Some Generalizations of Bessel Processes

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Contents

Introduction

Bessel processes are a one-parameter family of diffusion processes that appear in many financial problems and have remarkable properties. Following this introduction we recall the definition of Bessel processes and their properties in detail (see I.1 – I.6). One important property of Bessel processes is, that the transition densities are known explicitly (see I.2); apart from the case of Bessel processes they are essentially only known for Ornstein–Uhlenbeck processes and Brownian motion with drift. Further important properties are the scaling property (see I.4) and the additivity property of squared Bessel processes (see I.3), which allow us to reduce many problems to simpler ones.

As for the importance of Bessel processes in financial mathematics, let us first mention their contribution to the problem of pricing Asian options with arithmetic asset average. Using the fact that the exponential of Brownian motion with drift, i.e. geometric Brownian motion, is a time-changed Bessel process (see representation (0.4) in I.4), all moments of the arithmetic asset average can be calculated and an expression of the Laplace transform of the Asian option price is obtained, see Geman–Yor [7].

Furthermore, for the Cox–Ingersoll–Ross (CIR) model for interest rates [4], see (0.6), Bessel processes are playing an important role on which we will concentrate here. The CIR processes are time-space-transformed squared Bessel processes and can also be transformed by a time change to squared radial Ornstein-Uhlenbeck processes (for details see I.5 and I.6). Therefore, in order to obtain results for the CIR processes, we will consider (squared) Bessel processes respectively (squared) radial Ornstein-Uhlenbeck processes.

In financial mathematics, diffusion processes are often studied until they reach a certain level for the first time, as for instance in the case of barrier options (see e.g. Geman–Yor [8], Chesney–Geman–Jeanblanc-Picqué–Yor [3]). Also for statistical reasons, first hitting times are very interesting, e.g. if we want to condition a process on never hitting a given barrier. In order to obtain results about first hitting times of the CIR process, in the first chapter we are interested in (the law of) the first time a squared radial Ornstein–Uhlenbeck process hits an arbitrary level, especially level zero.

In Chapter 2 we introduce Bessel processes with negative dimensions. For instance, as can be seen from representation (0.4), these processes arise quite naturally when the exponential of Brownian motion with a negative drift is considered. We study their properties and extend the results to a wider class of processes.

In the last chapter we more deeply investigate time reversal which was an important tool in the previous chapters. We review general results for time-reversed diffusions and as an application we check a time reversal result for radial Ornstein–Uhlenbeck processes by Elworthy–Li–Yor [6]. Furthermore, we introduce a three-parameters-family of processes.

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Main properties of Bessel processes

Now we give the definition of Bessel processes and state some of their properties. As for the study of Bessel processes we refer to Revuz–Yor [30] and Pitman–Yor [27, 29].

I.1 For every $\delta \geq 0$ and $x \geq 0$, the solution to the equation

$$X_t = x + \delta\, t + 2\int_0^t \sqrt{|X_s|}\, dW_s$$

is unique and strong. In the case $\delta = 0, x = 0$, the solution X_t is identically zero and applying the comparison theorem we conclude $X_t \geq 0$ for all $\delta \geq 0$.

Definition 1 ($\mathbf{BESQ}^{\delta}$) *For every $\delta \geq 0$ and $x \geq 0$ the unique strong solution to the equation*

$$X_t = x + \delta\, t + 2\int_0^t \sqrt{X_s}\, dW_s$$

is called the square of a δ-dimensional Bessel process *started at x and is denoted by* $\mathrm{BESQ}^{\delta}(x)$.

Denote the law of $\mathrm{BESQ}^\delta(x)$ on $C(\mathbb{R}_+, \mathbb{R})$ by Q_x^δ. We call the number δ the *dimension* of BESQ^δ. This notation arises from the fact that a BESQ^δ process X_t can be represented by the square of the Euklidian norm of δ-dimensional Brownian motion B_t: $X_t = |B_t|^2$. The number $\nu \equiv \delta/2 - 1$ is called the *index* of the process BESQ^δ.

Definition 2 ($\mathbf{BES}^\delta$) *The square root of* $\mathrm{BESQ}^\delta(a^2)$, $\delta \geq 0$, $a \geq 0$ *is called the* Bessel process of dimension δ *started at* a *and is denoted by* $\mathrm{BES}^\delta(a)$.

Denote the law of $\mathrm{BES}^\delta(a)$ by P_a^δ.

In the case $\delta \geq 2$ $\mathrm{BES}^\delta(a)$, $a > 0$, will never reach 0. For $\delta > 1$ a $\mathrm{BES}^\delta(a)$ process Z_t satisfies $E[\int_0^t (ds/Z_s)] < \infty$ and is the solution to the equation

$$Z_t = a + \frac{\delta - 1}{2} \int_0^t \frac{1}{Z_s} ds + W_t\,.$$

For $\delta \leq 1$ the situation is less simple. For $\delta = 1$ we have with Itô-Tanaka's formula

$$Z_t = |W_t| = \tilde{W}_t + L_t\,,$$

where $\tilde{W}_t \equiv \int_0^t \mathrm{sgn}(W_s) dW_s$ is a standard Brownian motion, and L_t is the local time of Brownian motion. For a treatment of local times see e.g. Revuz–Yor [30], Chapter VI. For $\delta < 1$ we have

$$Z_t = a + \frac{\delta - 1}{2}\, \text{p.v.} \int_0^t \frac{ds}{Z_s} + W_t\,, \tag{0.1}$$

where the principal value is defined as

$$\text{p.v.} \int_0^t \frac{ds}{Z_s} \equiv \int_0^\infty x^{\delta-2}(L_t^x - L_t^0) dx$$

and the family of local times $(L_t^x, x \geq 0)$ is defined as

$$\int_0^t \varphi(Z_s) ds = \int_0^\infty \varphi(x) L_t^x x^{\delta-1} dx$$

for all Borel functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, see Yor [41], Chapter 10. The decomposition (0.1) was obtained using the fact that a power of a Bessel process is another Bessel process time-changed, see (0.5).

I.2 Transition densities

Bessel (squared) processes are Markov processes and their transition densities are known explicitly. For $\delta > 0$, the transition density for BESQ^δ is equal to

$$q_t^\delta(x, y) = \frac{1}{2t} \left(\frac{y}{x}\right)^{\frac{\nu}{2}} \exp\left\{-\frac{x+y}{2t}\right\} I_\nu\left(\frac{\sqrt{xy}}{t}\right)\,, \tag{0.2}$$

where $t > 0$, $x > 0$, $\nu \equiv \frac{\delta}{2} - 1$ and I_ν is the modified Bessel function of the first kind of index ν. For $x = 0$ we have

$$q_t^\delta(0, y) = (2t)^{-\frac{\delta}{2}} \Gamma\left(\delta/2\right)^{-1} y^{(\delta/2)-1} \exp\left\{-\frac{y}{2t}\right\} . \qquad (0.3)$$

The transition density for BES^δ is obtained from (0.2) resp. (0.3) and is equal to

$$p_t^\delta(x, y) = \frac{1}{t}\left(\frac{y}{x}\right)^\nu y \exp\left\{-\frac{x^2 + y^2}{2t}\right\} I_\nu\left(\frac{xy}{t}\right) ,$$

with $t > 0$, $x > 0$, and

$$p_t^\delta(0, y) = 2^{-\nu} t^{-(\nu+1)} \Gamma(\nu + 1)^{-1} y^{2\nu+1} \exp\left\{-\frac{y^2}{2t}\right\} .$$

I.3 Additivity property of Bessel squared processes

An important and well-known property of BESQ^δ processes with $\delta \geq 0$ is the following additivity property.

Theorem 1 (Shiga–Watanabe [35]) *For every δ, $\delta' \geq 0$ and x, $x' \geq 0$:*

$$Q_x^\delta * Q_{x'}^{\delta'} = Q_{x+x'}^{\delta+\delta'} ,$$

where $Q_x^\delta * Q_{x'}^{\delta'}$ denotes the convolution of Q_x^δ and $Q_{x'}^{\delta'}$.

For a proof see Shiga–Watanabe [35] or Revuz–Yor[30], p. 410.

I.4 Scaling property and representations

BES^δ processes have the Brownian scaling property, i.e. if X is a $\mathrm{BES}^\delta(x)$, then the process $c^{-1}X_{c^2t}$ is a $\mathrm{BES}^\delta(x/c)$ for any $c > 0$. BESQ^δ processes have the following scaling property: if X is a $\mathrm{BESQ}^\delta(x)$, then the process $c^{-1}X_{ct}$ is a $\mathrm{BES}^\delta(x/c)$.

An important result by Lamperti [19], see also Williams [38], is that the exponential of Brownian motion with drift can be represented as a time-changed Bessel process:

$$\exp(B_t + \nu t) = X^{(\nu)}\left(\int_0^t \exp 2(B_s + \nu s)\, ds\right) , \quad t \geq 0 , \qquad (0.4)$$

where $(X^{(\nu)}(u), u \geq 0)$ is a Bessel process with index ν. We remark that because of the Brownian scaling property this result can be extended to a representation of $\exp(a\, B_t + \nu t)$. There is also a representation of $\exp(a\, B_t +$

νt) in terms of BESQ processes (see Geman–Yor [7]), and in terms of arbitrary powers of BES processes. This result is closely related to the fact that a power of a Bessel process is another Bessel process time-changed

$$qZ_\nu^{1/q}(t) = Z_{\nu q}\left(\int_0^t \frac{ds}{Z_\nu^{2/p}(s)}\right), \tag{0.5}$$

where $\frac{1}{p} + \frac{1}{q} = 1, \nu > -\frac{1}{q}$, see e.g. Revuz–Yor [30], Proposition 11.(1.11).

I.5 Transformation: CIR process – squared Bessel process

Cox, Ingersoll and Ross [4] have proposed an interest rate model in which the process for the short term interest rate is the solution to the equation

$$X_t = x + \int_0^t (a + bX_s)\, ds + \sigma \int_0^t \sqrt{X_s}\, dW_s\,, \tag{0.6}$$

for $t \geq 0$, $a \geq 0$, $\sigma > 0$. In the finance literature this process is known as the Cox–Ingersoll–Ross (CIR) process. For a treatment of interest rate models including the CIR model we refer to Lamberton–Lapeyre [18], §6.

We remark that this process is also used in a population growth model, see Karlin–Taylor [15], p. 334.

The CIR process is a space-time transformed BESQ process, more explicitly: A BESQ$^\delta(y)$ process Y can be transformed to the CIR process X by

$$X_t = e^{bt}\, Y\left(\frac{\sigma^2}{4b}(1 - e^{-bt})\right),$$

where $\delta \equiv 4a/\sigma^2$.

I.6 Transformation: CIR process – squared radial Ornstein–Uhlenbeck process

A squared radial Ornstein–Uhlenbeck process is the solution to the equation

$$Y_t = y + \int_0^t (\delta - 2\lambda Y_s)\, ds + 2\int_0^t \sqrt{Y_s}\, dW_s$$

for $t \geq 0$, $\delta \geq 0$. Radial Ornstein–Uhlenbeck processes are studied in detail in the first chapter, see (1.1). A squared radial Ornstein–Uhlenbeck process can be transformed by the time transformation $g(t) = \sigma^2 t/4$ to the CIR process X, where $\delta \equiv 4a/\sigma^2$, $\lambda \equiv -2b/\sigma^2$, with the notation in (0.6).

The transformations in I.5 and I.6 show the strong relation between CIR processes and squared Bessel respectively squared radial Ornstein–Uhlenbeck processes. In Chapter 1 we are considering first hitting times of squared radial Ornstein–Uhlenbeck processes, and by means of the transformation in I.6 we see immediately that, once the law of the first hitting time of a squared radial Ornstein–Uhlenbeck process is found, we know the law of the first hitting time of the CIR process.

Chapter 1

First hitting times of radial Ornstein–Uhlenbeck processes

As motivated in the introduction (see I.6) we are interested in finding the law of first hitting times of squared radial Ornstein–Uhlenbeck processes. As for the subject of first hitting times we refer to Breiman [1], Novikov [24, 25, 26], Shepp [34] and Yor [39].

