A Methodology to Analyze Model Risk with an Application to Discount Bond Options in a Heath-Jarrow-Morton Framework

M. BOSSY, R. GIBSON, F.S. LHABITANT
N. PISTRE, D. TALAY[1]

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[1]M. Bossy and D. Talay are with INRIA Sophia-Antipolis (France). R. Gibson is with the University of Zurich (Switzerland). F.S. Lhabitant is with UBP (Switzerland) and Thunderbird, the American Graduate School of International Management (USA). N. Pistre is with ENSAE and INRIA Sophia-Antipolis (France). We wish to aknowledge financial support from RiskLab (Zurich). This work is part of the RiskLab project titled "Interest rate risk management and model risk". We wish to thank participants at the Mathematical Finance Conference in Hammamet (June 1999) and at the EFMA Conference in Paris (June 1999) for their valuable comments. All comments are welcome. Contact: rgibson@isb.unizh.ch, or Rajna Gibson, Swiss Banking Institute, University of Zurich, Plattenstrasse 14, CH 8032 Zurich (Switzerland); Phone: (0041) 1 6342969. Fax: (0041) 1 6344903

Abstract

In this paper, we propose a general methodology to analyze model risk for discount bond options within a unified Heath, Jarrow, Morton (1992) framework. We illustrate its applicability by focusing on the hedging of discount bond options and options portfolios. We show how to decompose the agent's "model risk" Profit and Loss, and emphasize the importance of the position's gamma in order to control it. We further provide mathematical results on the distribution of the forward Profit and Loss function for specific Markov univariate term structure models. Finally, we run numerical simulations for naked and combined options' hedging strategies in order to quantify the sensitivity of the forward Profit and Loss function with respect to the volatility of the forward rate curve, the shape of the term structure, and the characteristics of the position being hedged.

Introduction

Model risk has been an ongoing source of concern for financial institutions trading and managing interest rate sensitive positions. Indeed, the academic literature came up over the past decade with a variety of term structure modeling approaches that are used to price derivatives or manage global interest rate exposures of linear, convex or concave positions[1]. It is not trivial in such a context to choose the appropriate model since there have been very little studies dealing with the definition and formal analysis of model risk within nested and non-nested term structure model specifications. Should one use an equilibrium or a no-arbitrage based model, a one versus a multifactor model? Should one specify the state variables' process as being time-homogeneous or not? Furthermore, should the model be calibrated or not to the current term structure of interest rates to preserve the global consistency of the valuation process? These are some of the questions that arise at some point or another in any financial institution managing interest rate sensitive positions under several operational constraints such as the fact that it needs to rely on an accurate, preferably analytical, model that can also easily be extended to new derivative products both for trading and risk management purposes.

Several introductory remarks should be made at this point: first, it is very delicate to define any pricing or hedging strategy test as a meaningful indicator of model risk since these are always joint tests that can be rejected either because of the model's irrelevance or because markets are not efficient. Second, it is not necessarily the case that a good pricing model will also provide the best hedging or risk management alternative. The former focuses on absolute pricelevels, while the latter crucially depends on price variations and thus, is more sensitive to the gamma of a position; see Bakshi et al. (1997). Therefore, these two objectives may not lead to identical model ranking. Finally, the definition of model risk will always remain agent or institution's specific: just think about whether you want to buy or sell a given option, in the former case you are clearly less sensitive to any model that might undervalue the derivative while clearly this risk has to be minimized for the option seller. Similarly, a hedging strategy might be sensitive to intermediate losses if there are cash constraints – precluding negative wealth, for instance – or other intra-daily limits imposed on the agent while it might be totally path-independent in other institutions. Thus, the composition of the portfolio, the objectives and the constraints stemming from the environment in which model risk is examined will clearly have to be incorporated into the analysis.

