price functions (risk profiles) of the financial instruments and the real distributions of risk factors are not available in analytically closed forms. These components have to be approximated. In this work, a special class of *extremal measures* is employed for approximating not only the risk factor distribution but also the risk profiles, yielding an adequate approximation of P&L distributions.

3. Monotone Price Sensitivities (The Saddle Structure)

First, additional motivation is laid for investigating the derivatives of various value functions.

Formally, a function f whose values are real or $+/\Leftrightarrow\infty$ and whose domain is a subset S of $\mathbb{R}^M$ is *convex* precisely if the set $\{(\omega,\mu)|\omega \in S, \mu \in \mathbb{R}, \mu \geq f(\omega)\}$ is a convex subset of $\mathbb{R}^{M+1}$. Considering the simple case of one-dimensional, differentiable functions, it is known that in case these functions have increasing first order derivatives, they are *convex*; functions with decreasing first order derivatives are *concave*. Closed proper convex multi-dimensional functions may be characterized (up to an additive constant) by their subdifferentials (i.e., sets of subgradients). In case the subdifferentials are singletons, the subgradient coincides with the gradient, which in turn implies continuous differentiability of the convex function. The subdifferentials of convex functions completely describe the directional derivatives and may be characterized in terms of a monotonicity property (for details, see Rockafellar [11]). As functions are concave precisely if their negative is convex, and as bivariate functions are *saddle functions* precisely if these are convex in the one argument and concave in the other, it is possible to extend the above argumentation to the saddle case. It is this monotonicity property of subdifferential mappings of saddle functions which can be exploited by a specific class of *extremal probability measures* in case the arguments are stochastic (see [4],[5]).

These findings may be applied to our framework: In case of convex or concave value functions, the directional derivatives represent the price sensitivities along one-dimensional market moves. Monotonicity of the subdifferential mappings implies monotonicity of the directional derivatives and, hence, monotonicity of the price sensitivities. *Extremal measures* associated with the *barycentric approximation* of these value functions exploit this monotonicity and provide investors with a kind of *best discretizations* of the underlying stochastic risk factors, in a sense that will become more clear later.

In this section, particular attention is paid to the monotonicity of first order derivatives of standard value functions corresponding to bonds, interst rate swaps, and options priced with the Black-Scholes formula. It is useful to consider the value functions $g(\omega, t)$ dependent on time t. This helps illustrate the value change with respect to time t.

The value function of a *default-free zero coupon bond* is given by

$$g_B(\omega_1, t) = \frac{F}{(1+\omega_1)^{\tau}}, \tag{12}$$

where F is the face value of bond (payment at maturity), ω_1 is the annual interest rate associated with time to maturity $\tau = T \Leftrightarrow t$, stated in number of years. Considering the effect of time on the price of a bond, it is common to set $\tau = T \Leftrightarrow t$. Clearly, for fixed t, g_B has monotonously increasing first order derivatives in ω_1 and is herewith convex in ω_1.

The value of *default-free bonds with fixed coupon payments* C at T_m $(m = 1, ..., M)$ is the sum of the payments discounted with respect to the rates ω_m corresponding to maturities $\tau_m = T_m \Leftrightarrow t$ $(m = 1, ..., M)$. Setting $\omega = (\omega_1, \cdots, \omega_M)$, the value is given by

$$g_B(\omega, t) = \sum_{m=1}^{M} \frac{C}{(1+\omega_m)^{\tau_m}} + \frac{F}{(1+\omega_M)^{\tau_M}}. \tag{13}$$

Clearly in this case, the value function $g_B(\cdot, t)$ is a M-dimensional convex function in the term structure $\omega = (\omega_1, \cdots, \omega_M)$. It is noted that the value functions are linear in the coupon payments for fixed term structure; hence, in case the payments at T_m are uncertain (i.e., default is possible), price sensitivity is constant and given by $\frac{1}{(1+\omega_m)^{\tau_m}}$.

The value of an *interest rate swap* is the difference of the values of two bonds, one with fixed payments and one with floating payments. Let F denote the notional pincipal in the swap agreement. The value of a floating rate bond is equal to notional principal, F, immediately after a payment date. In our notation, the time until next payment date is T_1, so that

$$g_{B_{fl}}(\omega, t) := \frac{C_{fl}}{(1+\omega_1)^{\tau_1}} + \frac{F}{(1+\omega_1)^{\tau_1}}, \tag{14}$$

where C_{fl} is the known floating rate payment at time T_1. If a financial institution is paying floating rates and receiving fixed rates, then the value of the swap is

$$g_S(\omega, t) := g_B(\omega, t) \Leftrightarrow g_{B_{fl}}(\omega, t). \tag{15}$$

The value of the swap is 0 if first negotiated and 0 at the end of its life. During its life it may have a positive or negative value. It is observed that $g_S(\cdot, t)$ is convex in $(\omega_{\tau_2}, \cdots, \omega_{\tau_M})$ and concave in ω_{τ_1} as $F + C_{fl} \Leftrightarrow C \leq 0$. Clearly, the above argumentation holds also for discounting with continuously compounded rates.