First we recall the definition of (squared) radial Ornstein–Uhlenbeck processes. Let $\{W_t\}$ be a one-dimensional Brownian motion, $\lambda \in \mathbb{R}$, $\delta \geq 0$ and $z \geq 0$. The solution to the equation

$$Z_t = z + \int_0^t (\delta - 2\lambda Z_s)\, ds + 2 \int_0^t \sqrt{|Z_s|}\, dW_s$$

is unique and strong (see Revuz–Yor [30] Chapter IX §3). Since in the case $\delta = 0, z = 0$, the solution is $Z_t \equiv 0$, we deduce from the comparison theorem $Z_t \geq 0$ for all $\delta \geq 0$, and hence the absolute value can be omitted; the solution of

$$Z_t = z + \int_0^t (\delta - 2\lambda Z_s)\, ds + 2 \int_0^t \sqrt{Z_s}\, dW_s \tag{1.1}$$

is called a *squared δ-dimensional radial Ornstein–Uhlenbeck process with parameter $-\lambda$*. It is a Markov process; hence, the square root of this process is also a Markov process and is called a *δ-dimensional radial Ornstein–Uhlenbeck process with parameter $-\lambda$*. For $\delta \geq 2$ and $z > 0$, it almost surely does not hit zero. For $\delta > 1$ it is the solution to the equation

$$dR_t = \left(\frac{\delta - 1}{2R_t} - \lambda R_t \right) dt + dW_t\,, \quad R_0 = x = \sqrt{z}\,.$$

Our aim is to find the law of

$$T_y = \inf\{t \mid R_t = y\}\,, \tag{1.2}$$

the first time a radial Ornstein–Uhlenbeck process $\{R_t\}$ with parameter $-\lambda$ starting in $x > 0$ hits the level y, $0 < y < x$. For $\delta < 2$, we have ${}^{-\lambda}P_x^\delta(T_0 < \infty) > 0$, that is, the process $\{R_t\}$ may reach 0; if $\delta < 2$ and $\lambda > 0$, then ${}^{-\lambda}P_x^\delta(T_0 < \infty) = 1$, that is, $\{R_t\}$ reaches 0 a.s. and hence every y a.s., $0 < y < x$.

We remark, that since:

$$T_y = \inf\{t \mid R_t = y\} = \inf\{t \mid R_t^2 = y^2\}\,,$$

once we found the law of (1.2) we have solved our original problem: finding the law of the first time a squared radial Ornstein–Uhlenbeck process with $\delta < 2$ starting in $a\,(= x^2)$ hits $b\,(= y^2)$, $0 < b < a$.

Call ${}^{-\lambda}P_x^\delta$ the law of a δ-dimensional radial Ornstein–Uhlenbeck process with parameter $-\lambda$. The density ${}^{-\lambda}p_x^\delta(t)$ of the first hitting time of 0 of a radial Ornstein–Uhlenbeck process is calculated in Elworthy–Li–Yor [6] (see Cor. 3.10) by using a time reversal argument, that is, for $y = 0$ the problem is solved:

$${}^{-\lambda}p_x^\delta(t) = \frac{x^{2-\delta}}{2^\nu\Gamma(\nu)}\exp\left[\tfrac{\lambda}{2}(\delta t + x^2(1-\coth(\lambda t)))\right]\left[\frac{\lambda}{\sinh(\lambda t)}\right]^{\frac{4-\delta}{2}}, \tag{1.3}$$

where $\delta < 2$, $\lambda > 0$, $x > 0$ and $\nu = \frac{4-\delta}{2} - 1$.

In the following, let $T_{x\to y}$ denote the first time a radial Ornstein–Uhlenbeck process with dimension $\delta < 2$, starting in $x > 0$, hits y, $0 \le y < x$. Why calculating the density of $T_{x\to y}$ is more complicated than calculating the density of $T_{x\to 0}$ will be explained later. We have

$$T_{x\to 0} \overset{\text{(law)}}{=} T_{x\to y} + T_{y\to 0}\,,$$

where $T_{x\to y}$ and $T_{y\to 0}$ are independent because of the strong Markov property. Hence, we have for the Laplace transforms (LTs)

$${}^{-\lambda}E_x^\delta\left[\exp\left(-\mu T_{x\to 0}\right)\right] = {}^{-\lambda}E_x^\delta\left[\exp\left(-\mu T_{x\to y}\right)\right]\,{}^{-\lambda}E_y^\delta\left[\exp\left(-\mu T_{y\to 0}\right)\right]\,,$$

or equivalently,

$${}^{-\lambda}E_x^\delta\left[\exp\left(-\mu T_{x\to y}\right)\right] = \frac{\phi_x(\mu)}{\phi_y(\mu)}\,, \tag{1.4}$$

where

$$\phi_x(\mu) = {}^{-\lambda}E_x^\delta\left[\exp\left(-\mu T_{x\to 0}\right)\right] = \int_0^\infty \exp(-\mu t)\ {}^{-\lambda}p_x^\delta(t)\,dt.$$

We are interested in giving an explicit expression of the LT in (1.4). So far, we are not able to do this in general, but we can find explicit expressions of the LT in (1.4) for some examples in the case $\lambda = 0$, that is for Bessel processes. For that it is convenient to consider Bessel processes time-reversed. We refer to Chapter 3 where time reversal is studied in detail.

Let $\{R_t\}$ be a Bessel process with dimension $\delta < 2$. Then the time reversed process

$$\hat{R}_u \equiv R_{T_0 - u}\,, \quad u \geq 0,$$

is a $\hat{\delta}$-dimensional Bessel process starting in 0, with $\hat{\delta} \equiv (4-\delta)$. The process $(\hat{R}_u)$ is transient since $\hat{\delta} > 2$. We know by the time reversal theorem (3.7) that:

$$(R_{T_0-u},\ u \leq T_0) \overset{\text{(law)}}{=} \left(\hat{R}_u,\ u \leq \hat{L}_x\right)\,, \tag{1.5}$$

where

$$\hat{L}_x = \sup\{t \mid \hat{R}_t = x\}$$

is the last exit time of x by the process $\{\hat{R}_t\}$. In particular, as remarked in Getoor–Sharpe [9], Sharpe [33], $\hat{L}_x$ under $P_0^{\hat{\delta}}$ has the same law as T_0 under P_x^δ. We know (see Getoor [10], Pitman–Yor [27]):

$$\hat{L}_x \overset{\text{(law)}}{=} \frac{x^2}{2Z_{\hat{\nu}}}\,, \tag{1.6}$$

where $Z_{\hat{\nu}}$ is a Gamma variable with parameter $\hat{\nu} \equiv \frac{\hat{\delta}}{2} - 1$, i.e.:

$$P(Z_{\hat{\nu}} \in dt) = \frac{t^{\hat{\nu}-1}e^{-t}}{\Gamma(\hat{\nu})}\,dt\,, \tag{1.7}$$

and hence

$$P_x^\delta(T_0 \in dt) = P_0^{\hat{\delta}}(\hat{L}_x \in dt) = \frac{1}{t\Gamma(\hat{\nu})}\left(\frac{x^2}{2t}\right)^{\hat{\nu}} e^{-\frac{x^2}{2t}}\,dt\,. \tag{1.8}$$

Let $\hat{L}_{0\to x}$ resp. $\hat{L}_{0\to y}$ denote the last exit time of x resp. y by a $\hat{\delta}$-dimensional Bessel process $(\hat{R}_u)$ starting in 0. With $\hat{L}_{y\nearrow x}$ we denote the last exit time of x by $(\hat{R}_u)$ starting in y at the last exit time of y, i.e. we consider the process starting in y but never visiting y again; distinguish carefully between $\hat{L}_{y\nearrow x}$ and $\hat{L}_{y\to x}$, the last exit time of x by $(\hat{R}_u)$ starting in y. Then we have

$$\hat{L}_{0\to x} \stackrel{\text{(law)}}{=} \hat{L}_{0\to y} + \hat{L}_{y\nearrow x}\,,$$

where $\hat{L}_{0\to y}$ and $\hat{L}_{y\nearrow x}$ are independent because of the strong Markov property of last exit times (for a survey see Millar [21]). Since $\hat{L}_{y\nearrow x}$ has the same law as $T_{x\to y}$, our aim is to find an explicit expression of the LT of $\hat{L}_{y\nearrow x}$:

$$E_0^{\hat{\delta}}\left[\exp\left(-\tfrac{\alpha^2}{2}\hat{L}_{y\nearrow x}\right)\right] = \frac{E_0^{\hat{\delta}}\left[\exp\left(-\tfrac{\alpha^2}{2}\hat{L}_{0\to x}\right)\right]}{E_0^{\hat{\delta}}\left[\exp\left(-\tfrac{\alpha^2}{2}\hat{L}_{0\to y}\right)\right]}\,.$$

Now we can explain the difference between calculating the density of $T_{x\to 0}$ and the density of $T_{x\to y}$ for $y\neq 0$ of a Bessel process. The density of $T_{x\to 0}$ is obtained by considering the time reversed process started in 0 and using (1.6) and that $T_{x\to 0} \stackrel{\text{(law)}}{=} \hat{L}_{0\to x}$. The density of $T_{x\to 0}$ of a radial Ornstein–Uhlenbeck process (1.3) was obtained in a similar way, additionally using a Girsanov transformation. As for the density of $T_{x\to y}$ we cannot use the same time reversal argument since we are interested in the time reversed process *started at the last exit time of* y, and this is the time reversed process started in y conditioned on never hitting y again.

With (1.8) and with an integral representation of $K_{\hat{\nu}}$, i.e. the modified Bessel function of the second kind of order $\hat{\nu}$, (see e.g. Lebedev [20] p.119, (5.10.25)) we obtain the LT of $\hat{L}_{0\to y}$:

$$E_0^{\hat{\delta}}\left[\exp\left(-\tfrac{\alpha^2}{2}\hat{L}_{0\to y}\right)\right] = \frac{(\alpha y)^{\hat{\nu}}K_{\hat{\nu}}(\alpha\, y)}{2^{\hat{\nu}-1}\Gamma(\hat{\nu})}. \tag{1.9}$$

Thus (see also Getoor [10], Getoor–Sharpe [9], Kent [16]):

$$E_x^{\delta}\left[\exp\left(-\tfrac{\alpha^2}{2}T_y\right)\right] = E_0^{\hat{\delta}}\left[\exp\left(-\tfrac{\alpha^2}{2}\hat{L}_{y\nearrow x}\right)\right] = \left(\frac{x}{y}\right)^{\hat{\nu}}\frac{K_{\hat{\nu}}(\alpha\, x)}{K_{\hat{\nu}}(\alpha\, y)}\,.$$

Now we want to find a tractable form of $K_{\hat{\nu}}(\alpha\, x)/K_{\hat{\nu}}(\alpha\, y)$ to obtain the density explicitly. We cannot solve this problem in general but we have found explicit forms in some special cases.

a) First, we consider Brownian motion, that is, the case $\delta = 1$ (equivalently $\hat{\delta} = 3$ or $\hat{\nu} = \frac{1}{2}$). In this case we have

$$\hat{L}_{0\to x} \stackrel{\text{(law)}}{=} T_{x\to 0} \stackrel{\text{(law)}}{=} T_{y\to 0} + (\tilde{T}_{(x-y)\to 0}),$$

where $(\tilde{T}_{(x-y)\to 0})$ denotes an independent copy. Moreover, we know the density of $T_{x\to 0}$ for Brownian motion:

$$q_x(t) = \frac{x}{\sqrt{2\pi t^3}}\, e^{-\frac{x^2}{2t}}, \tag{1.10}$$

that is, we know the density of $T_{(x-y)\to 0}$ and therefore the density of $\hat{L}_{y\nearrow x}$. Note that

$$E_x\left[\exp\left(-\tfrac{\alpha^2}{2}T_{x\to 0}\right)\right] = \int_0^\infty e^{-\frac{\alpha^2 t}{2}} q_x(t)\, dt = e^{-\alpha x}. \tag{1.11}$$

b) In the cases $\delta = -1, -3, -5, \ldots$, or equivalently $\hat{\nu} = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$, we need to look precisely at

$$\frac{K_{n+\frac{1}{2}}(\alpha\, x)}{K_{n+\frac{1}{2}}(\alpha\, y)}, \quad n \in \mathbb{N}.$$

