Value At Risk
and
Maximum Loss Optimization

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Abstract:

A very condensed overview of risk measurement methods is given and the different techniques are classified. The risk measure "Value At Risk" (VAR) is presented from a new point of view and a general definition of VAR is derived. Next, "Maximum Loss" (ML) is formulated as a mathematical optimization problem and its modelling is described.
The techniques for calculating ML for linear and quadratic risk profiles are presented. Some theoretical relations between VAR and ML are demonstrated: ML is presented as a general framework including Delta–Normal VAR as well as Wilson's Delta–Gamma approach. It is also proven that ML is a worst case measure which is always more conservative than VAR.

Keywords: Risk Measurement — Value At Risk — Maximum Loss Optimization

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Chapter 1

Introduction

1.1 Problem Statement

One of the key issues of risk management is the measurement of market risk: What is the chance of loss in the portfolio if the market rates move in an adverse direction?

Mathematically, the setup of this problem can be formulated as follows: The risk factors $\omega_1, \ldots, \omega_M$ are translated market rates such that $\omega_i = 0$ corresponds to the actual value of market rate i. It is assumed that $\omega = (\omega_1, \ldots, \omega_M)^T$ are stochastic variates of the random space Ω, a connected set of $\mathbb{R}^M$. The joint density of ω is denoted by $f_t(\omega)$, where t is the holding period of the portfolio (i.e., the time required to liquidate the portfolio).

The *change* in portfolio value — called "profit and loss" (P&L) — is denoted by $v(\omega)$; the above definitions imply that $v(0) = 0$.

1.2 Overview of Risk Measurement Techniques

Two fundamentally different types of risk risk measurement techniques can be distinguished:

Correlation Based Methods

These methods consider correlations between individual risk factors and produce therefore a statistical netting effect (also called "aggregation"). Table 1.1 shows a classification of the common correlation based methods.

- If market rate innovations (e.g. absolute or relative differences of market rates at two consecutive points in time) are not stable, the risk factors cannot be modelled parametrically and scenario based simulation (e.g. using historical scenarios) has to be applied.

	portfolio positions: all linear	**portfolio positions: not all linear**
market rates: stable innovations stochastic model: parametric	Delta–Normal VAR	Delta–Gamma VAR Monte–Carlo Simulation Maximum Loss Optimization
market rates: non stable innovations	—	Scenario Based Simulation

Table 1.1: Classification of correlation based risk measurement techniques

- If the portfolio consists entirely of linear instruments and if the market rates are normally distributed, then the Delta–Normal VAR ("RiskMetrics method", described in [RiskMetrics]) is appropriate (cf. chapter 2.2).
- Several approaches exist for handling Gamma–risk in VAR calculations (cf. chapter 2.2): Wilson's *closed form* Delta–Gamma method (cf. [Wilson2]), as well as some proprietary techniques, first "linearizes" nonlinear instruments by incorporating the convexity into the linear coefficient, and then calculates standard Delta–Normal VAR. The approach of [Schaefer] uses a combination of chi–square distributions for estimating VAR.
- Parametric Monte–Carlo simulation can be used for estimating VAR of nonlinear portfolios. Unfortunately, simulation techniques are very time demanding. However, [Frauendorfer and Königsperger] suggest so–called "downside–approximations", which promise considerable accelerations.
- Maximum Loss Optimization (cf. chapter 3) comprises non–simulation based approaches to calculate nonlinear risks. One known representative of this category is Wilson's *numerical* Delta–Gamma method (cf. [Wilson1]).

Non Correlation Based Methods

These methods ignore statistical correlations of risk factors and show therefore no aggregation effect. Usually, these methods investigate the effects of predefined scenarios:

- Factor Push method
- Stress Testing (cf. Appendix D)

Chapter 2

Value At Risk

2.1 Concept of VAR

"Value At Risk" (cf. [Beckström and Campbell]) is defined as:

> The expected loss of a portfolio that will occur
> • with probability α,
> • over some time interval t.

Mathematically, VAR is the $(1-\alpha)$–quantile of the P&L distribution, i.e., it satisfies the relation:

$$\Pr(v(\omega) \leq \mathrm{VAR}) = 1 - \alpha, \tag{2.1}$$

where we assume that the P&L distribution is a continuous and strictly monotone function. A more general definition of VAR is given in Appendix B. The choice of α usually neglects the distribution in the tails; this aspect is discussed in [Embrechts]. In particular, an important question remains what value of α is a suitable confidence level.

2.2 Delta–Normal VAR

The most common method for calculating VAR is the so called Delta–Normal method, used by RiskMetrics (cf. [RiskMetrics]). It is based on the following assumptions (cf. Appendix A):

1. The change in portfolio–value depends linearly on the risk factors, i.e., $v(\omega) = \sum_{i=1}^{M} \delta_i \omega_i$.

2. The risk factors are multinormal variates with mean 0 and covariance matrix Σ_t.

These assumptions imply that P&L is normally distributed with mean 0 and variance $\delta^T \Sigma_t \delta$, where $\delta = (\delta_1, \ldots, \delta_M)^T$. Hence, VAR is just a quantile of a normal distributed variate (cf. figure 2.1):

$$\text{VAR} = -z_\alpha \sqrt{\delta^T \Sigma_t \delta}, \tag{2.2}$$

where, e.g., $z_\alpha = 1.64$ for $\alpha = 95\%$.