The value of a *forward contract* on a security ω_1 with known dividend yield q (paid continuously) is given by

$$g_f(\omega_1, \omega_2, t) := \omega_1 e^{-q\tau} \Leftrightarrow K e^{-\omega_2 \tau}, \tag{16}$$

where ω_2 denotes the continuously compounded rate corresponding to $\tau = T \Leftrightarrow t$ and K the forward price of the security. Clearly, $g_f(\cdot, t)$ is linear affine (i.e., convex and concave) in ω_1 and concave in ω_2.

European call options on non-dividend paying securities which follow a geometric Brownian motion are valued according to the Black-Scholes model. The risk factors are given by $\omega := (\omega_1, \omega_2, \omega_3)$, where ω_1 represents the value of the security at t, ω_2 is the interest rate for an investment with maturity $\tau := T \Leftrightarrow t$, and ω_3 is the volatility of the security price observed at t. Let X denote the exercise price and $\mathcal{N}(\cdot)$ the cumulative probability distribution function for a standardized normally distributed variable. Then, the value of the European call option is given by

$$g_c(\omega, t) = \omega_1 \mathcal{N}(d_1) \Leftrightarrow X e^{-\omega_2 \tau} \mathcal{N}(d_2), \tag{17}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(\frac{\omega_1}{X}) + (\omega_2 + \frac{1}{2}\omega_3^2)\tau}{\omega_3 \sqrt{\tau}} \\ d_2 &= d_1 \Leftrightarrow \omega_3 \sqrt{\tau}. \end{aligned}$$

It is known that the price sensitivities with respect to ω_1, ω_2, and ω_3 are positive. Moreover, due to the fact that the risk-neutral stochastic process for the security is lognormal, it is proven that $g_c(\cdot, t)$ is convex in ω_1. Whether convexity also holds for the price sensitivities with respect to ω_2 and ω_3 remains to be assessed. Similar results can easily be derived for European put options by employing the call-put parity.

It is noted that the value function g_c is convex in the strike price X, too, but as the strike price is determined by the investors and not by the market, this property is of less priority for risk measurement. This may have some influence on the design of adequate aggregation techniques. For further details, the reader is referred to Ingersoll [8].

Although some assumptions in the Black-Scholes model are not realistic this approach is viewed fundamental in option pricing theory and broadly applied for

European stock option. The less realistic assumptions are deterministic volatility and interest rate. Further, it has to be taken into account that most stock options are American. In literature, Black-Scholes is even applied to American stock option and options on interest rate, interest rate futures and caps. It has to be stressed that the assumptions within Black-Schole are not adequate for interest rate sensitive derivatives (see Hull [7]). Nevertheless, for our purposes, this model can be used as basic tool for investigating the structural properties of the corresponding value functions.

From the above, it becomes obvious that in case the value functions have monotonous derivatives, convexity and concavity depend on whether a position is held short or long. A comprehensive analysis of value functions with respect to the underlying risk factors remains to be carried out. A better understanding will certainly help aggregate the various instruments adequately and measure risk more reliably.

4. A Class of Extremal Probability Measures

In this section, t is kept fixed and suppressed for the ease of exposition. Therefore, the value function is written as $v(\omega) : \mathbb{R}^M \to \mathbb{R}$. Let ω be partitioned according to $\omega = (\eta, \xi)$ with $\eta \in \mathbb{R}^K$, $\xi \in \mathbb{R}^L$ and $\hat{\Omega}$ be represented as a Cartesian product of simplices $\Theta \in \mathbb{R}^K$ and $\Xi \in \mathbb{R}^L$, i.e., $\hat{\Omega} := \Theta \times \Xi$. Given a probability space $(\hat{\Omega}, \hat{\mathcal{B}}, \hat{P})$, we are interested in approximating the distribution function

$$\hat{F}(\hat{v}) = \hat{P}(v(\omega) \leq \hat{v}). \tag{18}$$

This will be achieved by deriving two sequences of discrete probability distributions $\{Q_1^J\}, \{Q_2^J\}$, $J = 1, 2, \cdots$, which converge weakly to $\hat{P}$ and allow for quantifying the goodness of the approximation with respect to each J. The following results have been derived in Frauendorfer ([4],[5]) for solving stochastic programs and form the basis for approximating the P&L distribution in the next section.