A large number of highly reputable banks and financial institutions have

[1] For a survey on interest rate models, see Gibson, Lhabitant and Talay (1997).

already suffered from extensive losses due to model risk. For instance, in 1992, an inadequate model of the prepayment function resulted in a 200 million USD in the mortgage-backed securities market for J.P. Morgan; in 1987, an incorrect yield curve model resulted in a 350 million USD loss with stripped securities at Merrill Lynch. More recently, in March 1997, improper volatility estimations for interest rate options costed 90 million GBP to NatWest Markets; similarly, Bank of Tokyo-Mitsubishi had to write off 83 million USD on its U.S. interest rate derivatives book due to the inappropriate application of an interest rate option pricing model. Yet, one must recognize that until now most academic studies have essentially defined model risk in terms of some basic statistics – such as the mean, the standard error or the root mean squared error – of the price differences computed in relative or absolute terms. As far as hedging or replicating strategies are concerned, the suitability of the model is usually defined in terms of the hedging mean excess return's significance and /or its residual variance. In both cases, such statistics for the error terms are clearly inappropriate if the latter are not i.i.d. distributed (which is seldom the case) unless we assume that the manager or the shareholder does indeed display a quadratic utility function.

Moreover, the theoretical literature on the subject is rather poor. Most papers are limited to a typology of model risks (see Crouhy, Galai and Mark (1998) and Gibson et al. (1999)) or attempt to estimate model risk numerically for some specific products and/or applied derivatives pricing models like the well known Black and Scholes model (see Figlewski and Green (1999), Jacquier and Jarrow (1996)).

The definition of a proper model risk metric has also gained regulators support as far as institutions aggregate banking books or trading portfolios are concerned. Thus, it is not surprising to observe that even in the context of capital adequacy rules for managing financial institutions, the BIS applies a multiple – ranging from three to four depending on whether or not the institution ranks well on the backtesting – to the daily VaR estimated by firms' internal models. This certainly represents an arbitrary but nevertheless indirect proof that, in the context of risk management, model risk is implicitly accounted for by the regulators.

The purpose of this study is to provide an analytical framework in which we formalize model risk as faced by a financial institution which acts either as a market maker – posting bid and ask prices and hedging the instrument bought or sold – or as an agent who takes the market price as given and hedges its transaction until a predetermined date (which does not necessarily extend until the maturity of his long or short position). We illustrate its applicability by focusing on the hedging of discount bond options and options portfolios. We first examine the replicating strategy, assuming that the agent trades a single derivative, namely a zero coupon risk-free bond written option. We then extend the approach to more complex derivatives positions such as spreads or straddles. We also assume that there are no market imperfections – for a discussion of how the latter market errors can be incorporated into the problem, see the study by Jacquier and Jarrow (1996) – so that the market price is indeed the fair value of the underlying contingent claim whenever it is used.

The study uses the univariate Markov Heath, Jarrow, Morton (1992) - hereafter HJM - class of contingent claims valuation models and thus encompasses several continuous time term structure specifications implicitly through adequate specifications of the forward rates volatility curve. Model risk will be analyzed by assuming that the "true" term structure is characterized by a model belonging to the univariate HJM class while the agent or the market maker chooses to rely on another model belonging to the same class. In reality, the "true" model is unknown and thus, it is more appropriate to consider model risk analysis as being performed with respect to "benchmark" models selected by the investor, the risk controller or the regulator. For instance, the agent could use a one factor Vasicek model while the benchmark is a univariate HJM model with a time-dependent volatility specification. The objective is to define the agent's profit and loss model risk function given that he uses a self-financing "pseudo-replicating" strategy and, to analytically (or numerically) solve and characterize its distribution at any time.

The first contribution of the study relies in an analytical decomposition of the P&L into three distinct terms: the first can be defined as the degree of freedom in initial pricing (date 0), the second term is the pricing error evaluated as of the current date t, and the final term is the cumulative replicating error. This last term is shown to be essentially determined by the agent's erroneous "gamma" multiplied by the squared deviation between the true and the agent's forward rate volatility curves specifications. This decomposition emphasizes the need to control the trader's or the institution's gamma. Given the inevitable nature of model risk, such a monitoring is required in order to minimize model risk without inducing volatility gaming strategies with respect to the benchmark model.

Second, we derive the analytical properties of the law of the forward P&L function for some simple forward rate volatility specifications. Finally, we conduct numerical simulations to illustrate and characterize the properties of model risk P&L function with respect to the moneyness, the time to maturity and the objective function chosen by the institution to evaluate the risk related to the replicating model misspecification with respect to a benchmark.