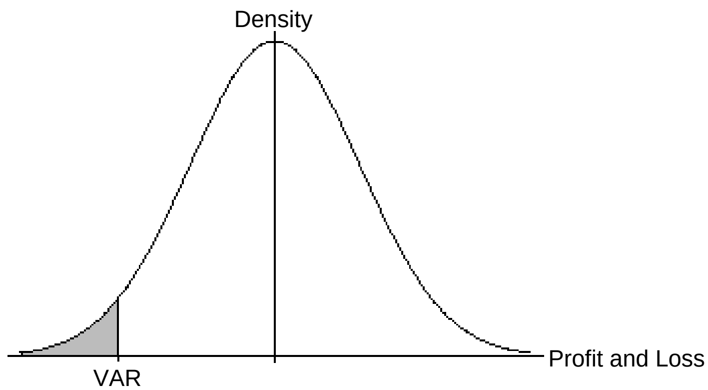

Figure 2.1: VAR as a quantile of the P&L distribution

For nonlinear portfolios, δ_i is the local sensitivity of the portfolio with respect to ω_i: $\delta_i = \frac{\partial v(\omega)}{\partial \omega_i}$. However, this approximation can lead to errors which may become dramatical for portfolios with high convexities.

Delta–Gamma Techniques

There exist several variants of the Delta–Normal method, which try to capture the nonlinearity of portfolios by incorporating the convexity into δ, e.g.:

$$\delta_i = \frac{\partial v(\omega)}{\partial \omega_i} + \kappa \frac{\partial^2 v(\omega)}{\partial \omega_i^2}, \tag{2.3}$$

where κ is a weight factor for convexity, whose value is based on experience. A similar, but mathematically more grounded idea is used in Wilson's closed form Delta–Gamma method (cf. [Wilson1], [Wilson2]).
The combination of delta and gamma risk described in [Schaefer] is based on the following idea: The P&L function is approximated by a second order polynomial: $v(\omega) = \frac{1}{2}\omega^T, \omega + \delta^T \omega$. After completion of the squares, this can be rewritten as

$$v(\omega) = \frac{1}{2}(e + \omega)^T, (e + \omega) + f, \tag{2.4}$$

where $e = \ ,\ ^{-1}\delta$ and $f = -\frac{1}{2}\delta^T,\ ^{-1}\delta$. The vector $(e + \omega)$ represents a normally distributed variate with mean e and variance Σ_t. Schaefer observes that for positive definite , the quantity $(e + \omega)^T,\ (e + \omega)$ is a weighted sum of independent non–central chi–squared variables, whose distribution can be calculated numerically.

Chapter 3

Maximum Loss Optimization

3.1 Definition of ML

Maximum Loss (ML) is defined as:

> The maximum loss
> - over a given trust region A_t of risk factors (A_t will be assumed a closed set with confidence level $\Pr(\omega \mid \omega \in A_t) = \alpha$)
> - for some holding period t.

This definition looks similar to the VAR definition of chapter 2.1. However, there is one important difference: Whereas for calculating VAR the *distribution* of P&L has to be known, ML is defined directly in the risk factor space Ω. The mathematical definition of Maximum Loss is:

$$\begin{aligned} \mathrm{ML} \;=\; & \min \quad v(\omega) \\ & \text{s.t.} \quad \omega \in A_t; \quad \text{where } \Pr(A_t) = \alpha, \end{aligned} \tag{3.1}$$

In contrast to VAR, which depends on the holding period t and the confidence level α, ML has a supplementary degree of freedom, called "trust region" A_t: any closed set in the risk factor space Ω with probability α is a valid trust region (cf. figure 3.1). Maximum Loss Optimization determines the worst case over such a trust region A_t.

3.2 Modelling Maximum Loss

What are the components of the Maximum Loss approach and how can they be modelled? This section gives some ideas on how the general ML framework might be applied (cf. [Studer1]).

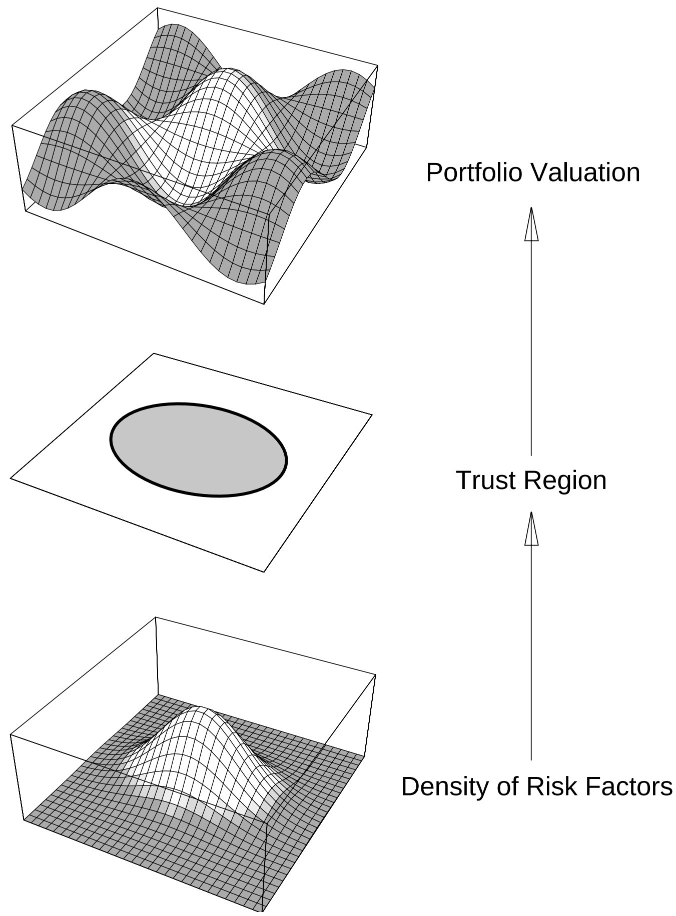

Figure 3.1: Modelling Maximum Loss

Risk Factors

In principle, risk factors are the parameters of the valuation models, namely:

- equity indices,
- commodity prices (spot and futures),
- foreign exchange rates (relative to some home currency, e.g., $\frac{\mathrm{CHF}}{\mathrm{DEM}}, \frac{\mathrm{CHF}}{\mathrm{FRF}}, \ldots$),
- interest rate curves for different currencies,
- expected future volatilities of all the above risk factors.