Let Δ denote the set of those probability measures Q on $(\hat{\Omega}, \hat{\mathcal{B}})$ which coincide in the first and joint moments with those of $\hat{P}$; i.e., Δ consists of those probability measures Q for which

$$\begin{aligned}\int_{\hat{\Omega}} \eta dQ &= \int_{\hat{\Omega}} \eta d\hat{P}, \\ \int_{\hat{\Omega}} \xi dQ &= \int_{\hat{\Omega}} \xi d\hat{P}, \\ \int_{\hat{\Omega}} \eta_k \xi_l dQ &= \int_{\hat{\Omega}} \eta_k \xi_l d\hat{P}, \qquad \forall k, l\end{aligned} \tag{19}$$

holds. Suppose that the value function $v(\cdot)$ is a saddle function (concave in η and convex in ξ). According to [5], a partial ordering $\leq^{(s)}$ for the set Δ may be defined with respect to the set $\mathcal{S}_{\hat{\Omega}}$ of continuous saddle functions relative to the $\times$-simplex $\hat{\Omega} = \Theta \times \Xi$:

$$Q_1 \leq^{(s)} Q_2 \Leftrightarrow \int_{\hat{\Omega}} v(\eta, \xi) dQ_1 \leq \int_{\hat{\Omega}} v(\eta, \xi) dQ_2, \quad \forall v(\cdot) \in \mathcal{S}_{\hat{\Omega}}. \tag{20}$$

The set of extremal probability measures of Δ taken with respect to $\leq^{(s)}$ is then defined according to

$$\begin{aligned}\inf{}^{(s)}\Delta &:= \{Q_l | Q_l \leq^{(s)} Q, \quad \forall Q, Q_l \in \Delta\}, \\ \sup{}^{(s)}\Delta &:= \{Q_u | Q \leq^{(s)} Q_u, \quad \forall Q, Q_u \in \Delta\},\end{aligned} \tag{21}$$

which represent the solutions of *generalized moment problems* (in the sense of Krein and Nudelman [9]). It is proven in [4] that the sets $\inf^{(s)}\Delta$ and $\sup^{(s)}\Delta$ are singletons. In this sense, these solutions, denoted $\hat{Q}_l$ and $\hat{Q}_u$, may be viewed as best discretization of the stochastic risk factors. The support of $\hat{Q}_l$ and $\hat{Q}_u$ is finite, whose elements may be viewed as *generalized barycenters* of the $\times$-simplex $\hat{\Omega} = \Theta \times \Xi$. Both barycenters and their probabilities are completely determined by the moments

$$\int_{\hat{\Omega}} \eta_k d\hat{P}, \int_{\hat{\Omega}} \xi_l d\hat{P}, \int_{\hat{\Omega}} \eta_k \xi_l d\hat{P}, \tag{22}$$

which characterize the set Δ. For the corresponding formulas of $\hat{Q}_l$ and $\hat{Q}_u$, refer to Frauendorfer ([4]).

The dual problems to the generalized moment problems (21) are *semiinfinite programs* which will be motivated next.

Let $\mathcal{L}$ denote the set of bilinear functions $L(\eta, \xi)$ (i.e., linear in η and ξ separately), for which $L(\cdot) \leq v(\cdot)$ on $\Theta \times \Xi$. Similarly, $\mathcal{U}$ denotes the set of bilinear functions $U(\eta, \xi)$ for which $U(\cdot) \geq v(\cdot)$ on $\Theta \times \Xi$. Hence, $L(\cdot)$ minorizes $v(\cdot)$ and $U(\cdot)$ majorizes $v(\cdot)$ on $\Theta \times \Xi$. Obviously,

$$\sup_{L\in\mathcal{L}} \int_{\hat\Omega} L(\eta,\xi)d\hat P \;\le\; \int v(\eta,\xi)d\hat P,$$
$$\inf_{U\in\mathcal{U}} \int_{\hat\Omega} U(\eta,\xi)d\hat P \;\ge\; \int v(\eta,\xi)d\hat P. \tag{23}$$

The lefthand side in (23) represents the semiinfinite programs which bound the expectation of the value function from below and above. The corresponding solutions, denoted $\hat L$ and $\hat U$, are completely determined by the first-order derivatives of $v(\cdot)$ at the barycenters.