We know

$$K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2}} P_n(z) e^{-z} z^{-\left(n+\frac{1}{2}\right)}, \tag{1.12}$$

where

$$P_n(z) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!}\, \frac{z^{n-k}}{2^k k!}, \quad n \in \mathbb{N}_0\,,$$

see Ismail–Kelker [12], p. 82. From (1.12) we deduce

$$\frac{K_{n+\frac{1}{2}}(\alpha\, x)}{K_{n+\frac{1}{2}}(\alpha\, y)} = \left(\frac{y}{x}\right)^{n+\frac{1}{2}} e^{-\alpha(x-y)}\, \frac{P_n(\alpha\, x)}{P_n(\alpha\, y)}\,.$$

Hence, the LT of $\hat{L}_{y\nearrow x}$ in $\frac{\alpha^2}{2}$ is

$$E_0^{\hat{\delta}}\left[\exp\left(-\tfrac{\alpha^2}{2}\hat{L}_{y\nearrow x}\right)\right] = \left(\frac{x}{y}\right)^{\hat{\nu}} \frac{K_{\hat{\nu}}(\alpha\, x)}{K_{\hat{\nu}}(\alpha\, y)} = e^{-\alpha(x-y)}\, \frac{P_n(\alpha\, x)}{P_n(\alpha\, y)} \tag{1.13}$$

in the case $\hat{\nu} = n + \frac{1}{2}$.

Let us look at the case $n = 1$. For the right term in (1.13) we have the following equalities:

$$\begin{aligned}
e^{-\alpha(x-y)}\, \frac{P_1(\alpha\, x)}{P_1(\alpha\, y)} &= e^{-\alpha(x-y)}\, \frac{\alpha x + 1}{\alpha y + 1} && (1.14)\\
&= \frac{\alpha x}{\alpha y + 1}\, e^{-\alpha(x-y)} + \frac{1}{\alpha y + 1}\, e^{-\alpha(x-y)} && (1.15)\\
&= \frac{x}{y}\left(\frac{(\alpha y + 1) - 1}{\alpha y + 1}\right) e^{-\alpha(x-y)} + \frac{1}{\alpha y + 1}\, e^{-\alpha(x-y)} && (1.16)\\
&= \underbrace{\tfrac{x}{y}\, e^{-\alpha(x-y)}}_{A} + \underbrace{\left(1 - \tfrac{x}{y}\right) \frac{1}{\alpha y + 1}\, e^{-\alpha(x-y)}}_{B}\,. && (1.17)
\end{aligned}$$

We know that A is the LT in $\frac{\alpha^2}{2}$ of $\frac{x}{y}\, q_{x-y}(t)$, see (1.10) and (1.11). As for the second term B we have

$$B = \left(1-\tfrac{x}{y}\right) e^{-\alpha(x-y)} \int_0^\infty \exp[-(1+\alpha y)\, u]\, du \tag{1.18}$$

$$= \left(1-\tfrac{x}{y}\right) \int_0^\infty e^{-u} e^{-\alpha(yu+x-y)}\, du, \tag{1.19}$$

which is the LT in $\frac{\alpha^2}{2}$ of

$$\left(1-\tfrac{x}{y}\right) \int_0^\infty e^{-u}\, q_{(yu+x-y)}(t)\, du.$$

And since

$$\hat{q}(t) \equiv \tfrac{x}{y}\, q_{x-y}(t) + \left(1-\tfrac{x}{y}\right) \int_0^\infty e^{-u}\, q_{(yu+x-y)}(t)\, du$$

is a convex combination of densities we have that

$$e^{-\alpha(x-y)} \frac{P_1(\alpha x)}{P_1(\alpha y)}$$

is the LT in $\frac{\alpha^2}{2}$ of $\hat{q}(t)$.

As mentioned, so far we are not able to invert the LT of $T_{x\to y}$ of a δ-dimensional radial Ornstein–Uhlenbeck process R with $\delta < 2$ in general. But considering the time reversed radial Ornstein–Uhlenbeck process $\hat{R}$ started in 0, we can write the process after $\hat{L}_{0\to y}$, the last exit of y, as a diffusion, that is, as the solution of a stochastic differential equation. This we obtain by using a specific technique, the "enlargement of filtration". For a treatment see e.g. Jeulin [14] and Yor [41] §12. Heuristically spoken, we enlarge the original filtration progressively, so that the last exit time $\hat{L}_{0\to y}$ becomes a stopping time. Applying Theorem 12.4 in Yor [41] we obtain

$$\tilde{R} \equiv \hat{R}_{(\hat{L}_{0\to y}+u)} = y + \int_0^u b(\tilde{R}_v) dv - \int_0^u \frac{s'(\tilde{R}_v)}{s(y)-s(\tilde{R}_v)} 1_{(\tilde{R}_v > y)} dv + \tilde{W}_u\,, \tag{1.20}$$

where $u \geq 0$, b is the drift and s is the scale function of the transient diffusion $\hat{R}$. Thus we have written the process after $\hat{L}_{0\to y}$, that is, the time reversed process started in y conditioned on never hitting y again, as a diffusion $\tilde{R}$. As an illustration consider the process $\hat{R}$ to be a transient BES process $\hat{X}$, i.e. a BES process with index $\nu > 0$, started in 0. Its scale function may be chosen as $s(x) = -x^{-2\nu}$ and we obtain from (1.20)

$$\tilde{X}_u \equiv \hat{X}_{(\hat{L}_{0\to y}+u)} = y + \int_0^u \frac{1}{\tilde{X}_v} \frac{(\nu+\frac{1}{2})\tilde{X}_v^{2\nu} + (\nu-\frac{1}{2}) y^{2\nu}}{\tilde{X}_v^{2\nu} - y^{2\nu}}\, dv + \tilde{W}_u. \tag{1.21}$$

For a $\mathrm{BES}^3(0)$ process $\hat{X}$ (1.21) reduces to

$$\tilde{X}_u \equiv \hat{X}_{(\hat{L}_{0\to y}+u)} = y + \int_0^u \frac{dv}{\tilde{X}_v - y} + \tilde{W}_u .$$

Hence for a $\mathrm{BES}^3(0)$ process X we have

$$(X_{L_y+u} - y,\, u \geq 0) \overset{\text{(law)}}{=} (X_u,\, u \geq 0)\,.$$

In general, the transition density $\tilde{p}$ of the diffusion $\tilde{R}$ in (1.20) is unknown. Note that if it were known, we would obtain the density of the last hitting time of x by the process $\tilde{R}$ immediately from the formula

$$P_y(\tilde{L}_x \in dt) = -\frac{1}{2s(x)}\,\tilde{p}_t(y,x)\,dt\,, \tag{1.22}$$

see Borodin–Salminen [2] IV.43, Revuz–Yor [30] VII.(4.16), where $\tilde{R}_0 = y$ and s is the scale function of $\tilde{R}$ with $\lim_{a\downarrow 0} s(a) = -\infty$ and $s(\infty) = 0$.

Chapter 2

BESQ processes with negative dimensions and extensions

Bessel processes with nonnegative dimension $\delta \geq 0$ and starting point $x \geq 0$ are well-studied, see Introduction I.1-I.4 and e.g. Revuz–Yor [30], Chapter XI. It seems to be quite natural also to consider Bessel processes with nonnegative dimension and negative starting point. As will be developed later this case is strongly related to Bessel processes with negative dimensions. Therefore we are motivated to extend the definition of BESQ_x^δ processes (see I.1) to BESQ_x^δ processes with arbitrary δ, $x \in \mathbb{R}$.

Definition 3 *The solution to the stochastic differential equation*

$$dX_t = \delta\, dt + 2\sqrt{|X_t|}\, dW_t\,, \quad X_0 = x\,, \tag{2.1}$$

where $\{W_t\}$ is a one-dimensional Brownian motion, $\delta \in \mathbb{R}$ and $x \in \mathbb{R}$, is called the square of a δ-dimensional Bessel process*, starting in x, and is denoted by* BESQ_x^δ.

Equation (2.1) has a unique strong solution (see Revuz–Yor [30], Chapter IX §3). Denote its law by Q_x^δ. First, we want to investigate the behaviour of a BESQ_x^δ process, starting in $x > 0$ with dimension $\delta \leq 0$. In the case $\delta = 0$ the process reaches 0 in finite time and stays there. As for the case $\delta < 0$, we deduce from the comparison theorem that this process is smaller than the process with $\delta = 0$, hence, that 0 is also reached in finite time. Let us consider the behaviour of a BESQ_x^δ process $\{X_t\}$ with $\delta < 0$ and $x > 0$ after it reached 0; we find:

$$\tilde{X}_u \equiv X_{T_0+u} = \delta\, u + 2\int_{T_0}^{T_0+u} \sqrt{|X_s|}\, dW_s\,, \quad u \geq 0\,, \tag{2.2}$$

where T_0 denotes the first time the process $\{X_t\}$ hits 0

$$T_0 = \inf\{t \mid X_t = 0\}.$$

With the notation $\gamma \equiv -\delta$ we obtain from (2.2)

$$-\tilde{X}_u = \gamma\, u + 2\int_0^u \sqrt{|\tilde{X}_s|}\, d\tilde{W}_s\,, \quad u \geq 0\,,$$

where $\tilde{W}_s \equiv -(W_{s+T_0} - W_{T_0})$, that is, after the $\mathrm{BESQ}_x^{-\gamma}$ process $\{X_t\}$ hits 0, it behaves as a $-\mathrm{BESQ}_0^{\gamma}$, $\gamma > 0$.

From the above discussion we deduce that a BESQ_x^{δ} process with $\delta < 0$ and $x \leq 0$ behaves as a $-\mathrm{BESQ}_{-x}^{-\delta}$, especially it never becomes positive. Now we return to the case which motivated us to study Bessel processes with arbitrary δ, $x \in \mathbb{R}$:
For a BESQ_x^{δ} process with dimension $\delta \geq 0$ and starting point $x \leq 0$, we obtain with the same argument as above, that it behaves as a $-\mathrm{BESQ}_{-x}^{-\delta}$ process; this means, until it hits 0 for the first time it behaves as a $-\mathrm{BESQ}_{-x}^{-\delta}$ process, and after that it behaves as a BESQ_0^{δ}.

An important and well-known property of squared Bessel processes with non-negative dimensions is the additivity property, see Introduction I.3. We show that the additivity property is no longer true for BESQ^{δ} processes with $\delta \in \mathbb{R}$ arbitrary. Consider the BESQ_x^{β} process Z^{β} and the $\mathrm{BESQ}_y^{\tilde{\beta}}$ process $Z^{\tilde{\beta}}$, where $\beta > 0$, $\tilde{\beta} \equiv -\gamma < 0$ with $\beta \geq \gamma$, $x \geq 0$ and $y \leq 0$. Assuming the additivity property holds, would yield:

$$Z^{\beta} + Z^{\tilde{\beta}} \overset{\text{(law)}}{=} Z^{\beta} - Z^{\gamma} \overset{\text{(law)}}{=} Z^{\beta-\gamma} \geq 0,$$

since $\beta \geq \gamma$. Z^{β} and Z^{γ} are independent processes and for the following simple reason $Z^{\beta} \geq Z^{\gamma}$ cannot be true: Let X, Y be two independent real-valued random variables with $F_Y(x) < 1$ for all $x \in \mathbb{R}$. Then, if $1 = P(X \geq Y) = E[F_Y(X)]$, we have $F_Y(X) = 1$ a.s. and hence, X has to be infinite, which is a contradiction.