Many of these variables are elements of J.P. Morgan's RiskMetrics data set (cf. [RiskMetrics]).
There exist various ways to model changes in interest rate curves (cf. [Duffie]):

- the simplest models divide the maturity axis into a set of time buckets. Then, risk factors represent changes of the interest rate at well defined maturity vertices.
- changes of the curve can be represented with the help of a "basis", e.g. as combinations of shifts, tilts and humps (cf. figure 3.2).

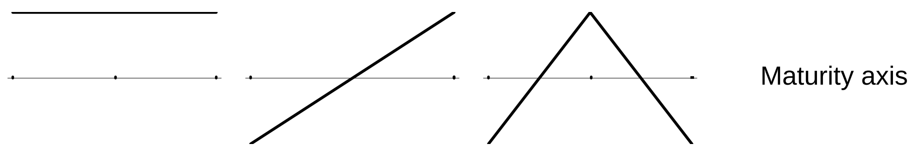

Figure 3.2: Elementary changes in the interest rate curve

- changes in the term structure can be decomposed by statistical methods (e.g. principal component analysis or factor analysis; cf. [Kärki and Reyes], [Schaefer]) into a set of factors building together some kind of basis.

The last two models have the advantage that the complete curve can be represented by a small number of variables, i.e., the dimensions of the risk factor space can be dramatically reduced. Of course, this kind of representation can also be used for modelling the term structure of volatilities.

P&L Surface

The P&L surface is constructed with the help of valuation models: In principle, the complete portfolio has to be re–valuated for every point in the risk factor space. However, if we assume additivity of the P&L function $v(\omega)$ in the risk factors $\omega_1, \ldots, \omega_M$, the P&L surface can be approximated by the following procedure:

1. For every risk factor ω_i, the complete portfolio is marked–to–market at points $\omega_i^j (j = 1, \ldots, N_i)$ — the other risk factors are held constantly zero:

$$v_i^j = v(0, \ldots, 0, \omega_i^j, 0, \ldots, 0). \tag{3.2}$$

   This process can be seen as a portfolio valuation on a one–dimensional grid (cf. figure 3.3):

2. The one–dimensional grid values are extrapolated onto a multi–dimensional grid by assuming strict additivity:

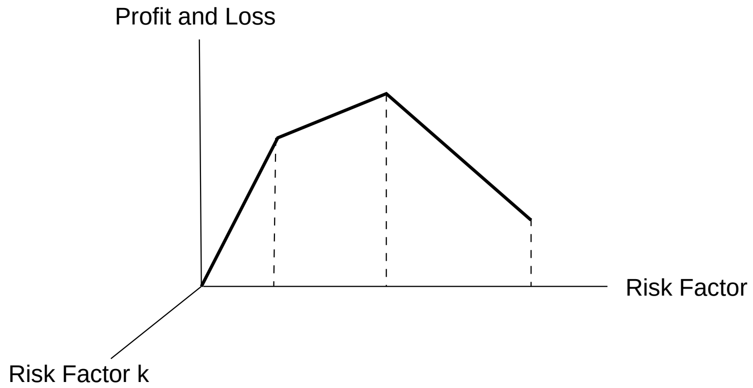

Figure 3.3: One–dimensional portfolio valuation

$$v(\omega_1^{j_1}, \ldots, \omega_M^{j_M}) = \sum_{i=1}^{M} v_i^{j_i}. \tag{3.3}$$

3. The P&L of points which do not lie on the multi–dimensional grid is obtained by interpolation procedures (e.g., polynomial interpolation, splines).

If sufficient computational power is available for the exact valuation of the portfolio on every point of the multi–dimensional grid, then the approximation in formula (3.3) can be avoided. However, this requires $\prod_{i=1}^{M} N_i$ valuations of the complete portfolio instead of $\sum_{i=1}^{M} N_i$.
For well behaved situations, the number of portfolio valuations might be further reduced by using local approximations (Taylor series) to obtain appropriate approximations of the P&L surface.

3.3 Modelling Trust Regions

Trust regions are primarily defined with the help of the risk factor density $f_t(\omega)$. In the following we explain the construction for multinormal distributions: Assume that the risk factors $\omega_1, \ldots, \omega_M$ are multinormal variates with mean 0 and covariance matrix Σ_t. The joint density function is

$$f_t(\omega) = \frac{1}{(2\pi)^{M/2}\sqrt{\det \Sigma_t}} \exp\left(-\frac{1}{2}\omega^T \Sigma_t^{-1} \omega\right). \tag{3.4}$$

The goal is to find a trust region A_t which covers a probability of α and includes the scenario $\omega = 0$. One possible choice is to search a constant c such that $\Pr\{\omega \mid f_t(\omega) \geq c\} = \alpha$. This makes sense since $f_t(\omega)$ attains its maximum at

$\omega = 0$ and leads to the trust region of minimal volume. By eliminating the constants, the problem is reduced to the following: Find c_α such that

$$\Pr(\omega \mid \omega^T \Sigma_t^{-1} \omega \leq c_\alpha) = \alpha. \tag{3.5}$$

Write

$$\omega^T \Sigma_t^{-1} \omega = \omega^T U^{-1} U^{-T} \omega = (U^{-T}\omega)^T (U^{-T}\omega), \tag{3.6}$$

where $\Sigma_t = U^T U$ is the Cholesky decomposition of the covariance matrix. But

$$\begin{aligned} \mathrm{Var}(U^{-T}\omega) &= \mathrm{E}\left[(U^{-T}\omega)(U^{-T}\omega)^T\right] \\ &= U^{-T}\mathrm{Var}(\omega)U^{-1} \\ &= U^{-T}U^T U U^{-1} \\ &= \mathbb{1}. \end{aligned} \tag{3.7}$$

Hence, $(U^{-T}\omega) \sim \mathcal{N}(0, \mathbb{1})$ and

$$\omega^T \Sigma_t^{-1} \omega = \sum_{i=1}^{M} X_i^2, \tag{3.8}$$

where X_i are independent standard normal variates. Thus, $\sum_{i=1}^{M} X_i^2$ is χ^2 distributed with M degrees of freedom. Consequently, a valid trust region is obtained by choosing:

$$A_t = \{\omega \mid \omega^T \Sigma_t^{-1} \omega \leq c_\alpha), \tag{3.9}$$

where c_α is the α quantile of a χ^2_M distribution. This is the equation of an ellipsoid centered at the origin. Supplementary conditions might eventually be introduced to restrict the trust region further; chapter 3.4 shows how this can be done using the triangular relationship of foreign exchange rates.