Due to strong duality it is proven that

$$\int_{\hat\Omega} \hat L(\eta,\xi)d\hat P \;=\; \int_{\hat\Omega} g(\eta,\xi)d\hat Q_l,$$
$$\int_{\hat\Omega} \hat U(\eta,\xi)d\hat P \;=\; \int_{\hat\Omega} g(\eta,\xi)d\hat Q_u. \tag{24}$$

Further, as the integral of the bilinear functions are completely determined by the moments (22), it also holds that

$$\int_{\hat\Omega} \hat L(\eta,\xi)d\hat P \;=\; \int_{\hat\Omega} \hat L(\eta,\xi)d\hat Q_l,$$
$$\int_{\hat\Omega} \hat U(\eta,\xi)d\hat P \;=\; \int_{\hat\Omega} \hat U(\eta,\xi)d\hat Q_u. \tag{25}$$

This way one obtains approximations for the value functions $v(\cdot)$ as well as for the probability measure $\hat P$.

These approximations can be improved with refinements of $\hat\Omega$. Let a $\times$-simplicial partition of $\hat\Omega$ be denoted by $\mathcal{P}^J$; i.e., $\mathcal{P}^J := \{\hat\Omega^j;\ j = 1,\cdots,J\}$, where the subcells are mutually disjoint, Cartesian products of simplices whose union equals $\hat\Omega$. Applying the above statements to each of these subcells $\hat\Omega^j$ $(j = 1,\cdots,J)$ yields improved piecewise bilinear approximations $\hat L^J(\cdot),\hat U^J(\cdot)$ and $\hat Q_l^J,\hat Q_u^J$ for the value functions $v(\cdot)$ and for the probability measure $\hat P$, respectively.

In particular, let $\mathcal{L}^J$ denote the set of functions which are piecewise linear with respect to the partition $\mathcal{P}^J$ and which minorize the value function. Analogously, $\mathcal{U}^J$ denotes the set of piecewise linear majorants of $v(\cdot)$. Then $\hat L^J(\cdot),\hat U^J(\cdot)$ solve the semiinfinite programs

$$\sup_{L\in\mathcal{L}^J} \int_{\hat\Omega} L(\eta,\xi)d\hat P \;\le\; \int v(\eta,\xi)d\hat P,$$
$$\inf_{U\in\mathcal{U}^J} \int_{\hat\Omega} U(\eta,\xi)d\hat P \;\ge\; \int v(\eta,\xi)d\hat P. \tag{26}$$

Again, these semiinfinite programs are duals of the corresponding generalized moment problems with unique solutions $\hat Q_l^J,\hat Q_u^J$. In this sense, $\hat Q_l^J$ and $\hat Q_u^J$ may be viewed as best discretization of the stochastic risk factors with respect to the partition. The support of $\hat{Q}_l^J$ and $\hat{Q}_u^J$ is finite, whose elements may be viewed as *generalized barycenters* of the $\times$-simplices $\hat{\Omega}^j$ $(j = 1, \cdots, J)$. Both barycenters and their probabilities are completely determined by the corresponding conditional first-order and joint moments. From the dual viewpoint, $\hat{L}^J(\cdot)$ and $\hat{U}^J(\cdot)$ are completely determined by the first-order derivatives at the barycenters. Due to the characteristic features of the methodology, $\hat{L}^J(\cdot)$ and $\hat{U}^J(\cdot)$ are called *barycentric approximations* of the value function. Being aware of the piecewise bilinearity of $\hat{L}^J(\cdot)$ and $\hat{U}^J(\cdot)$ with respect to the partition,

$$\begin{aligned} \int_{\hat{\Omega}} \hat{L}^J(\eta, \xi) d\hat{P}_l &= \int_{\hat{\Omega}} \hat{L}^J(\eta, \xi) d\hat{Q}_l^J, \\ \int_{\hat{\Omega}} \hat{U}^J(\eta, \xi) d\hat{P}_l &= \int_{\hat{\Omega}} \hat{U}^J(\eta, \xi) d\hat{Q}_u^J \end{aligned} \tag{27}$$

hold and due to strong duality,

$$\begin{aligned} \int_{\hat{\Omega}} \hat{L}^J(\eta, \xi) d\hat{P}_l &= \int_{\hat{\Omega}} v(\eta, \xi) d\hat{Q}_l^J, \\ \int_{\hat{\Omega}} \hat{U}^J(\eta, \xi) d\hat{P}_l &= \int_{\hat{\Omega}} v(\eta, \xi) d\hat{Q}_u^J \end{aligned} \tag{28}$$

hold. Moreover, in case the diameters of all subcells tend towards 0 for $J \to \infty$, convergence of $\hat{L}^J(\cdot), \hat{U}^J(\cdot)$ to $v(\cdot)$ and weak convergence of $\hat{Q}_l^J, \hat{Q}_u^J$ to $\hat{P}$ is ensured.