Aside from providing a fairly general yet conceptual framework for assessing model risk for interest rate sensitive claims, this approach has two interesting properties: first, it can be applied to a fairly large class of univariate Markov term structure models (nested in the HJM general specification). Second, it shows that model risk does indeed encompass three well-defined steps, that is, the identification of the factors, their specification and the estimation of the model's parameters. The elegance of the HJM term structure characterization is that those three steps can all be recast in terms of the specification and the estimation of the "proper" forward volatility curve function. Finally, the analysis conducted for univariate Markov term structure models could easily be extended to non-Markov HJM models through a proper redefinition of the state space and the relevant definition of the hedging parameters.

The structure of the paper is the following. In section 1, we briefly describe the HJM term structure framework and define the basic principles underlying contingent claims replicating strategies. In section 2, we define the agent's hedging strategy under model risk, and analytically characterize the forward model risk Profit and Loss (P&L) function in section 3. Section 4 provides mathematical results characterizing the law of the P&L function for specific term structure models and contingent claims' payoffs. We then rely on numerical simulations in section 5 to compute the moments and quantiles of the model risk P&L function for simple and more complex derivatives' positions.

1 A review of the HJM contingent claim pricing framework

We suppose that the real economy can be described by one of the univariate Markov HJM models, and that the price of any discount bond satisfies the valuation equation as provided by this model. In this context, we analyze interest rate model risk from the perspective of an agent such as, for instance, a market maker who needs to post bid/ask prices, or a financial institution that takes the prices as given and is interested in the risk management of its P&L. We rule out other market imperfections such as market frictions, information asymmetries, etc., that a market maker will face. Thus, his bid ask spread is also a markup in his quotes for bearing model risk[2]. Furthermore, we do not explicitly attempt to model the operational factors that may create model risk (human skills, moral hazard, partially observable data, etc.).

[2]See in particular Figlewski and Green (1999).

1.1 The HJM framework: an overview

We consider a continuous-time economy, with a complete and perfect market in the Harrison and Kreps (1979) and Harrison and Pliska (1981) sense. We take as given a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $F = \{\mathcal{F}_t; t \in [0, T^*]\}$ where $T^* > 0$ is a finite time horizon. The set $\mathcal{F}_t$ represents the whole information available at time t. We make the usual assumption that $\mathcal{F}_0$ is trivial and $\mathcal{F}_{T^*} = \mathcal{F}$.

To begin with, we suppose that, in the real world, the term structure dynamics is characterized by a univariate Markov model belonging to the Heath-Jarrow-Morton (1992) (hereafter: HJM) general framework. That is, at time t, the instantaneous forward rate with maturity T satisfies the following equality under $\mathbb{P}$:

$$f(t,T) = f(0,T) + \int_0^t \mu_f(s,T)ds + \int_0^t \sigma_f(s,T)dW_s, \tag{1}$$

where W is a standard one dimensional Brownian Motion and where $\sigma_f(\cdot, T)$ is a bounded function on $[0,T]$. We furthermore limit mathematical treatment of the problem to the Markov univariate HJM models[3] specifying $\sigma_f(\cdot, T)$ at most as a deterministic function of t and T. While the proposed formal treatment of model risk could be extended to path-dependent multivariate HJM models, its treatment would however be much more complex and difficult to interpret in terms of the usual hedging parameters.

The major contribution of Heath, Jarrow and Morton was to show that in order to allow model (1) to be consistent with arbitrage-free pricing, the parameters $\mu_f(t,T)$ and $\sigma_f(t,T)$ cannot be freely specified. More precisely, in order to avoid arbitrage opportunities, there must exist an adapted process $\lambda(t)$ which is independent of the maturity T such that under $\mathbb{P}$

$$\mu_f(s,T) = \sigma_f(s,T)\left[\int_s^T \sigma_f(s,u)du + \lambda(s)\right] \tag{2}$$

It is possible to interpret $\lambda(s)$ as a unitary interest rate risk premium, which is independent of the maturity T and has to be exogenously specified (while controlling for its consistency with an equivalent risk-neutral martingale measure).