3.4 Modelling FX–Restrictions

The triangular relationship of foreign exchange rates leads to supplementary restrictions upon the trust region (cf. [Allen]): Consider the 3 currencies A, B and C, where C is the home currency (reporting currency). Then, the cross rate $\frac{A}{B}$ is derived from the triangular relationship $\frac{A}{B} = \frac{\frac{C}{B}}{\frac{C}{A}}$. Consequently, the model has only two risk factors

$$\omega_1 = \frac{C}{A} - k_A \tag{3.10}$$

$$\omega_2 = \frac{C}{B} - k_B, \tag{3.11}$$

where k_A and k_B are the actual values of the exchange rates. The assumption of a multinormal distribution results in an ellipsoidal trust region. However, the triangular relationship not only implies the value of the cross rate $\frac{A}{B}$, but also its volatitlity $\hat{\sigma}$ (cf. Appendix C). Formulating the condition that the cross rate has to lie in the interior of a confidence band leads to

$$\left|\frac{\omega_2 + k_B}{\omega_1 + k_A} - \frac{k_B}{k_A}\right| \leq z_{\frac{1+\alpha}{2}}\hat{\sigma}, \tag{3.12}$$

where $z_{\frac{1+\alpha}{2}}$ is the $\frac{1+\alpha}{2}$–quantile of the standard normal distribution. This condition can be rewritten as two linear restrictions:

$$\begin{aligned} (-k_B - k_A z_{\frac{1+\alpha}{2}}\hat{\sigma})\omega_1 + k_A\omega_2 &\leq k_A^2 z_{\frac{1+\alpha}{2}}\hat{\sigma} && (3.13) \\ (k_B - k_A z_{\frac{1+\alpha}{2}}\hat{\sigma})\omega_1 - k_A\omega_2 &\leq k_A^2 z_{\frac{1+\alpha}{2}}\hat{\sigma} && (3.14) \end{aligned}$$

These equations represent two hyperplanes, intersecting each other at $(\omega_1, \omega_2) = (-k_A, -k_B)$, the absolute zero of the exchange rates $\frac{C}{A}$ and $\frac{C}{B}$ (cf. figure 3.4). The result is a more realistic model, which leads to ML figures that might be less conservative.

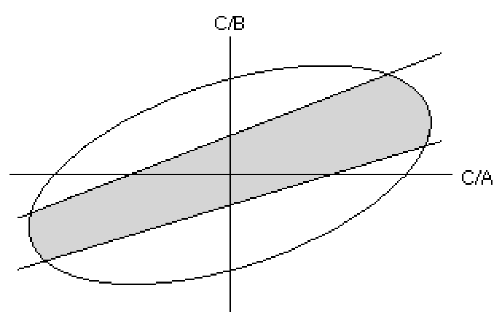

Figure 3.4: Effect of supplementary FX–restrictions

Chapter 4

Comparison of VAR with ML

4.1 Relationship between VAR and ML

To compare VAR and ML we fix some confidence level α. The risk factor distribution, as well as the holding period t, are supposed to be identical for both models. Furthermore, it is assumed that

- the change in portfolio value $v(\omega)$ is a continuous function,
- the joint density $f_t(\omega)$ is strictly positive on Ω (not necessarily normal).

The general definition of VAR given in formula (B.1) becomes:

$$\begin{aligned} \text{VAR} = \quad & \min \quad \hat{v} \\ & \text{s.t.} \quad \Pr(\omega \mid v(\omega) \leq \hat{v}) \geq 1 - \alpha. \end{aligned} \tag{4.1}$$

The "active risk factor area" for VAR is defined as $B_t = \{\omega \mid v(\omega, t) \leq \text{VAR}\}$; the continuity of $v(\omega)$ implies that this is a closed set. From the definition of B_t it follows that

$$v(\omega) \leq \text{VAR}, \qquad \forall \omega \in B_t. \tag{4.2}$$

For calculating ML, a closed trust region A_t with probability α has to be chosen. Then, ML is then defined by formula (3.1) as:

$$\begin{aligned} \text{ML}_{A_t} = \quad & \min \quad v(\omega) \\ & \text{s.t.} \quad \omega \in A_t; \quad \text{where } \Pr(A_t) = \alpha. \end{aligned} \tag{4.3}$$

The set $C_t = (A_t \cup B_t)^C$ is open. Assume — ad absurdum — that $A_t \cap B_t = \emptyset$. The relation $\Pr(A_t) + \Pr(B_t) \geq \alpha + (1 - \alpha) = 1$ implies that $\Pr(A_t \cup B_t) = 1$

and consequently $\Pr(C_t) = 0$. The density function $f_t(\omega)$ is strictly positive on Ω, hence $C_t = \emptyset$ and $A_t \cup B_t = \Omega$. But Ω was supposed to be connected and the sets A_t and B_t to be closed: this contradicts the assumption that $A_t \cap B_t = \emptyset$. Consequently $A_t \cap B_t \neq \emptyset$ and

$$\mathrm{ML}_{A_t} \leq \mathrm{ML}_{A_t \cap B_t} \leq \mathrm{VAR}, \tag{4.4}$$

which means that ML is always more conservative than VAR. Figure 4.1 shows how different choices of trust regions A_t can produce different values of ML: take a portfolio consisting of one linear instrument. The underlying risk factor has a standard normal distribution, the confidence level is $\alpha = 95\%$.