Defining the associate approximate distribution functions with respect to $v(\cdot)$ according to

$$\begin{aligned} \hat{F}_l^J(\hat{v}) &= \hat{Q}_l^J(v(\eta, \xi) \leq \hat{v}), \\ \hat{F}_u^J(\hat{v}) &= \hat{Q}_u^J(v(\eta, \xi) \leq \hat{v}), \end{aligned} \tag{29}$$

implies $\forall \hat{v} \in \mathbb{R}$

$$\lim_{J \to \infty} \hat{F}_l^J(\hat{v}) = \lim_{J \to \infty} \hat{F}_u^J(\hat{v}) = \hat{F}(\hat{v}), \tag{30}$$

which confirms the pointwise convergence of the approximate distribution functions. Hence, $\hat{F}_l^J(\hat{v})$, $\hat{F}_u^J(\hat{v})$ may be accepted as approximation of $F(\hat{v})$. It is emphasized that due to the weak convergence of $\hat{Q}_l^J, \hat{Q}_u^J$ to $\hat{P}$, the convergence in (30) holds for value functions $v(\cdot)$ that are continuous on $\hat{\Omega}$. If $v(\cdot)$ is a saddle function in addition, the corresponding error can be quantified at each stage J via the bound

$$\int_{\hat{\Omega}} \hat{U}^J(\eta, \xi) d\hat{Q}_u^J \Leftrightarrow \int_{\hat{\Omega}} \hat{L}^J(\eta, \xi) d\hat{Q}_l^J > \; 0. \tag{31}$$

It is emphasized that $\hat{F}_l^J(\hat{v})$, $\hat{F}_u^J(\hat{v})$ and, hence, the error bound become available by evaluating $v(\cdot)$ at the generalized barycenters. If the sequence of partitions represents successive refinements with the diameters of the subcells becoming arbitrarily small, the error bound converges monotonously towards 0.

5. Approximating the P&L Distribution

Let the stochasticity of the risk factors ω with respect to the end of some fixed holding period [0,t] be modelled by the probability space $(\Omega, \mathcal{B}, P)$. Ω is an arbitrary measurable subset of $\mathbb{R}^M$ and may cover the entire space $\mathbb{R}^M$. The value function $v(\cdot) : \mathbb{R}^M \rightarrow \mathbb{R}$ of the portfolio, whose risk has to be assessed by means of the VaR approach, is given implicitly via some pricing model.

Take a measurable $\times$-simplex $\hat{\Omega} := \Theta \times \Xi$ and an adequate $\times$-simplicial partition $\mathcal{P}^J := \{\hat{\Omega}^j;\ j = 1, \cdots, J\}$ with the property

$$P(\hat{\Omega}) \geq 1 \Leftrightarrow \epsilon \tag{32}$$

for some positive ϵ that is sufficiently small and define for any $B \in \mathcal{B}$,

$$\hat{P}(B) = \frac{P(B \cap \hat{\Omega})}{P(\hat{\Omega})}. \tag{33}$$

Recalling $F(\hat{v}) = P(v(\omega) \leq \hat{v})$, $\hat{F}(\hat{v}) = \hat{P}(v(\omega) \leq \hat{v})$ and (32), the following relation holds:

$$\hat{F}(\hat{v}) \Leftrightarrow \epsilon \leq F(\hat{v}) \leq \hat{F}(\hat{v}) + \epsilon. \tag{34}$$

Hence, whenever one manages to approximate $\hat{F}$ with sufficient accuracy, this approximation may also be accepted as sufficiently accurate with respect to F.

One may apply the methodology outlined above to derive the approximations $\hat{L}^J(\cdot), \hat{U}^J(\cdot)$ and $\hat{Q}_l^J, \hat{Q}_u^J$ for the value function $v(\cdot)$ and the probability measure $\hat{P}$, respectively, which yield the desired approximations $\hat{F}_l^J(\hat{v}), \hat{F}_u^J(\hat{v})$ for $\hat{F}(\hat{v})$ and, hence, for $F(\hat{v})$ due to (30) and (34). The corresponding VaR-estimates of the first type with respect to level α are denoted $\hat{v}_{J,\alpha}^{I,l}$, $\hat{v}_{J,\alpha}^{I,u}$ and given by

$$\hat{F}_l^J(\hat{v}_{J,\alpha}^{I,l}) = \hat{F}_u^J(\hat{v}_{J,\alpha}^{I,u}) = \alpha. \tag{35}$$

Being aware of the dimensionality of the underlying portfolio value functions and asking for the capital at risk, it is useful to concentrate on loss events, i.e., on those ω for which $v(\omega) \leq 0$. The numerical effort associated with approximating $F(\cdot) : \mathbb{R}_- \to [0,1]$ heavily depends on how 'tricky' the sequence of partitions is constructed. At the beginning of this section, it was emphasized that only continuity of $v(\cdot)$ was supposed and the saddle property was relaxed. However, having in mind that the diameters of the subcells tend towards 0, it makes sense to provide additional concepts for identifying the saddle property in case of its existence, at least, relative to some subcell $\hat{\Omega}^j$. Then, error bounds will be available locally on $\hat{\Omega}^j$, and help identify efficient refinement strategies, that allow for approaching F sufficiently fast and reliable. As long as the saddle property cannot be identified on $\hat{\Omega}^j$, the difference in (31) may be accepted as an estimate of inaccuracy.