Now we apply the time reversal result (3.7) to a $\mathrm{BESQ}_x^{-\gamma}$ process $\{X_t\}$ with $\gamma \geq 0$, $x \geq 0$,

$$(X_t,\, t \leq T_0) \overset{\text{(law)}}{=} (\tilde{X}_{\tilde{L}_x - t},\, t \leq \tilde{L}_x), \tag{2.3}$$

where $(\tilde{X}_t,\, t \geq 0)$ denotes a $\mathrm{BESQ}_0^{4+\gamma}$ process and $\tilde{L}_x = \sup\{t \mid \tilde{X}_t = x\}$. Our aim is to find the semigroup of a $\mathrm{BESQ}_x^{-\gamma}$ process $\{X_t\}$ with $\gamma \geq 0$, $x \geq 0$. We have to decompose the process (X_t) before and after it hits 0 for the first time separately:

$$E_x^{-\gamma}[f(X_t)] = \underbrace{E_x^{-\gamma}[f(X_t)\, 1_{(t<T_0)}]}_{A} + \underbrace{E_x^{-\gamma}[f(X_t)\, 1_{(t\geq T_0)}]}_{B}\,.$$

By means of (2.3) we obtain for the first term A

$$A = E_0^{4+\gamma}[f(X_{\tilde{L}_x - t})\, 1_{(t<\tilde{L}_x)}]\,,$$

where X is a $\mathrm{BESQ}_0^{4+\gamma}$ process. Since we know (see (1.6))

$$\tilde{L}_x \stackrel{\text{(law)}}{=} \frac{x}{2Z_\nu}\,,$$

where Z_ν is a Gamma variable with parameter $\nu \equiv \frac{4+\gamma}{2} - 1$, see (1.7), we have

$$Q_0^{4+\gamma}(\tilde{L}_x \in dt) = \frac{1}{t\,\Gamma(\nu)} \left(\frac{x}{2t}\right)^\nu e^{-\frac{x}{2t}}\, dt \equiv q_x(t)\, dt\,. \tag{2.4}$$

We obtain

$$\begin{aligned} E_0^{4+\gamma}[f(X_{\tilde{L}_x - t})\, 1_{(t<\tilde{L}_x)}] &= \int_t^\infty E_0^{4+\gamma}[f(X_{s-t})]\, q_x(s)\, ds \\ &= \int_t^\infty \left(\int_0^\infty f(y)\, p_{s-t}^{4+\gamma}(0,y)\, dy\right) q_x(s)\, ds\,, \end{aligned}$$

where $p_t^\delta(0,y)$, with $\delta > 0$, is the semigroup density $p_t^\delta(x,y)$ in y of BESQ^δ for $x = 0$

$$p_t^\delta(0,y) = \left(\frac{1}{2t}\right)^{\frac{\delta}{2}} \frac{1}{\Gamma(\frac{\delta}{2})}\, y^{\frac{\delta}{2}-1} \exp\left(\frac{-y}{2t}\right)\,.$$

Analogously, for the second term B we have since $(-X_{t-T_0})$ is a BESQ_0^γ

$$B = \int_0^t \left(\int_0^\infty f(-y)\, p_{t-s}^\gamma(0,y)\, dy\right) q_x(s)\, ds\,.$$

Putting A and B together we obtain

$$\begin{aligned} E_x^{-\gamma}[f(X_t)] &= \int_t^\infty \left(\int_0^\infty f(y)\, p_{s-t}^{4+\gamma}(0,y)\, dy\right) q_x(s)\, ds \\ &\quad + \int_0^t \left(\int_0^\infty f(-y)\, p_{t-s}^\gamma(0,y)\, dy\right) q_x(s)\, ds\,, \end{aligned}$$

and hence, the semigroup density $\hat{p}_t^{(-\gamma)}$ in y of a $\mathrm{BESQ}^{-\gamma}$ process is equal to

$$\hat{p}_t^{(-\gamma)}(x,y) = \int_t^\infty p_{s-t}^{4+\gamma}(0,y)\, q_x(s)\, ds\,,$$

for $y > 0$, and

$$\hat{p}_t^{(-\gamma)}(x,y) = \int_0^t p_{t-s}^\gamma(0,|y|)\, q_x(s)\, ds\,, \tag{2.5}$$

for $y < 0$. We want to find the semigroup density $\hat{p}_t^{(-\gamma)}$ more explicitly. Consider the case $y > 0$:

$$\begin{aligned} \hat{p}_t^{(-\gamma)}(x,y) &= g(x,y,\gamma) \int_t^\infty (s(s-t))^{-(2+\frac{\gamma}{2})} \exp\left(-\frac{1}{2}\left(\frac{y}{s-t}+\frac{x}{s}\right)\right) ds \\ &= g(x,y,\gamma) \int_1^\infty \frac{t^{-3-\gamma}}{(u(u-1))^{(2+\frac{\gamma}{2})}} \exp\left(-\frac{1}{2}\left(\frac{y}{t\,(u-1)}+\frac{x}{tu}\right)\right) du\,, \end{aligned}$$

where

$$g(x,y,\gamma) = \Gamma^{-2}\left(1+\frac{\gamma}{2}\right) \frac{(xy)^{1+\frac{\gamma}{2}}}{2^{(2+\gamma)}(2+\gamma)}\,.$$

With

$$h(x,y,\gamma,t) \equiv g(x,y,\gamma)\, t^{-3-\gamma},\ m \equiv 2+\frac{\gamma}{2},\ a \equiv \frac{y}{2t},\ b \equiv \frac{x}{2t}$$

we obtain

$$\begin{aligned} \hat{p}_t^{(-\gamma)}(x,y) &= h(x,y,\gamma,t) \int_1^\infty \frac{du}{(u(u-1))^m} \exp\left(-\left(\frac{a}{u-1}+\frac{b}{u}\right)\right) \\ &= h(x,y,\gamma,t) \int_0^1 \frac{s^{2(m-1)}}{(1-s)^m} \exp\left(-bs-\frac{as}{1-s}\right) ds\,, \end{aligned}$$

and hence we find a more explicit formula for $\hat{p}_t^{(-\gamma)}$:

$$\hat{p}_t^{(-\gamma)}(x,y) = h(x,y,\gamma,t)\, e^{a-b} \int_0^1 \frac{(1-w)^{2(m-1)}}{w^m} \exp\left(bw-\frac{a}{w}\right) dw\,.$$

Analogously we obtain in the case $y < 0$:

$$\hat{p}_t^{(-\gamma)}(x,y) = k(x,y,\gamma,t) e^{-a-b} \int_0^\infty \frac{(w+1)^{2m}}{w^m} \exp\left(-bw-\frac{a}{w}\right) dw\,,$$

where

$$k(x,y,\gamma,t) \equiv \Gamma^{-2}\left(\frac{\gamma}{2}\right) \frac{2^{-\gamma}}{\gamma}\, x^{\frac{\gamma}{2}+1}\, |y|^{\frac{\gamma}{2}-1}\, t^{-\gamma-1},$$

and

$$m \equiv \frac{\gamma}{2}\,,\ a \equiv \frac{|y|}{2t}\,,\ b \equiv \frac{x}{2t}\,.$$

As an application of the time reversal result (2.3) we make some computations concerning the law $Q_x^{-\gamma}$ of a $\mathrm{BESQ}_x^{-\gamma}$ process $(Z_t)_{t\geq 0}$ with $\gamma, x > 0$. Since

$(-Z_{T_0+t}, t \geq 0)$ is a BESQ_0^γ process, independent of the past of (Z_t) up to T_0, we have

$$Q_x^{-\gamma}\left[\exp(-\int_0^\infty f(u)\, Z_u\, du)\right] = \int_0^\infty Q_x^{-\gamma}\left[\exp\,(-\int_0^t f(u)\, Z_u\, du)\Big| T_0 = t\right]$$

$$\times\, Q_0^\gamma\left[\exp(\int_0^\infty f(t+v)\, X_v\, du)\right]\, Q_x^{-\gamma}[T_0 \in dt]\,,$$

with a BESQ_0^γ process $(X_v)_{v\geq 0}$ and a Borel function f. From the time reversal result (2.3) we obtain

$$Q_x^{-\gamma}\left[\exp(-\int_0^t f(u)\, Z_u\, du)\,\Big|\, T_0 = t\right] = Q_0^{4+\gamma}\left[\exp(-\int_0^t f(t-u) X_u du)\,\Big|\, L_x = t\right],$$

and with the equalities

$$\begin{aligned} &Q_0^{4+\gamma}\left[\exp\,(-\int_0^t f(t-u) X_u du)\,\Big|\, L_x = t\right] \\ &\qquad = Q_0^{4+\gamma}\left[\exp\,(-\int_0^t f(t-u) X_u du)\,\Big|\, X_t = x\right] \\ &\qquad = Q_x^{4+\gamma}\left[\exp\,(-\int_0^t f(t-u) X_{t-u} du)\,\Big|\, X_t = 0\right] \\ &\qquad = Q_x^{4+\gamma}\left[\exp\,(-\int_0^t f(u) X_u du)\,\Big|\, X_t = 0\right] \end{aligned}$$

finally, we have with (2.4)

$$Q_x^{-\gamma}\left[\exp(-\int_0^\infty f(u)\, Z_u\, du)\right] = \int_0^\infty Q_x^{4+\gamma}\left[\exp\,(-\int_0^t f(u) X_u du)\,\Big|\, X_t = 0\right]$$

$$\times\, Q_0^\gamma\left[\exp(\int_0^\infty f(t+v)\, X_v\, dv)\right]\, q_x(t)\, dt\,.$$

As an interesting example consider

$$f(u) \equiv \frac{\lambda^2}{2}\, 1_{[0,a]}(u)\,, \quad a,\, \lambda > 0\,.$$

We know (see Pitman–Yor [29], p. 432 (2.m))

$$\begin{aligned} &Q_x^{4+\gamma}\left[\exp(-\frac{\lambda^2}{2}\int_0^t X_u du)\Big|\, X_t = 0\right] \\ &\qquad = \left[\frac{\lambda t}{\sinh(\lambda t)}\right]^{\frac{4+\gamma}{2}} \exp\left[-\frac{x}{2t}(\lambda t \coth(\lambda t) - 1)\right]\,, \end{aligned}$$

and hence we have

$$Q_x^{-\gamma}\left[\exp(-\frac{\lambda^2}{2}\int_0^a Z_t dt)\right] = \int_0^a \left[\frac{\lambda t}{\sinh(\lambda t)}\right]^{\frac{4+\gamma}{2}} \exp\left[-\frac{x}{2t}(\lambda t\coth(\lambda t)-1)\right]$$
$$\times Q_0^{\gamma}\left[\exp(\frac{\lambda^2}{2}\int_0^{a-t} X_v dv)\right] q_x(t)\, dt$$
$$+\int_a^\infty Q_x^{4+\gamma}\left[\exp(-\frac{\lambda^2}{2}\int_0^a X_u du)\bigg| X_t = 0\right] q_x(t)\, dt\,.$$

If in addition $\lambda(a-t) < \frac{\pi}{2}$,

$$Q_0^{\gamma}\left[\exp(\frac{\lambda^2}{2}\int_0^{a-t} X_v dv)\right] = \cos(\lambda(a-t))^{-\frac{\gamma}{2}}\,,$$

and we have (with $\nu = \frac{\gamma}{2}+1$)

$$Q_x^{-\gamma}\left[\exp(-\frac{\lambda^2}{2}\int_0^a Z_t dt)\right]$$
$$= \left(\frac{x}{2}\right)^{\nu}\frac{\lambda^{\nu+1}}{\Gamma(\nu)}\int_0^a \exp\left[-\frac{x}{2}\lambda\coth(\lambda t)\right]\frac{\cos(\lambda(a-t))^{-\nu+1}}{\sinh(\lambda t)^{\nu+1}}\, dt$$
$$+\left(\frac{x}{2}\right)^{\nu}\frac{1}{\Gamma(\nu)}\int_a^\infty Q_x^{4+\gamma}\left[\exp(-\frac{\lambda^2}{2}\int_0^a X_u du)\bigg| X_t = 0\right]\frac{e^{-\frac{x}{2t}}}{t^{\nu+1}}\, dt\,.$$

Extension

Now we extend the results for squares of Bessel processes with negative dimensions to a wider class of processes. We generalize the definition of a squared δ-dimensional radial Ornstein-Uhlenbeck process with parameter $-\lambda$ and starting point z in (1.1), by considering arbitrary δ, $z \in \mathbb{R}$.