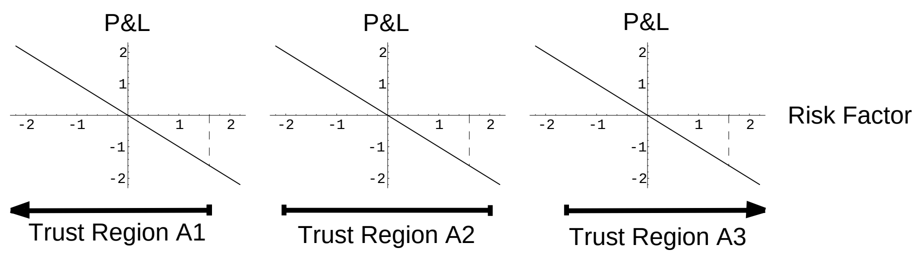

Figure 4.1: Different choices of trust regions

VAR of this portfolio is -1.64; ML however depends heavily on the choice of the trust region:

- If the trust region $A1 =]-\infty, 1.64]$ is chosen: $\mathrm{ML}_{A1} = -1.64$.
- If the trust region $A2 = [-1.96, 1.96]$ is chosen: $\mathrm{ML}_{A2} = -1.96$.
- If the trust region $A3 = [-1.64, \infty[$ is chosen: $\mathrm{ML}_{A3} = -\infty$.

In any case, the relation ML $\leq$ VAR holds. It will be shown in chapter 4.3 that *for linear portfolios with normally distributed risk factors* it is always possible to "adjust" the confidence level $\tilde{\alpha}$ of ML, such that — for the standard choice (3.9) of A_t — ML and VAR become equal.

4.2 Example: Nonlinear Portfolio

Figure 4.2 shows an example of a nonlinear portfolio with 2 foreign exchange rates $\frac{C}{A}$ and $\frac{C}{B}$ (cf. risk profiles 3/4 in [Allen et al.]); the risk factors are ω_1 and ω_2 respectively (measured in standard deviations).
Figure 4.3 shows the ellipsoidal trust region together with the cross–currency restriction described in chapter 3.4. Note that the cross–currency restriction has no effect on neither the Maximum Loss nor the Maximum Profit.

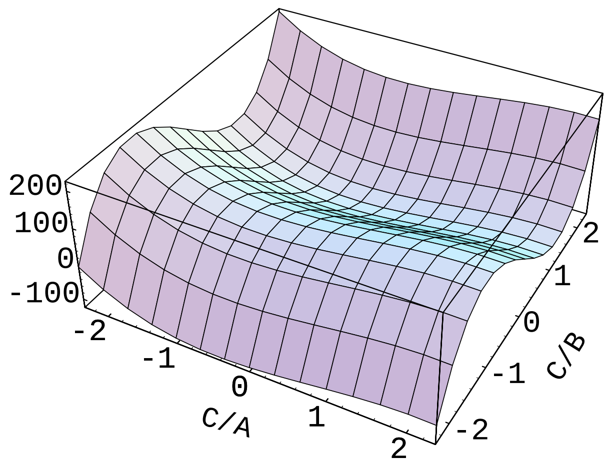

Figure 4.2: P&L surface of FX–portfolio

Table 4.1 compares different risk measures for this nonlinear portfolio. Notice that the value of the Factor Push method (cf. [Wilson2]) is not only higher than ML, it also exceeds VAR.

Confidence level: 95%	Location	Risk
VAR (Monte–Carlo simulation)	—	-84.44
ML (numerical optimization)	$\begin{pmatrix} -0.36 \\ 1.16 \end{pmatrix}$	-103.27
Factor push	$\begin{pmatrix} -1.64 \\ 1.64 \end{pmatrix}$	-36.07

Table 4.1: Risk measures for nonlinear portfolio

4.3 VAR and ML for Multinormal Risk Factors and Linear Risk Profiles

Delta–Normal VAR is a methodology for calculating analytically the risk of a portfolio if the risk factors are *multinormal* variates (i.e., $\omega \sim \mathcal{N}(0, \Sigma_t)$) and if P&L is a *linear* function (i.e., $v(\omega) = \sum_{i=1}^{M} a_i \omega_i$). If these two conditions both hold, it is also possible to derive an analytical expression for ML:

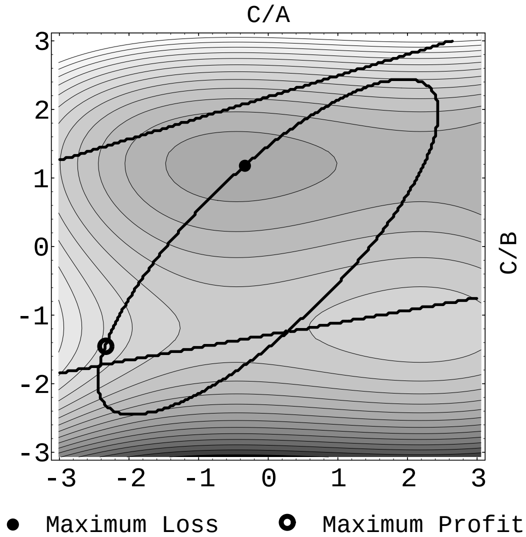

Figure 4.3: Trust region and cross–currency restrictions

$$\begin{aligned} \text{ML} \;=\; & \min \quad a^T\omega \\ & \text{s.t.} \quad \omega^T\Sigma_t^{-1}\omega \le c_\alpha, \end{aligned} \tag{4.5}$$

where the standard choice of the trust region is used (cf. chapter 3.3). Defining the functions $f(\omega) = a^T\omega$ and $g(\omega) = \omega^T\Sigma_t^{-1}\omega - c_\alpha$, the problem can be rewritten as

$$\begin{aligned} \text{ML} \;=\; & \min \quad f(\omega) \\ & \text{s.t.} \quad g(\omega) \le 0. \end{aligned} \tag{4.6}$$