For determining a $\times$-simplicial partition $\mathcal{P}^{J+1}$ by means of a refinement of $\mathcal{P}^J$, a subcell in $\mathcal{P}^J$ and an adequate edge are required to be chosen, subject to which the subcell is split. The experiences made with barycentric approximation for solving stochastic programs (see [4], [5]) make one aware of the fact that the choice of both subcell and edge is one of the key steps for an appealing convergence. In the following section, we report on the software package currently available for approximating the P&L distribution and state first numerical results.

6. First Numerical Results

The current version of the package considers a simplex $\hat{\Omega}$ and an adequate sequence of simplicial partitions, taking into account one of the following strategies for subcell and edge selection. The longest edge of a subcell $\hat{\Omega}^j$ is denoted ℓ_j, the maximum loss with respect to the barycenters and vertices of $\hat{\Omega}^j$ is denoted v_j. The subcell to partition may be chosen according to:

S1: Select that subcell $\hat{\Omega}^j$ for which $\ell_j \cdot P(\hat{\Omega}^j)$ attains the maximum.

S2: Select that subcell $\hat{\Omega}^j$ for which $\Leftrightarrow v_j \cdot P(\hat{\Omega}^j)$ attains the maximum.

In S1, the longest edge is taken into account to enforce that the diameters of the subcells decrease, which finally ensures weak convergence of the extremal measures and herewith convergence of the approximate distribution functions to the P&L distribution. In S2, we focus on improving the accuracy at the downside of the P&L distribution.

Having selected the subcell $\hat{\Omega}^j$ an adequate edge remains to be chosen:

$J=50$	*S1/E1*	*S1/E2*	*S1/E3*	*S2/E1*	*S2/E3*	*BM*
$v_{1,6}$	-4.129	-12.749	-12.749	-3.219	-17.526	-4.562
$v_{2,6}$	-86.875	-165.952	-75.341	-66.247	-75.086	-65.995
$v_{3,4}$	-92.293	-85.950	-85.950	-87.967	-87.227	-84.443

Table 1: VaR-estimates with respect to $\alpha = 0.05$

E1: Select the longest edge of $\hat{\Omega}^j$ for which $\ell_j \cdot P(\hat{\Omega}^j)$ attains the maximum

E2: Select that edge along which the portfolio values differ most.

E3: Select the edge along which the first order approximation of v is most inaccurate. In case the first order approximation coincides with the value function on $\hat{\Omega}^j$, take the longest edge.

E1 aims at decreasing diameters of the partitioned subcells and herewith at approximating the P&L distribution. E2 focuses on the downside approximation of the P&L distribution due to the structural property of a simplex. E3 aims at improving approximations of the risk profile by Δ-hedging. It is noted that only E3 incorporates the first order information, the other strategies only require the evaluation of v at the barycenters and vertices of the subcells.

The following three two-dimenional risk profiles have been taken in normalized polynomial form from a large FX-portfolio [2] for approximating their associated P&L distribution by means of extremal measures.

$$\begin{aligned} v_{1,6}(\omega_1, \omega_2) &= 18.74\omega_1^2 \Leftrightarrow 8\omega_2^3 \\ v_{2,6}(\omega_1, \omega_2) &= 43.29\omega_1 \Leftrightarrow 8\omega_2^3 \\ v_{3,4}(\omega_1, \omega_2) &= \Leftrightarrow 5.34(\omega_1 \Leftrightarrow 1)^3 \Leftrightarrow 2.67\omega_1^2 + 32.04\omega_1 \\ &\quad +31.96\omega_2^3 \Leftrightarrow 128.7\omega_2 \Leftrightarrow 5.34 \end{aligned}$$

The two-dimensional risk factors are distributed normally with mean 0 and a variance-covariance matrix

$$\Sigma_{1,6} := \begin{bmatrix} 1 & 0.95 \\ 0.95 & 1 \end{bmatrix}, \ \Sigma_{2,6} := \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \ \Sigma_{3,4} := \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}.$$