Definition 4 *The solution to the stochastic differential equation*

$$dX_t = (a + b\, X_t)\, dt + 2\sqrt{|X_t|}\, dW_t, \quad X_0 = x\,, \tag{2.6}$$

where a, b, $x \in \mathbb{R}$ and $\{W_t\}$ is a one-dimensional Brownian motion, is called a squared radial Ornstein–Uhlenbeck process.

In the following, we consider the case $a < 0$ and $x > 0$ which corresponds for $b = 0$ to a BESQ_x^a process with negative dimension a. We call ${}^bQ_x^a$ the law on $C(\mathbb{R}_+, \mathbb{R})$ and for $b = 0$, as before, simply Q_x^a. Via Girsanov transformation we obtain the relationship

Lemma 1

$$
{}^{b}Q_x^a\Big|_{\mathcal{F}_t} = \exp\left[\frac{b}{2}\int_0^t \sigma(X_s)\,dW_s - \frac{b^2}{8}\int_0^t |X_s|\,ds\right]\, Q_x^a|_{\mathcal{F}_t}\,, \tag{2.7}
$$

where $\sigma(x) = \sqrt{|x|}\,\mathrm{sgn}\,(x)$.

Proof: Consider the Q_x^a-process

$$
dX_t = a\,dt + 2\sqrt{|X_t|}\,dW_t\,, \quad X_0 = x\,.
$$

Via Girsanov's theorem $\tilde{W}_t = W_t - \frac{b}{2}\int_0^t \mathrm{sgn}\,(X_s)\sqrt{|X_s|}\,ds$ is a Brownian motion under the absolutely continuous probability with density

$$
D_t \equiv \exp\left[\frac{b}{2}\int_0^t \mathrm{sgn}\,(X_s)\sqrt{|X_s|}\,dW_s - \frac{b^2}{8}\int_0^t |X_s|\,ds\right],
$$

with respect to Q_x^a, from which the result follows. □

We may also write the stochastic integral $\int_0^t \mathrm{sgn}\,(X_s)\sqrt{|X_s|}\,dW_s$ in a simpler form, since we have from Itô's formula:

$$
|X_t| = |x| + \int_0^t \mathrm{sgn}\,(X_s)\,(a\,ds + 2\sqrt{|X_s|}\,dW_s) + L_t^0(X)\,,
$$

where $L_t^0(X)$ is the semimartingale local time of X in 0. For $L_t^0(X)$ we obtain:

$$
\begin{aligned}
L_t^0(X) &= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t 1_{[0,\varepsilon]}(X_s)\,d\langle X,X\rangle_s \qquad \text{([30], p.212, Cor. 1.9)}\\
&= \lim_{\varepsilon\downarrow 0}\frac{4}{\varepsilon}\int_0^t |X_s|\,1_{[0,\varepsilon]}(X_s)\,ds\\
&\le \lim_{\varepsilon\downarrow 0}\left(4\int_0^t 1_{[0,\varepsilon]}(X_s)\,ds\right) \;=\; 0\,.
\end{aligned}
$$

Hence we have

$$
\int_0^t \mathrm{sgn}\,(X_s)\sqrt{|X_s|}\,dW_s = \frac{1}{2}\left[|X_t| - |x| - a\int_0^t \mathrm{sgn}\,(X_s)\,ds\right].
$$

Thus, (2.7) takes the form

$$
\left.\frac{{}^{b}Q_x^a}{Q_x^a}\right|_{\mathcal{F}_t} \equiv \exp\left[\frac{b}{4}\left(|X_t| - |x| - a\int_0^t \mathrm{sgn}\,(X_s)\,ds\right) - \frac{b^2}{8}\int_0^t |X_s|\,ds\right]. \tag{2.8}
$$

Now we can apply formula (2.8) to obtain some conditional expectation formula:

$$\begin{aligned} {}^b q_t^a(x,y) &= q_t^a(x,y)\,\exp\left(\frac{b}{4}\,(|y|-|x|)\right) \\ &\cdot Q_x^a\left[\exp\left(-\frac{ab}{4}\int_0^t \mathrm{sgn}(X_s)\,ds-\frac{b^2}{8}\int_0^t |X_s|\,ds\right)\Bigg|\; X_t=y\right]. \end{aligned} \tag{2.9}$$

The case $y>0$ corresponds to $t<T_0$, and $y\leq 0$ to $t\geq T_0$. For $y>0$, (2.9) reduces to

$$\begin{aligned} {}^b q_t^a(x,y) &= q_t^a(x,y)\,\exp\left(\frac{b}{4}\,(y-x)-\frac{abt}{4}\right) \\ &\cdot Q_x^a\left[\exp\left(-\frac{b^2}{8}\int_0^t X_s\,ds\right)\Bigg|\; X_t=y\right]. \end{aligned}$$

Using the time-space transformation from a BESQ^a process (X_t^a) to a squared radial Ornstein–Uhlenbeck process $({}^bX_t^a)$

$${}^bX_t^a = e^{bt}\,X^a_{\left(\frac{1-e^{-bt}}{b}\right)},$$

we also have together with the relationship (2.9) the following

$${}^b q_t^a(x,y) = e^{-bt} q^a_{\left(\frac{1-e^{-bt}}{b}\right)}\,(x,\,e^{-bt}y)\,, \tag{2.10}$$

from which ${}^b q_t^a(x,y)$ is obtained since we know $q_t^a(x,y)$, see (2.5). Hence we know the conditional expectation $Q_x^a\,[\ldots|X_t=y]$ in formula (2.9).

Chapter 3

Time reversal

In the previous chapters we used results on time reversed diffusions as an essential tool, e.g. in the first chapter as for the distribution of first hitting times by Bessel processes, or in Chapter 2 as for the transition densities of BESQ processes with negative dimensions. In this chapter we investigate these time reversal results more deeply. As for time reversal we refer to Nagasawa [22, 23] and Revuz–Yor [30], Chapter VII. First, some results on time reversed diffusions in general are reviewed. In section 3.1 we study Doob's h-transform and consider some applications. Using these results, in section 3.2 we check a time reversal theorem by Elworthy–Li–Yor [6], and based on this theorem, we introduce a three-parameters-family of processes in section 3.3.

Consider a transient diffusion (X_t), living on $\mathbb{R}_+$, with $X_0 = x_0 \geq 0$. Denote its last exit time of $a \geq 0$ by $L_a = \sup\{u \mid X_u = a\}$, where $\sup \emptyset = 0$. For a fixed, L_a is finite a.s., and for $x_0 < a$, $L_a > 0$ a.s. We want to identify the time reversed process $(\tilde{X}_t)$, where

$$\tilde{X}_t(\omega) \equiv \begin{cases} X_{L_a(\omega)-t}(\omega)\,, & \text{if } 0 < t < L_a(\omega)\,, \\ \partial & \text{if } L_a(\omega) \leq t \text{ or } L_a(\omega) = \infty\,, \end{cases} \tag{3.1}$$

where ∂ denotes the cemetary, and $\tilde{X}_0(\omega) = X_{L_a(\omega)}(\omega)$, if $0 < L_a(\omega) < \infty$, else $\tilde{X}_0(\omega) = \partial$. We remark that a diffusion cannot only be time reversed at a last exit time, but more generally at a cooptional time, see Nagasawa [22, 23], Revuz–Yor [30], Chapter VII.4. For our purposes it is reasonable to restrict ourselves to last exit times.

We will need the following (see also Itô–McKean [13], p. 149ff)

Lemma 2 *Let (p_t) be the transition density of a diffusion (X_t) with respect to the speed measure m. Then we have for all x, y the symmetry*

$$p_t(x, y) = p_t(y, x)\,.$$

Proof: Let A denote the infinitesimal generator of the diffusion and let the functions ψ_α and ϕ_α be the fundamental solutions of the generalized differential equation

$$Au = \alpha u, \quad \alpha > 0\,.$$

Call f^+ the right derivative of a function f with respect to the scale function s of the diffusion,

$$f^+(x) = \lim_{h\downarrow 0} \frac{f(x+h) - f(x)}{s(x+h) - s(x)}\,.$$

Then the so called Wronskian w_α is defined as

$$w_\alpha = \psi_\alpha^+(x)\phi_\alpha(x) - \psi_\alpha(x)\phi_\alpha^+(x)$$

and is independent of x. The function

$$g_\alpha(x,y) \equiv \int_0^\infty e^{-\alpha t} p_t(x,y)\,dt \tag{3.2}$$

is called the Green function and

$$g_\alpha(x,y) = \begin{cases} w_\alpha^{-1}\psi_\alpha(x)\phi_\alpha(y), & x \le y, \\ w_\alpha^{-1}\psi_\alpha(y)\phi_\alpha(x), & x \ge y. \end{cases} \tag{3.3}$$

We obtain $g_\alpha(x,y) = g_\alpha(y,x)$, and hence, from the injectivity of the Laplace transform the result follows. $\square$

Furthermore, some assumptions and notations are needed. Call (P_t) the semigroup of (X_t). The time reversed process $(\tilde{X}_t)$ is a Markov process (see Nagasawa [22] and [23], III.3.2, p. 66ff) and we call its semigroup $(\tilde{P}_t)$. The resolvent G_α of order α, $\alpha > 0$, of the semigroup (P_t), and its analog $\tilde{G}_\alpha$ for $(\tilde{P}_t)$, is

$$G_\alpha(f)(x) = \int_0^\infty e^{-\alpha t}\, P_t f(x)\,dt\,,$$

for all positive Borel functions f. The semigroup (P_t) has a transition density (p_t) with respect to the speed measure m: $P_t(x,dy) = p_t(x,y)\,m(dy)$ (see Itô–McKean [13], p. 149ff). We have

$$\begin{aligned} G_\alpha(f)(x) = \int_0^\infty e^{-\alpha t} P_t f(x)\,dt \;&=\; \int_0^\infty e^{-\alpha t} \left(\int p_t(x,y) f(y)\, m(dy) \right) dt \\ &=\; \int g_\alpha(x,y) f(y)\, m(dy), \end{aligned}$$

for all positive Borel functions f, with g_α in (3.2). G_0 is the potential kernel of X, and we assume that there is a probability measure μ such that the potential $\nu = \mu\, G_0$ is a Radon measure, and additionally, that the resolvents $\tilde{G}_\alpha$ and G_α are in duality with respect to ν, that is

$$\langle G_\alpha f, g\rangle_\nu = \langle f, \tilde{G}_\alpha g\rangle_\nu\,,$$

for all $\alpha > 0$ and for all positive Borel functions f and g. Then we have the duality between the semigroups (P_t) and $(\tilde{P}_t)$:

$$\langle P_t f, g\rangle_\nu = \langle f, \tilde{P}_t g\rangle_\nu \tag{3.4}$$

for all positive Borel functions f and g, see Nagasawa [23], §3.2, Revuz–Yor [30], Th. 4.5, p. 292.