Since $f(\omega)$ and $g(\omega)$ are convex functions, the solution ω^* must satisfy the 3 Kuhn–Tucker conditions:

$$\nabla f(\omega^*) = -\lambda\nabla g(\omega^*) \tag{4.7}$$

$$\lambda g(\omega^*) = 0 \tag{4.8}$$

$$\lambda \ge 0. \tag{4.9}$$

Equation (4.7) implies that

$$a = -2\lambda\Sigma_t^{-1}\omega^*, \tag{4.10}$$

and therefore $\lambda \neq 0$. It follows that

$$\omega^* = -\frac{1}{2\lambda}\Sigma_t a. \tag{4.11}$$

Considering formula (4.8), this results in

$$\frac{1}{\lambda} = \frac{2\sqrt{c_\alpha}}{\sqrt{a^T \Sigma_t a}}. \tag{4.12}$$

Hence, (4.11) allows to identify the worst case scenario

$$\omega^* = -\frac{\sqrt{c_\alpha}}{\sqrt{a^T \Sigma_t a}}\Sigma_t a, \tag{4.13}$$

and the corresponding loss is

$$\text{ML} = -\sqrt{c_\alpha}\sqrt{a^T \Sigma_t a}. \tag{4.14}$$

This expression is very similar to Delta–Normal VAR ($-z_\alpha\sqrt{a^T\Sigma_t a}$); the only difference lies in the scaling factor: c_α is the α–quantile of a χ^2 distribution with M degrees of freedom (cf. chapter 3.3), whereas z_α is the α–quantile of a standard normal distribution. Contrarily to VAR, ML depends on the number of risk factors used in the model (cf. table 4.2).

	$M = 2$	$M = 5$	$M = 10$	$M = 50$
$\alpha = 90.0\%$	1.67	2.37	3.12	6.20
$\alpha = 95.0\%$	1.49	2.02	2.60	5.00
$\alpha = 97.5\%$	1.39	1.83	2.31	4.31
$\alpha = 99.0\%$	1.30	1.67	2.07	3.75

Table 4.2: Relation ML/VAR

However, the choice of a different confidence level $\tilde{\alpha} \neq \alpha$ such that $\sqrt{c_{\tilde{\alpha}}} = z_\alpha$ leads to identical values for both measures. Since all quantities of formula (4.13) — except the constant c_α — are are known from the calculation of Delta–Normal VAR, it is possible to determine the worst case scenario ω^* in every Delta–Normal VAR implementation without additional costs.

4.4 Example: Linear Portfolio

The following example points out the fundamental difference between VAR and ML: Whereas VAR is an expected value for the loss, ML is more conservative and represents the value of the worst case that may occur in looking at α percent of all possible situations. Consider a linear portfolio consisting of 2 commodities:

$$\delta = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \qquad \Sigma_t = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 2 \end{pmatrix}, \tag{4.15}$$

where δ is the price sensitivity and Σ_t is covariance matrix of the normally distributed price changes. The risk measures for a confidence level of $\alpha = 95\%$ are shown in table 4.3:

	Formula	Value
VAR	$-z_\alpha\sqrt{\delta^T\Sigma_t\delta}$	-7.69
ML	$-\sqrt{c_\alpha}\sqrt{\delta^T\Sigma_t\delta}$	-11.48

Table 4.3: Comparison of VAR and ML

Besides the value of the Maximum Loss, the methodology gives also information about the worst case scenario $\omega^* = \frac{-\sqrt{c_\alpha}}{\sqrt{\delta^T\Sigma_t\delta}}\Sigma_t\delta = \begin{pmatrix} -1.30 \\ -3.39 \end{pmatrix}$.

4.5 ML for Multinormal Risk Factors and Quadratic Risk Profiles

Consider a portfolio with a quadratic risk profile: $v(\omega) = \frac{1}{2}\omega^T G\omega + a^T\omega$. Calculating ML means solving the problem

$$\begin{aligned} \mathrm{ML} \;=\; \min \quad & \frac{1}{2}\omega^T G\omega + a^T\omega \\ \text{s.t.} \quad & \omega^T\Sigma_t^{-1}\omega \le c_\alpha. \end{aligned} \tag{4.16}$$

Since Σ_t is positive definite, there exists a Cholesky–decomposition

$$\Sigma_t = U^T U. \tag{4.17}$$

Writing $\omega = U^T\hat{\omega}$ leads to an equivalent formulation to (4.16):

$$\begin{aligned} \mathrm{ML} \;=\; \min \quad & \frac{1}{2}\hat{\omega}^T\hat{G}\hat{\omega} + \hat{a}^T\hat{\omega} \\ \text{s.t.} \quad & \hat{\omega}^T\hat{\omega} \le c_\alpha, \end{aligned} \tag{4.18}$$

where $\hat{a} = Ua$ and $\hat{G} = UGU^T$. Again, the objective function is quadratic, but this time the constraint represents a sphere, centered at the origin. The

Levenberg–Marquardt method (cf. [Fletcher]) allows to solve this kind of problem numerically by searching in an iterative process $\nu \in \mathbb{R}$ and $\hat{\omega}^{(k)} \in \mathbb{R}^M$ satisfying

$$\begin{aligned} (\hat{G} + \nu I)\hat{\omega}^{(k)} &= -\hat{a} \\ \nu\{c_\alpha - (\hat{\omega}^{(k)})^T \hat{\omega}^{(k)}\} &= 0 \\ \nu &\geq 0. \end{aligned} \tag{4.19}$$

The key idea of the algorithm is to make a one–dimensional parameter search for $\nu \geq 0$, such that $(\hat{G} + \nu I)$ is positive definite, until $\hat{\omega}^{(k)} := -(\hat{G} + \nu I)^{-1}\hat{a}$ satisfies the condition $\nu\{c_\alpha - (\hat{\omega}^{(k)})^T \hat{\omega}^{(k)}\} = 0$. Once such an $\hat{\omega}^{(k)}$ is found, it is also a solution to (4.18). Astonishing is the fact that this algorithm is time polynomial, i.e. ML can be calculated very efficiently.