The achieved VaR-estimates with respect to a level of $\alpha = 0.05$ and the associated Monte-Carlo simulation benchmarks BM (due to [2]) are summarized in Table 1. For the ease of exposition we listed the VaR-estimate

$$\hat{v}^I_{J,\alpha} := \frac{\hat{v}^{I,l}_{J,\alpha} + \hat{v}^{I,l}_{J,\alpha}}{2}, \tag{36}$$

which is achieved after 50 refinements (i.e., $J = 50$). It is observed that the efficiency of some combination Si/Ej is strongly dependent on the degree of non-linearity and on the monotonicity of the price sensitivities. In the current version, no such structural properties are taken into account in the sense that these structural properties are not monitored during the refinement process. Merely the fact is exploited, that sooner or later with the subcells becoming sufficiently small the saddle property with respect to these subcells arises, and the approximation of the P&L distribution benefits from the characteristic features of the incorporated extremal measures.

Nevertheless, the presented methodology considerably outperforms the current practical approaches by their accuracy and the Monte-Carlo simulation by its numerical effort (see in [2]). The corresponding approximations of the P&L distribution are displayed in Figures 1-3. The current version of the package requires 50 evaluations at the barycenters and 52 at the vertices of the subcells for $J = 50$ refinements. For strategy $E3$, further 52 first order evaluations are needed. It may be expected that this numerical effort can even be reduced in case the above mentioned structural properties are exploited.

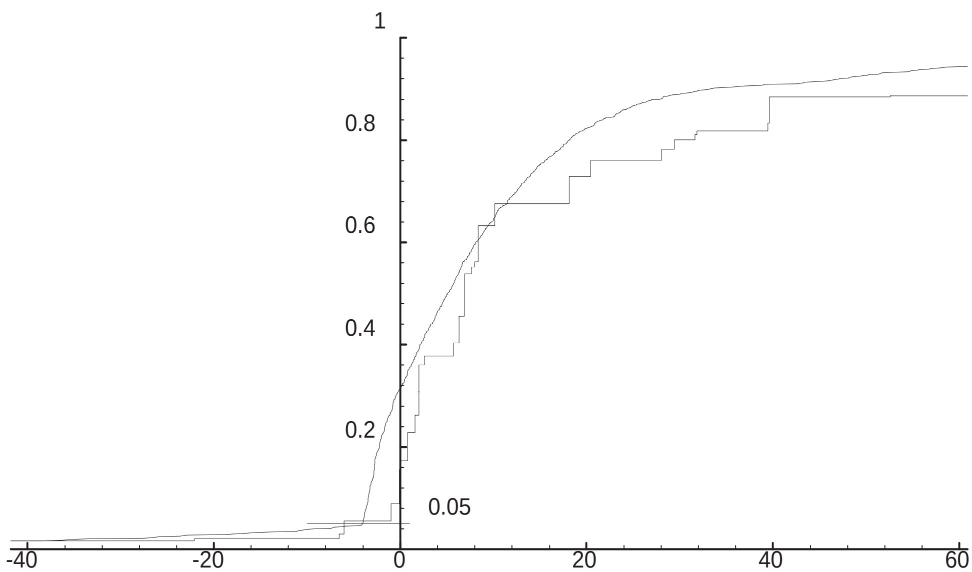

Figure 1: Approximation of the P&L distribution (risk profile 1/6, s.t. S1/E1)

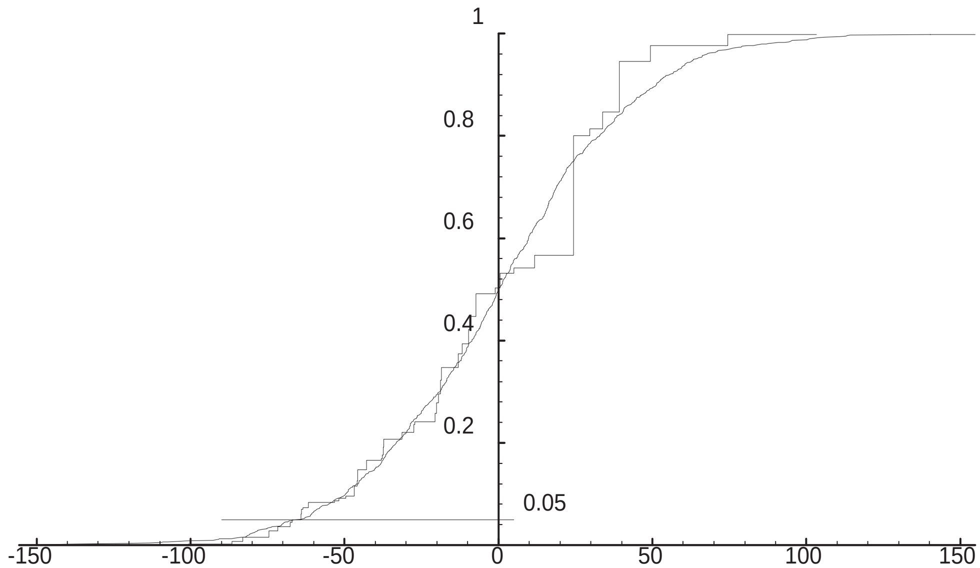

Figure 2: Approximation of the P&L distribution (risk profile 2/6, s.t. S2/E1)