From

$$\mu\, G_0(A) = \int G_0(x,A)\mu(dx) = \int_A \left(\int g_0(x,y)\mu(dx)\right) m(dy),$$

for every Borel set A, we obtain $\nu = \mu\, G_0 \ll m$. With the notation

$$h(y) \equiv \frac{d(\mu\, G_0)}{dm}\,(y) = \int g_0(x,y)\,\mu(dx) \tag{3.5}$$

for m-almost all y we have

$$\begin{aligned}
\langle P_t f, g\rangle_\nu &= \int \left(\int f(y)\, p_t(x,y)\, m(dy)\right) g(x)h(x)m(dx)\\
&= \int \left(\frac{1}{h(y)}\int g(x)h(x)\, p_t(x,y)m(dx)\right) f(y)h(y)\, m(dy)\\
&= \int \frac{1}{h(y)} P_t(gh)(y)\, f(y)h(y)\, m(dy)\,,
\end{aligned}$$

by Lemma 2, and from duality (3.4) we deduce

$$\tilde{P}_t g(x) = \frac{1}{h(x)} P_t(gh)(x) \tag{3.6}$$

for m-almost all x.

We state a time reversal result proven with the results above (see Sharpe [33], Getoor–Sharpe [9]), which was a useful tool in the previous chapters.

Theorem 2 *For X a transient diffusion, living on $\mathbb{R}_+$, starting at 0, $\tilde{X}$ the time reversed process (3.1), starting at a, and $T_0 \equiv \inf\{u|\tilde{X}_u = 0\}$ we have*

$$\{X_u,\ u \le L_a\} \overset{\text{(law)}}{=} \{\tilde{X}_{T_0-u},\ u \le T_0\}\,. \tag{3.7}$$

This implies that the law of L_a for the process X starting at 0 is identical to the law of T_0 for the process $\tilde{X}$ starting at a.

We want to derive a more explicit formula for h in (3.5). We know, see (1.22):

$$P_{x_0}(L_a \in dt) = -\frac{1}{2s(a)}\, p_t(x_0, a)\, dt\,, \tag{3.8}$$

where $X_0 = x_0$, and s is the scale function with $\lim_{y\downarrow 0} s(y) = -\infty$ and $s(\infty) = 0$. Consider the two cases $x_0 \le a$ and $x_0 > a$. If $x_0 \le a$, then

$$-\frac{1}{2\,s(a)} \int_0^\infty p_t(x_0, a)\, dt = 1\,,$$

hence $-2\,s(a) = g_0(x_0, a)$. If $x_0 > a$, then the law of L_a has mass at 0

$$P_{x_0}(L_a = 0) = P_{x_0}(T_a = \infty) = 1 - \frac{s(x_0)}{s(a)}\,,$$

hence,

$$P_{x_0}(0 < L_a) = -\frac{1}{2s(a)} \int_0^\infty p_t(x_0, a)\, dt = \frac{s(x_0)}{s(a)}\,,$$

and $-2\,s(x_0) = g_0(x_0, a)$. We conclude

$$g_0(x_0, a) = -2\,s(x_0 \vee a) \tag{3.9}$$

for $x_0, a \in \mathbb{R}^+$ arbitrary.

Now, assume $x_0 = 0$, that is, the transient process (X_t) starts at 0 and μ is the Dirac measure at 0. Then from (3.5) and (3.9) we obtain

$$h(a) = g_0(0, a) = -2\,s(a). \tag{3.10}$$

As we will see in the next section, the process $(\tilde{X}_t)$ is the Doob's h-transform of (X_t).

3.1 Doob's h-transform

Consider a one-dimensional diffusion X, with sample space $(I^{\partial,\infty}, \mathcal{F}^c_\infty)$, where $I \subseteq [-\infty, \infty]$, ∂ denotes the cemetary and $I^{\partial,\infty} := \{\omega : [0,\infty) \mapsto I \cup \{\partial\}\}$, $\mathcal{F}^c_\infty := \sigma\{\omega(t) \,|\, t \ge 0\}$.

Definition 5 *A non-negative measurable function $h : I \mapsto \mathbb{R} \cup \{\infty\}$ is called α-excessive, $\alpha \ge 0$, for X, if* *a)* $e^{-\alpha t} E_x(h(X_t)) \le h(x)$, *for all* $x \in I$, $t \ge 0$,

b) $e^{-\alpha t} E_x(h(X_t)) \to h(x)$, *for all* $x \in I$ *as* $t \downarrow 0$.

A 0-excessive function is simply called excessive.

Let h be an α-excessive function for a diffusion X. The life time of a path $\omega \in I^{\partial,\infty}$ is defined by $\zeta(\omega) := \inf\{t \,|\, \omega_t = \partial\}$. We construct a new probability measure P^h by

$$P_x^h\Big|_{\mathcal{F}_t} = e^{-\alpha t} \frac{h(\omega(t))}{h(x)} \, P_x|_{\mathcal{F}_t} \,, \tag{3.11}$$

for $t < \zeta$ and $x \in I$. The process under the new measure P^h is a regular diffusion and is called *Doob's h-transform* of X. Examples of α-excessive functions are ψ_α and ϕ_α, see Lemma 2, and $g_\alpha(x, \cdot)$, see (3.2). We remark, that all α-excessive functions h (except for $h \equiv 0$ or $h \equiv \infty$) can be expressed in terms of a linear combination of these three functions.

As an application we consider a Doob's h-transform (3.11) with the excessive function h in (3.10) for the transient diffusion X. As seen in (3.11) the factor -2 cancels, so we simply consider $h \equiv s$, with s the scale function of X. Let us denote the probability measure associated to X by $P^{(\uparrow)}$. We obtain:
The Doob's h-transform of the transient diffusion X with $h \equiv s$ the scale function of X, that is, the process under the new measure P^h, is a process which reaches 0 almost surely. For clarity we write $P^{(0)} \equiv P^h$. With this notation we have

$$P_x^{(\uparrow)}\Big|_{\mathcal{F}_t} = \frac{\left(\frac{1}{s}\right)(X_{t\wedge T_0})}{\left(\frac{1}{s}\right)(x)} \; P_x^{(0)}\Big|_{\mathcal{F}_t} \,. \tag{3.12}$$

Note that we obtain

$$\frac{s\,(X_t)}{s(x)} \; P_x^{(\uparrow)}\Big|_{\mathcal{F}_t} \equiv 1_{(t<T_0)} \, P_x^{(0)}\Big|_{\mathcal{F}_t} \,,$$

and hence,

$$Q_t^{(0)} f(x) \equiv \frac{1}{s(x)} \, E_x\left[(fs)(X_t)\right] \equiv \frac{1}{s(x)} P_t^{(\uparrow)}(fs)(x) \,, \tag{3.13}$$

is the semigroup of the process under $P^{(0)}$ killed when it reaches 0. In other words, we have the following time reversal result: the Doob's h-transform of the transient process under $P^{(\uparrow)}$ with $h = s$ is the process under $P^{(0)}$ killed at T_0; this is the process $\tilde{X}$ in our former notation.

From formula (3.12) we can obtain explicit formulae of the diffusion processes via Girsanov's theorem. Assume, a process (X_t) under $P_x^{(0)}$ has the form

$$X_t = x + \int_0^t b(X_s)\, ds + \int_0^t a(X_s)\, dB_s, \quad t \le T_0\,.$$

Via Girsanov's theorem we have with (3.12)

$$B_t = \int_0^t \frac{\left(\frac{1}{s}\right)'(X_u)}{\left(\frac{1}{s}\right)(X_u)}\, du + \hat{B}_t = \int_0^t - \left(\frac{s'}{s}\right)(X_u)\, du + \hat{B}_t\,,$$

where $(\hat{B}_t)$ is Brownian motion under $P_x^{(\uparrow)}$. Hence, the process under $P_x^{(\uparrow)}$ has the form

$$X_t = x + \int_0^t \left(b - \left(a\,\frac{s'}{s}\right)\right)(X_u)\, du + \int_0^t a(X_u)\, d\hat{B}_u\,.$$

Note, that if we assume, a process (X_t) under $P_x^{(\uparrow)}$ has the form

$$X_t = x + \int_0^t \beta(X_u)\, du + \int_0^t \alpha(X_u) d\hat{B}_u\,,$$

then analogously we obtain that the process under $P_x^{(0)}$ has the form

$$X_t = x + \int_0^t \left(\beta + \left(\alpha\,\frac{s'}{s}\right)\right)(X_u)\, du + \int_0^t \alpha(X_u)\, dB_u, \quad t \le T_0\,.$$

Let $A^{(0)}$ resp. $A^{(\uparrow)}$ denote the infinitesimal generator of the process under $P^{(0)}$ resp. $P^{(\uparrow)}$. If we want to avoid a Girsanov argument, we can compute the infinitesimal generator of the time-reversed process from (3.13)

$$A^{(0)}(f) = \frac{1}{h} A^{(\uparrow)}(hf)\,,$$

with h the scale function of $X^{(\uparrow)}$. The term $A(hf)$ can be written as

$$A(hf) = hA(f) + fA(h) + \Gamma(h,f)\,,$$

where Γ is the 'Opérateur Carré du Champ', see Kunita [17] and Revuz–Yor [30], p. 326. For instance, if the infinitesimal generator takes the form $A(f) = \frac{1}{2}f'' + bf'$, then $\Gamma(f,g) = f'g'$ for $f, g \in C^2$.

3.2 Checking a time reversal theorem in Elworthy–Li–Yor

As an application of the previously stated results, we want to check Theorem 3.7 in Elworthy–Li–Yor [6]. Let ${}^{-\lambda}P^{\delta}_x$ denote the law of a δ-dimensional radial Ornstein–Uhlenbeck process with parameter $-\lambda$, starting in x. We have the following time reversal result:

Theorem (Elworthy–Li–Yor, [6]): *Let $\lambda \geq 0$ and $\delta > 2$. For every bounded measurable function F on path space,*

$$ {}^{\lambda}E^{\delta}_0[F(Y_{L_a-t},\, t \leq L_a)] = \frac{1}{g(a)}\; {}^{-\lambda}E^{(4-\delta)}_a[F(Y_t,\, t \leq T_0)\, e^{-2\lambda T_0}]\,, \tag{3.14} $$

where $g(a) = {}^{-\lambda}E^{(4-\delta)}_a(e^{-2\lambda T_0})$.

Let (R_t) be a radial Ornstein–Uhlenbeck process with law ${}^{\lambda}P^{\delta}_0$ and denote by σ its scale function and by A its infinitesimal generator. We know that the infinitesimal generator of the time-reversed process is

$$ \begin{aligned} \frac{1}{\sigma}A(\sigma f) &= \frac{1}{\sigma}\left[\sigma A(f) + f A(\sigma) + \Gamma(\sigma, f)\right] \\ &= A(f) + \frac{1}{\sigma}\Gamma(\sigma, f) = A(f) + \frac{\sigma'}{\sigma} f'\,. \end{aligned} \tag{3.15} $$

Let us check whether this is coherent with Theorem 3.7. From

$$ \begin{aligned} {}^{\lambda}E^{\delta}_0[F(Y_{L_a-t},\, t \leq L_a)] &= \frac{1}{g(a)}\; {}^{-\lambda}E^{(4-\delta)}_a[F(Y_t,\, t \leq T_0)\, e^{-2\lambda T_0}] \\ &\equiv \hat{E}_a[F(\hat{Y}_t,\, t \leq \hat{T}_0)] \end{aligned} $$

we obtain

$$ \hat{P}_a\Big|_{\mathcal{F}_t} = D_t \cdot\; {}^{-\lambda}P^{(4-\delta)}_a\Big|_{\mathcal{F}_t}\,, \tag{3.16} $$

with

$$ D_t \equiv \frac{g(Y_{t\wedge T_0})}{g(a)}\, e^{-2\lambda(t\wedge T_0)}\,. $$

Writing the process $(Y_t,\, t \leq T_0)$ simply as

$$ Y_t = \int_0^t b(Y_s)ds + B_t\,, $$

where B is Brownian motion under ${}^{-\lambda}P^{(4-\delta)}_a$, then we obtain by Girsanov's theorem

$$ Y_t = \int_0^t \left(b + \frac{g'}{g}\right)(Y_s)\, ds + \hat{B}_t\,, $$

where $\hat{B}$ is Brownian motion under $\hat{P}_a$. Let A^* denote the infinitesimal generator of the process (Y_t) under ${}^{-\lambda}P_a^{(4-\delta)}$. We have for the new infinitesimal generator $\hat{A}$ of the process $(\hat{Y}_t)$ under $\hat{P}_a$:

$$\hat{A}(f) = A^*(f) + \frac{g'}{g}f' .$$

Now, all we have to show is (see (3.15))

$$A(f) + \frac{\sigma'}{\sigma}f' = A^*(f) + \frac{g'}{g}f' ,$$

that is, we have to identify the drift coefficients:

$$\frac{\delta - 1}{2r} + \lambda r + \frac{\sigma'}{\sigma} = \frac{3-\delta}{2r} - \lambda r + \frac{g'}{g} . \tag{3.17}$$

The problem is that we do not know g explicitly. But since D_t is a martingale, the following two lemmas will help us.