4.6 Assessment of ML

ML provides a very general framework for risk measurement (cf. figure 4.4): As has been shown in chapter 4.1, ML gives results identical to Delta–Normal VAR for linear portfolios with multinormally distributed risk factors (if a corrected confidence level $\tilde{\alpha}$ is chosen). On the other hand, Wilson's Delta–Gamma approach is just one particular implementation of Maximum Loss Optimization. A summary of the qualities of Maximum Loss is given in table 4.4.

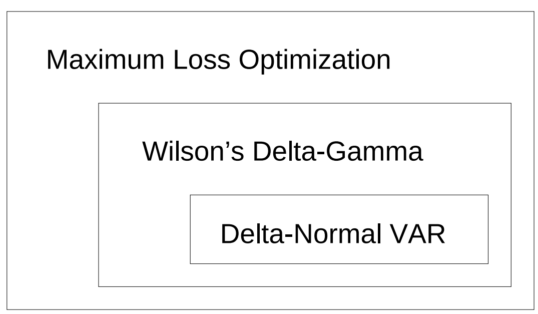

Figure 4.4: Hierarchy of risk measurement techniques

4.7 Outlook

The results obtained so far are on the way of gaining insight into what configuration of risk factors are necessary to cover the whole range of market risks. As pointed out in chapter 4.5, the computation of ML for quadratic (not necessarily convex) risk profiles is possible and, even more, can be computed efficiently.

Pros	Cons
• correlation based • handles nonlinearity • quadratic risk profiles can be analyzed efficiently • any level of aggregation possible • risk factors not restricted to normal distribution • no assumptions on P&L distribution required • marginal contributions (shadow prices) can be computed • identification of worst case	• stable market rate innovations required • modelling of trust region presumes normally distributed risk factors • computationally demanding in general case

Table 4.4: Pros and cons of ML

Using this feature for a family of expanding trust regions allows to generate a "worst case scenario path" in the risk space. This path provides rich information on the exposure of a given portfolio — and may lead to determine risk reducing strategies. Even in case of non quadratic P&L functions, the quadratic concept described above may be appropriate to investigate the "local" behavior and, hence, to guide the search process for computing ML.
Furthermore, we intend to cope with the high dimensionality of the problem by making additional structural assumptions on the P&L functions as described in chapter 3.2, especially regarding additivity and piecewise linear approximations. Towards that end we started to set up a test environment to make comparisons for different portfolios including the BIS portfolio. We expect to report numerical results in the near future and to run "in–house" test cases together with the collaborating institutions. In these implementations, the concepts of "risk–mapping" and "aggregation of risk factors" become important topics: Not only we expect to derive some guidelines for "good" mappings (such that accurate results can be obtained efficiently), but also to investigate in more depth how worst case scenarios from several portfolios can be aggregated in the overall risk space.

Appendix A

Stochastic Models of Market Rate Innovations

The modelling of market rates has a great influence on the risk measurement model itself: market rates r such as commodity prices, equity indices or FX rates are generally supposed to follow geometric Brownian motion (cf. [Hull]):

$$\frac{dr}{r} = \mu dt + \sigma dz, \tag{A.1}$$

where μ is the drift factor and σ the volatility; t is time and dz is a Wiener process (i.e., $dz \sim \mathcal{N}(0, dt)$). The application of Itô's lemma to the function $\log r$ leads to

$$\log\left(\frac{r + dr}{r}\right) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz. \tag{A.2}$$

If the constant drift rate $(\mu - \frac{\sigma^2}{2})dt$ is eliminated, it follows that

$$\log\left(\frac{r + dr}{r}\right) \sim \mathcal{N}(0, \sigma^2 dt), \tag{A.3}$$

which means that the driftless returns are lognormally distributed. A first order Taylor approximation results in

$$\begin{aligned}
\log\left(\frac{r + dr}{r}\right) &= \log(r + dr) - \log(r) \\
&\approx \left[\log(r) + dr\frac{1}{r}\right] - \log(r) \\
&= \frac{dr}{r},
\end{aligned} \tag{A.4}$$

which implies that the driftless relative returns are approximately normally distributed:

$$\frac{dr}{r} \sim \mathcal{N}(0, \sigma^2 dt). \tag{A.5}$$

If the value V of an instrument depends linearly on r (cf. [RiskMetrics]), i.e.,

$$V = Pr, \tag{A.6}$$

where P is the position of the instrument, then

$$dV = Pdr = V\frac{dr}{r}. \tag{A.7}$$

Hence, the change in value is normally distributed with mean 0 and volatility $V\sigma\sqrt{dt}$.