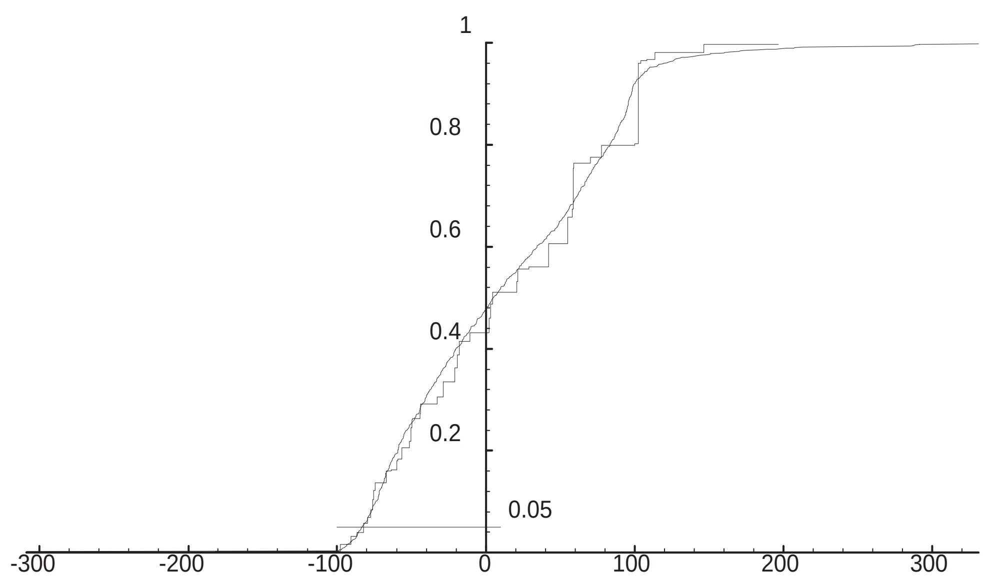

Figure 3: Approximation of the P&L distribution (risk profile 3/4, s.t. S2/E1)

7. Conclusions and Outlook

The introduced methodology utilizes a special class of extremal measures for approximating the P&L distribution. Associated with the *barycentric approximation* of the risk profiles monotonicity of the subdifferential mappings is exploited and provide investors with a kind of best discretization of the underlying stochastic risk factors. In case the value functions have monotonous derivatives, the payoff-functions are convex or concave depending on whether a position is held short or long. Standard functions corresponding to bonds, interest rate swaps and Black-Scholes option prices have been roughly discussed for motivating a comprehensive analysis of common value functions with respect to the saddle property in the risk factors. A better understanding will certainly help derive appealing *aggregations techniques* for keeping the VaR-evaluations numerically managable. It is observed that the structural properties of $\times$-simplices and their partitions motivate the aggregation of short and long positions separately. In a next step, such an analysis is intended to be applied to the value functions of instruments in the BIS-portfolio.

The basics of the methodology has been outlined by introducing a pair of dual problems, the *generalized moment problem* and the *semiinfinite program.* The associated pair of dual solutions, the extremal majors and the piecewise bilinear functions provide adequate approximations of both the risk profiles and the risk factor distributions, termed *barycentric approximations.* In case a proper refinement process is installed convergence is ensured by weak convergence of the extremal measures, implying the convergence of the VaR-estimates to the actual VaR values. If the saddle property is identified, error bounds help assess the inaccuracy of the approximations. This motivates the investigation of efficient procedures for *saddle property identification* with respect to various risk factor regions and for controlling the refinement process, such that convergence is appealing and numerically manageable

The current implemetation of this methodology considers a simplex and an adequate sequence of simplicial partitions, taking into account useful strategies for subcell and edge selection. It is observed that the efficiency of selection strategies is strongly dependent on the degree of nonlinearity and the monotonicity of price sensitivities. In the current version no such structural properties are taken into account explicitly, only implicitly by the characteristic features of extremal measures. In this sense, these structural properties are not exploited explicitly within the refinement process. Incorporating the above-mentioned $\times\Leftrightarrow$*simplices*, *aggregation techniques*, and *saddle property identifying procedures* into the package will certainly increase the efficiency of the methodology and its applicability to portfolios of larger dimensions.

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