Lemma 3 *$g(Y_t)\, e^{-2\lambda t}$ is a ${}^{-\lambda}P_a^{(4-\delta)}$ martingale iff $g \in D(A)$ and $A(g) = 2\lambda g$.*

Proof: If $g(Y_t)e^{-2\lambda t}$ is a martingale, then $P_h g(Y_t)e^{-2\lambda(t+h)} - g(Y_t)e^{-2\lambda t} = 0$ for all $h > 0$, hence $e^{-2\lambda t}P_t g(x) = g(x)$ with $Y_0 = x$ and we have $\frac{d}{dt}P_t g(x)|_{t=0} = 2\lambda g(x)$, that is $g \in D(A)$.

For $g \in D(A)$, the process

$$M_t^g = g(Y_t) - g(Y_0) - \int_0^t Ag(Y_s)ds$$

is a martingale, see Revuz–Yor [30] Prop.7.1.6. By Itô's lemma we obtain

$$g(Y_t)e^{-2\lambda t} = \int_0^t e^{-2\lambda s} d(g(Y_s)) - 2\lambda \int_0^t e^{-2\lambda s} g(Y_s)ds .$$

With the notation: $U \sim V \Leftrightarrow U_t - V_t$ is a martingale, we obtain:

$$g(Y_t)e^{-2\lambda t} \sim \int_0^t e^{-2\lambda s}(Ag)(Y_s)ds - 2\lambda \int_0^t e^{-2\lambda s} g(Y_s)ds ,$$

and the claim follows. □

Lemma 4 *$(\frac{g}{\sigma})(Y_t)e^{-2\lambda t}$ is a martingale under ${}^{-\lambda}P_a^{(4-\delta)}$.*

Proof: Consider the Doob's h-transform of the process under ${}^{\lambda}P_0^{\delta}$ with $h = \sigma$ its scale function, or put differently (see (3.12))

$$ {}^{\lambda}P_0^{\delta}\Big|_{\mathcal{F}_t} = M_t \cdot \hat{P}_a\Big|_{\mathcal{F}_t} \, , $$

where

$$ M_t \equiv \frac{\left(\frac{1}{\sigma}\right)(\hat{Y}_{t\wedge T_0})}{\left(\frac{1}{\sigma}\right)(a)} $$

is a martingale (with respect to $\hat{P}_a$). Together with (3.16) we know, M_t is a martingale under $\hat{P}_a$ if and only if $M_t D_t$ is a martingale under ${}^{-\lambda}P_a^{(4-\delta)}$, that is $((\frac{1}{\sigma})g)(Y_t)e^{-2\lambda t}$ is a martingale under ${}^{-\lambda}P_a^{(4-\delta)}$. □

We deduce $A^*(g) = 2\lambda g$ and $A^*(\frac{g}{\sigma}) = 2\lambda(\frac{g}{\sigma})$. From

$$ A^*\left(\frac{g}{\sigma}\right) = gA^*\left(\frac{1}{\sigma}\right) + \frac{1}{\sigma}A^*(g) - \frac{\sigma'}{\sigma^2}g' $$

we obtain with $\frac{1}{\sigma}A^*(g) = 2\lambda\frac{g}{\sigma} = A^*(\frac{g}{\sigma})$

$$ gA^*\left(\frac{1}{\sigma}\right) = \frac{\sigma'}{\sigma^2}g' \, . \tag{3.18} $$

By Itô's lemma we have

$$ A(f(u)) = \frac{1}{2}f''(u)(u')^2 + f'(u)A(u) $$

and hence

$$ A^*\left(\frac{1}{\sigma}\right) = \frac{(\sigma')^2}{\sigma^3} - \frac{1}{\sigma^2}A^*(\sigma) \, . $$

Together with (3.18) we have

$$ \frac{\sigma'}{\sigma} - \frac{1}{\sigma'}A^*(\sigma) = \frac{g'}{g} \, , $$

from which we obtain

$$ g(r) = c\,\frac{\sigma(r)}{\sqrt{\sigma'(r)}}\,r^{-\frac{3-\delta}{2}}e^{\frac{\lambda}{2}r^2} \, , $$

with a constant $c > 0$. Our aim is to derive (3.17), that is,

$$ \frac{A^*(\sigma)}{\sigma'} = \frac{3-\delta}{2r} - \frac{\delta-1}{2r} - 2\lambda r \, , $$

and since $A^*(\sigma) = (\frac{3-\delta}{2r} - \lambda r)\sigma' + \frac{1}{2}\sigma''$, it remains to show

$$\frac{1}{2}\frac{\sigma''}{\sigma'} = -\frac{\delta-1}{2r} - \lambda r\,. \tag{3.19}$$

Remember, σ is the scale function of (R_t). Let b^R denote the drift of (R_t)

$$b^R(r) = \frac{\delta-1}{2r} + \lambda r\,,$$

then $b^R\sigma' + \frac{1}{2}\sigma'' = 0$ and hence (3.19).

3.3 The three-parameters-family of processes with law ${}^{\lambda}P^{\delta,\mu}$

Based on Theorem 3.7 in Elworthy–Li–Yor [6] (see here p. 28) we introduce a three-parameters-family of processes. From (3.14) we deduce

$${}^{\lambda}E_0^{\delta}[F(Y_{L_a-t},\, t\le L_a)\, e^{-\frac{\mu^2}{2}L_a}] = \frac{1}{g(a)}\; {}^{-\lambda}E_a^{(4-\delta)}[F(Y_t,\, t\le T_0)\, e^{-(2\lambda+\frac{\mu^2}{2})T_0}]\,,$$

where $\lambda\ge 0$, $\delta>2$, $\mu\ge 0$ and $g(a) = {}^{-\lambda}E_a^{(4-\delta)}(e^{-2\lambda T_0})$, and with the notation $2\theta\equiv 2\lambda+\frac{\mu^2}{2}$ we have

$$\frac{1}{g_\theta(a)}\; {}^{-\lambda}E_a^{(4-\delta)}[F(Y_t,\, t\le T_0)\, e^{-2\theta T_0}] = \frac{g(a)}{g_\theta(a)}\; {}^{\lambda}E_0^{\delta}[F(Y_{L_a-t},\, t\le L_a)\, e^{-\frac{\mu^2}{2}L_a}]\,,$$

where $g_\theta(a) = {}^{-\lambda}E_a^{(4-\delta)}(e^{-2\theta T_0})$. This motivates us to define a three-parameters-family of diffusion processes with laws ${}^{\lambda}P_0^{\delta,\mu}$ as

$${}^{\lambda}P_0^{\delta,\mu}\Big|_{\mathcal{F}_t} = \frac{\psi(Y_t)}{\psi(0)}\, e^{-\frac{\mu^2}{2}t}\cdot {}^{\lambda}P_0^{\delta}\Big|_{\mathcal{F}_t}\,,$$

or more generally for arbitrary $x\ge 0$

$${}^{\lambda}P_x^{\delta,\mu}\Big|_{\mathcal{F}_t} = \frac{\psi(Y_t)}{\psi(x)}\, e^{-\frac{\mu^2}{2}t}\cdot {}^{\lambda}P_x^{\delta}\Big|_{\mathcal{F}_t}$$

for some increasing function ψ and $A\psi = \frac{\mu^2}{2}\psi$, $\psi\in C^2$, where A is the infinitesimal generator of ${}^{\lambda}P_x^{\delta}$. Indeed, with Itô's formula we see that $\frac{\psi(Y_t)}{\psi(x)}\, e^{-\frac{\mu^2}{2}t}$ is a martingale with respect to ${}^{\lambda}P_x^{\delta}$ and $(\mathcal{F}_t)$.

First, let us consider the case $\lambda = 0$ where we simply write $P_x^{\delta,\mu}$ and P_x^δ instead of ${}^0P_x^{\delta,\mu}$ and ${}^0P_x^\delta$,

$$P_x^{\delta,\mu}\Big|_{\mathcal{F}_t} = \frac{\psi(Y_t)}{\psi(x)}\, e^{-\frac{\mu^2}{2}t} \cdot P_x^\delta\Big|_{\mathcal{F}_t} \,. \tag{3.20}$$

By means of the optional stopping theorem (see Revuz–Yor [30] § 2.3, p. 65) we obtain

$$\frac{\psi(a)}{\psi(x)} E_x^\delta[e^{-\frac{\mu^2}{2}T_a}] = 1\,. \tag{3.21}$$

For $\nu = \frac{\delta}{2} - 1 > -1$, $\infty > a > x > 0$ and $\mu > 0$ we know (see e.g. Kent [16] Th. 3.1, Pitman–Yor [27] Prop. 2.3)

$$E_x^\delta[e^{-\frac{\mu^2}{2}T_a}] = \left(\frac{a}{x}\right)^\nu \frac{I_\nu(\mu x)}{I_\nu(\mu a)}\,,$$

thus we have

$$\psi(a) = \frac{I_\nu(\mu a)}{a^\nu}\,, \tag{3.22}$$

and

$$\psi(0) = \lim_{\varepsilon \downarrow 0} \frac{I_\nu(\mu\varepsilon)}{\varepsilon^\nu} = \left(\frac{\mu}{2}\right)^\nu \Gamma^{-1}(\nu+1)\,.$$

$(P_x^{\delta,\mu})$ is the family of laws of a diffusion with infinitesimal generator

$$\frac{1}{2}D^2 + \left(\frac{\psi'}{\psi} + \frac{\delta-1}{2y}\right) D$$

Watanabe [36] calls a diffusion process determined by such an infinitesimal generator a *Bessel diffusion process in the wide sense with index* $(\delta, \frac{\mu^2}{2})$. These diffusions arise as the Euclidian norm of a δ-dimensional Brownian motion with vector drift starting in zero. Denoting the norm of δ-dimensional Brownian motion with drift starting *at a nonzero vector* by R_t, we remark that R_t is not a Markov process, whereas $(R_t, \int_0^t \frac{ds}{R_s^2})$ has the Markovian property (see Pitman–Yor [27, 28] or Rogers–Pitman [32]). From (3.22) we have

$$\frac{\psi'}{\psi}(y) = \mu \frac{I_{\nu+1}(\mu y)}{I_\nu(\mu y)}\,. \tag{3.23}$$

For example, in the case $\delta = 3$ we obtain by (3.22)

$$\frac{\psi(a)}{\psi(x)} = \frac{\sinh(\mu a)\, x}{a\, \sinh(\mu x)} \quad \text{and} \quad \frac{\psi(a)}{\psi(0)} = \frac{\sinh(\mu a)}{\mu a}\,. \tag{3.24}$$

Hence, $(P^{3,\mu})$ is the family of laws of a diffusion with infinitesimal generator

$$\frac{1}{2}D^2 + \frac{1}{y}D + (\mu \coth(\mu y) - \frac{1}{y})D = \frac{1}{2}D^2 + \mu \coth(\mu y) D\,.$$

This diffusion is the radial part of hyperbolic Brownian motion in dimension 3 (see Rogers–Pitman [32], Gruet [11]). As will be discussed below, see p. 34, the reversed process at L_a is Brownian motion with drift $(-\mu)$ killed the first time it reaches zero.