For interest rates, however, the situation is somewhat different: The present value $\tilde{V}$ of a zero coupon bond of maturity n is

$$\tilde{V} = \tilde{P}(1+r_n)^{-n}, \tag{A.8}$$

where $\tilde{P}$ is the face value of the bond and r_n is the zero coupon rate of an n year investment. It follows that

$$\begin{aligned} d\tilde{V} &= -\frac{n}{1+r_n}\tilde{P}(1+r_n)^{-n}dr_n \\ &= -D\tilde{V}dr_n \qquad (A.9) \\ &= -D\tilde{V}r_n\frac{dr_n}{r_n}. \qquad (A.10) \end{aligned}$$

The term $D = \frac{n}{1+r_n}$ is called "modified duration" (cf. [RiskMetrics]). If the interest rate r_n is supposed to follow geometric Brownian motion, equation (A.10) implies that $d\tilde{V}$ is normally distributed:

$$d\tilde{V} \sim \mathcal{N}(0, D^2\tilde{V}^2 r_n^2 \sigma^2 dt). \tag{A.11}$$

Most often, however, interest rates r_n are assumed to follow arithmetic Brownian motion (i.e., $dr_n = \mu dt + \sigma dz$), therefore

$$d\tilde{V} \sim \mathcal{N}(0, D^2\tilde{V}^2\sigma^2 dt), \tag{A.12}$$

by equation (A.9). Note that the variance of formula (A.11) is r_n^2 times the variance of (A.12).

Appendix B

General Definition of VAR

Consider the portfolio of figure B.1:

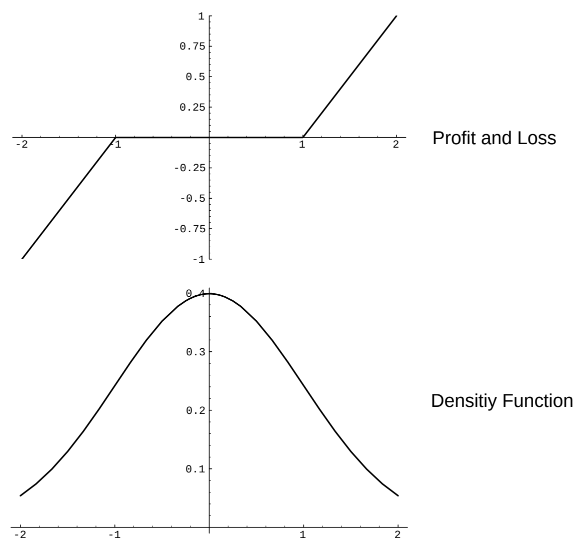

Figure B.1: Value of portfolio with underlying normal density

Its P&L distribution is shown in figure B.2. The VAR definition of formula (2.1) can not be applied here because the P&L distribution is no longer continuous. However, it is possible to generalize the VAR definition:

$$\begin{aligned} \text{VAR} = \; & \min \quad \hat{v} \\ & \text{s.t.} \quad \Pr(\omega \mid v(\omega) \leq \hat{v}) \geq 1 - \alpha, \end{aligned} \tag{B.1}$$

For continuous and strictly monotone P&L distributions, the solution of this problem is also the unique solution to $\Pr(\omega \mid v(\omega) \leq \hat{v}) = 1 - \alpha$, the defining equation of chapter 2.1. Hence, the generalized definition (B.1), which is used in the proof the ML$\leq$VAR (cf. chapter 4.1), is consistent with the usual definition of VAR.

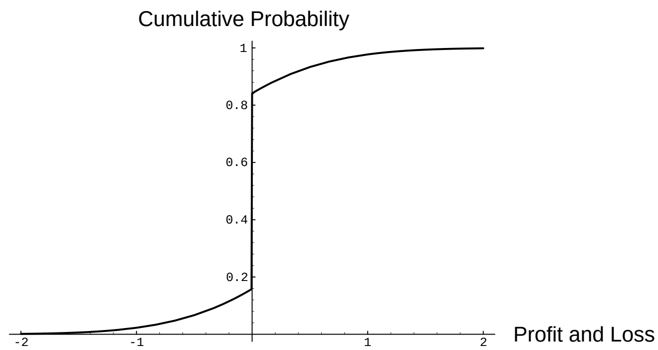

Figure B.2: P&L distribution of the portfolio

Appendix C

Implied Volatilities of Cross–Currency Rates

Consider 3 related currency rates $x = \frac{C}{A}$, $y = \frac{C}{B}$ and $z = \frac{A}{B}$. Clearly

$$z = \frac{A}{B} = \frac{\frac{C}{B}}{\frac{C}{A}} = \frac{y}{x}. \tag{C.1}$$

Differentiating this expression leads to:

$$dz = \frac{1}{x}dy - \frac{y}{x^2}dx, \tag{C.2}$$

and consequently

$$\frac{dz}{z} = \frac{1}{z}\frac{y}{x}\frac{dy}{y} - \frac{1}{z}\frac{y}{x}\frac{dx}{x} = \frac{dy}{y} - \frac{dx}{x}. \tag{C.3}$$

Assuming the relative rate innovations to be normally distributed (cf. Appendix A), i.e., $\frac{dx}{x} \sim \mathcal{N}(0, \sigma_x^2)$ and $\frac{dy}{y} \sim \mathcal{N}(0, \sigma_y^2)$, it follows that

$$\frac{dz}{z} \sim \mathcal{N}(0, \sigma_x^2 + \sigma_y^2 - 2\rho_{x,y}\sigma_x\sigma_y), \tag{C.4}$$

where $\rho_{x,y}$ is the correlation between $\frac{dx}{x}$ and $\frac{dy}{y}$.

Appendix D

Stress Testing

"Stress Testing" means investigating the impacts of improbable market conditions with the help of extreme scenarios. Stress testing is not a risk measure by itself: it is a meta–method which has to be used in conjunction with quantitative analyses such as:

- portfolio valuation,
- P&L for subclasses of instruments,
- greeks (i.e., $\Delta, , , \Theta, \rho, \upsilon$),
- Delta–Normal VAR,
- duration,
- cashflows,

which have to be determined for every scenario of a given set. Contrasting these values may help to get a better understanding of the qualities of a complex portfolio; in particular, risky market conditions may be identified.
The main problem of stress testing is the definition of a meaningful set of scenarios: A good a priori knowledge of the portfolio structure is required in order to define a reasonably small set of scenarios covering all eventualities.

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