Furthermore, we know, see Yor [40] p. 55,

Theorem 3 *Consider two transient diffusions with laws P_x and Q_x such that*

$$Q_x|_{\mathcal{F}_t} = D_t \cdot P_x|_{\mathcal{F}_t}\ .$$

Then

$$Q_x|_{\mathcal{F}_{L_a}} = (h(a)D_{L_a}) \cdot P_x|_{\mathcal{F}_{L_a}}\ ,$$

where $h(a) = \frac{\sigma'(a)}{\sigma(a)}\frac{s(a)}{s'(a)}\frac{\alpha(a)}{\beta(a)}$; σ and s denote the respective scale functions for Q_x and P_x and α and β are the respective diffusion coefficients for Q_x and P_x.

Continuing our discussion we apply Theorem 3 to (3.20). With

$$D_t \equiv \frac{\psi(Y_t)}{\psi(x)}\, e^{-\frac{\mu^2}{2}t}$$

we deduce from

$$h(a)\frac{\psi(a)}{\psi(x)}E_x^\delta(e^{-\frac{\mu^2}{2}L_a}) = 1 \tag{3.25}$$

the equality

$$E_x^{\delta,\mu}[F(Y_{L_a-t},\, t \le L_a)] = (E_x^\delta(e^{-\frac{\mu^2}{2}L_a}))^{-1}\, E_x^\delta[F(Y_{L_a-t},\, t \le L_a)\, e^{-\frac{\mu^2}{2}L_a}]\,,$$

where for $\delta > 2$ arbitrary and for $x > 0$

$$E_x^\delta(e^{-\frac{\mu^2}{2}L_a}) = 2\nu\left(\frac{a}{x}\right)^\nu I_\nu(\mu(x\wedge a))K_\nu(\mu(x \vee a))\,,$$

see e.g. Pitman–Yor [27], p. 330, (7.ẽ), and where (see (1.9))

$$E_0^\delta(e^{-\frac{\mu^2}{2}L_a}) = \frac{(\mu a)^\nu K_\nu(\mu\, a)}{2^{\nu-1}\Gamma(\nu)}\,.$$

Note that by means of (3.25) we obtain for $x = a$

$$E_a^\delta(e^{-\frac{\mu^2}{2}L_a}) = (h(a))^{-1}\,.$$

For instance, for the case $\delta = 3$ and $x = 0$ we have

$$D_t \equiv \left(\frac{\sinh(\mu Y_t)}{\mu Y_t}\right) e^{-\frac{\mu^2}{2}t}\,,$$

and obtain

$$E_0^{3,\mu}[F(Y_{L_a-t},\, t \le L_a)] = e^{\mu a}\, E_0^3[F(Y_{L_a-t},\, t \le L_a)\, e^{-\frac{\mu^2}{2}L_a}]\,,$$

since, see also (1.11),

$$E_0^3(e^{-\frac{\mu^2}{2}L_a}) = e^{-\mu a}\,.$$

Now let us look at the diffusion process with law $P_0^{3,\mu}$ time-reversed at L_a. As we remarked above (see p. 33) it will turn out that this is Brownian motion with drift. Let P_x^0 denote the law of Brownian motion and P_x^3 the law of a 3-dimensional Bessel process starting in $x > 0$. From Doob's h-transform (3.12) we know

$$P_x^3\Big|_{\mathcal{F}_t} = \frac{X_{t\wedge T_0}}{x} \cdot P_x^0\Big|_{\mathcal{F}_t}\,,$$

and together with (3.20) and (3.24) we have

$$P_x^{3,\mu}\Big|_{\mathcal{F}_t} = \frac{\sinh(\mu X_{t\wedge T_0})}{\sinh(\mu x)}\, e^{-\frac{\mu^2}{2}(t\wedge T_0)} \cdot P_x^0\Big|_{\mathcal{F}_t}\,. \tag{3.26}$$

Let $P_x^{(0)(\nu)}$ denote the law of Brownian motion with drift $\nu < 0$. Its scale function s is

$$s(x) = \frac{1}{2\nu}(1 - e^{-2\nu x})\,.$$

Via Doob's h-transform (3.12) we have

$$P_x^{(\uparrow)(\nu)}\Big|_{\mathcal{F}_t} = \frac{e^{-2\nu X_{t\wedge T_0}} - 1}{e^{-2\nu x} - 1} \cdot P_x^{(0)(\nu)}\Big|_{\mathcal{F}_t}\,.$$

The Doob's h-transform (3.11) of Brownian motion with $h(x,t;\nu) \equiv e^{\nu x - \frac{\nu^2}{2}t}$ is identical in law to Brownian motion with drift ν, see e.g. Revuz–Yor [30], VIII.3, p. 327, or Borodin–Salminen [2], IV.28, that is

$$P_x^{(0)(\nu)}\Big|_{\mathcal{F}_t} = \frac{e^{\nu X_{t\wedge T_0} - \frac{\nu^2}{2}(t\wedge T_0)}}{e^{\nu x}} \cdot P_x^{(0)}\Big|_{\mathcal{F}_t}\,. \tag{3.27}$$

And finally, since

$$\frac{\sinh(\mu X_{t\wedge T_0})}{\sinh(\mu x)}\, e^{-\frac{\mu^2}{2}(t\wedge T_0)} = \frac{e^{2\mu X_{t\wedge T_0}}-1}{e^{2\mu x}-1}\, \frac{e^{-\mu X_{t\wedge T_0}-\frac{\mu^2}{2}(t\wedge T_0)}}{e^{-\mu x}}\,,$$

we obtain from (3.26) and from (3.27) with $\nu = -\mu$

$$P_x^{3,\mu}\Big|_{\mathcal{F}_t} = \frac{e^{2\mu X_{t\wedge T_0}}-1}{e^{2\mu x}-1} \cdot P_x^{(0)(-\mu)}\Big|_{\mathcal{F}_t}\,,$$

this means, a diffusion process with law $P^{3,\mu}$ time-reversed is a Brownian motion with drift $(-\mu)$ killed the first time it reaches zero.

So far, we treated the case $\lambda = 0$ in full generality and as an application looked at the case $\delta = 3$. Now let us consider the general case $\lambda \geq 0$, i.e. we want to determine a ${}^{\lambda}P_x^{\delta}$-martingale D_t with

$${}^{\lambda}P_x^{\delta,\mu}\Big|_{\mathcal{F}_t} = D_t \cdot {}^{\lambda}P_x^{\delta}\Big|_{\mathcal{F}_t}\,.$$

We know for $\lambda = 0$ (see (3.20))

$$P_x^{\delta,\mu}\Big|_{\mathcal{F}_t} = \frac{\psi(Y_t)}{\psi(x)}\, e^{-\frac{\mu^2}{2}t} \cdot P_x^{\delta}\Big|_{\mathcal{F}_t}\,, \tag{3.28}$$

with ψ in (3.22) and via Girsanov's theorem we know

$${}^{\lambda}P_x^{\delta}\Big|_{\mathcal{F}_t} = \exp\left\{\int_0^t \lambda Y_s dW_s - \frac{\lambda^2}{2}\int_0^t Y_s^2 ds\right\} \cdot P_x^{\delta}\Big|_{\mathcal{F}_t}\,. \tag{3.29}$$

In order to obtain D_t, first we write the P_x^{δ}-martingale $\frac{\psi(Y_t)}{\psi(x)}\, e^{-\frac{\mu^2}{2}t}$ in (3.28) in a different form. We will tackle this problem more generally.

Let us assume, that H_t is a continuous, strictly positive semimartingale, and A_t is a continuous process with bounded variation such that $H_t e^{-A_t}$ is a local martingale. We will write $H_t e^{-A_t}$ in a different way. The continuous semimartingale H_t can be decomposed uniquely

$$H_t = m_t + a_t\,, \tag{3.30}$$

where m is a continuous local martingale and a a continuous adapted process of finite variation. With Itô's lemma we obtain

$$\begin{aligned} H_t e^{-A_t} &= \exp\{\ln H_t - A_t\} \\ &= \exp\left\{\int_0^t \frac{dH_s}{H_s} - \frac{1}{2}\int_0^t \frac{d\langle H,H\rangle_s}{H_s^2} - A_t\right\}\,, \end{aligned}$$

where the process $\langle H, H\rangle$ is the quadratic variation of H. With (3.30) and since $\langle H, H\rangle = \langle m, m\rangle$ we obtain

$$H_t e^{-A_t} = \exp\left\{\int_0^t \frac{dm_s}{H_s} - \frac{1}{2}\int_0^t \frac{d\langle m, m\rangle_s}{H_s^2} + \int_0^t \frac{da_s}{H_s} - A_t\right\} .$$

Since $H_t e^{-A_t}$ is a martingale, or equivalently

$$\int_0^t e^{-A_s} da_s - \int_0^t \frac{dA_s}{ds} e^{-A_s} H_s ds = 0$$

(see also the proof of Lemma 3), finally we have

$$H_t e^{-A_t} = \exp\left\{\int_0^t \frac{dm_s}{H_s} - \frac{1}{2}\int_0^t \frac{d\langle m, m\rangle_s}{H_s^2}\right\} . \tag{3.31}$$

Denoting $M_t \equiv \int_0^t \frac{dm_s}{H_s}$, equation (3.31) has the simple form

$$H_t e^{-A_t} = \exp\left\{M_t - \frac{1}{2}\langle M, M\rangle_t\right\} . \tag{3.32}$$

For example in the case $\delta = 3$, $x = 0$, we obtain

$$\begin{aligned} \frac{\sinh(\mu Y_t)}{\mu Y_t} e^{-\frac{\mu^2}{2}t} = \exp\Big\{ & \int_0^t (\mu \coth(\mu Y_s) - \frac{1}{Y_s}) dW_s \\ & - \frac{1}{2}\int_0^t (\mu \coth(\mu Y_s) - \frac{1}{Y_s})^2 ds\Big\} . \end{aligned}$$

With the notation $N_t \equiv \int_0^t \lambda Y_s dW_s$, equation (3.29) can be written as

$${}^{\lambda}P_x^{\delta}\Big|_{\mathcal{F}_t} = \exp\left\{N_t - \frac{1}{2}\langle N, N\rangle_t\right\} \cdot P_x^{\delta}\Big|_{\mathcal{F}_t} ,$$

and hence, together with (3.32) we have

$$D_t = \exp\left\{M_t + N_t - \frac{1}{2}\langle M + N, M + N\rangle_t\right\} .$$

In the case $\delta = 3$, $x = 0$, we obtain

$$\begin{aligned} D_t = \exp\Big\{ & \int_0^t (\mu \coth(\mu Y_s) - \frac{1}{Y_s} + \lambda Y_s) dW_s \\ & - \frac{1}{2}\int_0^t (\mu \coth(\mu Y_s) - \frac{1}{Y_s} + \lambda Y_s)^2 ds\Big\} , \end{aligned}$$

and the infinitesimal generator of the diffusion process under ${}^{\lambda}P_x^{3,\mu}$ is

$$\frac{1}{2}D^2 + (\mu \coth(\mu y) + \lambda y)\, D\,.$$

For arbitrary $\delta > 0$ analogous calculations together with (3.23) lead to

$$D_t = \exp\left\{ \int_0^t (\mu \frac{I_{\nu+1}(\mu Y_s)}{I_\nu(\mu Y_s)} + \lambda Y_s)\, dW_s - \frac{1}{2}\int_0^t (\mu \frac{I_{\nu+1}(\mu Y_s)}{I_\nu(\mu Y_s)} + \lambda Y_s)^2\, ds \right\}\,,$$

and the infinitesimal generator of the diffusion process under ${}^{\lambda}P_x^{\delta,\mu}$ is

$$\frac{1}{2}D^2 + \left(\mu \frac{I_{\nu+1}(\mu y)}{I_\nu(\mu y)} + \frac{\delta - 1}{2y} + \lambda y \right) D\,,$$

and hence, we described the three-parameters-family of diffusion processes with law ${}^{\lambda}P^{\delta,\mu}$ in full generality. We remark that this family of diffusion processes is closely related with Ornstein–Uhlenbeck processes built from Brownian motion with drift.

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