Since we primarily want to focus on the specification rather than on estimation, we will abstract from the rather delicate task of having to specify the risk premium. Thus, for simplicity, we analyze model risk directly under

[3] See Jeffrey (1995).

the risk-neutral equivalent martingale measure, and thus set $\lambda \equiv 0$[4]. In such a case, the dynamics of the yield curve is completely described by the function $\sigma_f(s,T)$, since the function $\mu_f(s,T)$ is uniquely related to $\sigma_f(s,T)$:

$$\mu_f(s,T) = \sigma_f(s,T)\sigma_f^*(s,T), \text{ with } \sigma_f^*(s,T) := \int_s^T \sigma_f(s,u)du. \tag{3}$$

In the following developments, we will suppose that the agent does not know the forward rate volatility function $\sigma_f(s,T)$, although the model that he uses still belongs to the HJM univariate Markov class of models. The current yield curve is furthermore observable by all agents in the economy.

The price at any time $0 \leq t \leq T$ of a zero coupon bond maturing at date T is given by

$$B(t,T) = \exp\left(-\int_t^T f(t,s)ds\right). \tag{4}$$

We denote by r_t the instantaneous rate $f(t,t)$:

$$r_t = f(0,t) + \int_0^t \sigma_f(s,t)\sigma_f^*(s,t)ds + \int_0^t \sigma_f(s,t)dW_s. \tag{5}$$

Thus, from (4) and (5), one can show that the process $(B(t,T), t \leq T)$ solves the stochastic differential equation

$$\begin{cases} dB(t,T) &= r_t B(t,T)dt - \sigma_f^*(t,T)B(t,T)dW_t, \\ B(T,T) &= 1. \end{cases} \tag{6}$$

In order to simplify calculations, for $T^O \leq T$, we will also introduce the T^O-forward price $B^F(t,T)$ of the bond of maturity T. This price is defined by a change of numeraire:

$$B^F(t,T) := \frac{B(t,T)}{B(t,T^O)}. \tag{7}$$

1.2 The pricing of a European call option

We now briefly recall how to compute the no-arbitrage price of a European option on a zero coupon bond in the absence of model risk and in a complete market setting. If we denote by $T^O < T$, the maturity of the option, by K its strike price, and suppose that its payoff at maturity is of the type

[4]This setting is also valid under $\mathbb{P}$ for instance if the local expectations hypothesis prevails in the economy. See for instance Cox, Ingersoll and Ross (1981).

$\phi(B(T^O, T))$, where ϕ is a given function, then it is well known that the agent can replicate the option by a continuous trading strategy involving holding H_t^O units of the discount bond of maturity T^O (whose price at time $t \leq T^O$ is $B(t, T^O)$) and H_t units of the discount bond of maturity T (whose price at time $t \leq T$ is $B(t, T)$).

Let V_t be the value at time t of this self-financing strategy. Then,

$$V_t := H_t^O B(t, T^O) + H_t B(t, T)$$

Let V_t^F be the T^O-forward value at time t of the self-financing strategy. Then,

$$V_t^F := \frac{V_t}{B(t, T^O)} = H_t^O + H_t B^F(t, T). \tag{8}$$

It can be shown (see Appendix 1) that dV_t^F satisfies

$$dV_t^F = d\left(H_t^O + H_t B^F(t, T)\right) = H_t dB^F(t, T). \tag{9}$$

In this setting, it is well known that the forward value of the replicating portfolio is a martingale under the forward risk adjusted probability.

$$V_t^F = \mathbb{E}^F[\phi(B^F(T^O, T))|\mathcal{F}_t], \ \ a.s., \tag{10}$$

where $\mathbb{E}^F$ stands for the expectation under the forward risk adjusted measure $\mathbb{P}^F$ defined by[5]

$$\left.\frac{d\mathbb{P}^F}{d\mathbb{P}}\right|_{\mathcal{F}_t} := \frac{B(t, T^O)}{\exp(\int_0^t r_\theta d\theta) B(0, T^O)}. \tag{11}$$

Under $\mathbb{P}^F$, the process $(W_t^F, \ 0 \leqslant t \leqslant T^O \leqslant T)$ defined by

$$W_t^F = W_t + \int_0^t \sigma_f^*(\theta, T^O) d\theta$$

is a Brownian motion, and one obtains

$$dB^F(t, T) = B^F(t, T)(\sigma_f^*(t, T^O) - \sigma_f^*(t, T)) dW_t^F. \tag{12}$$

Suppose now that the function ϕ is smooth, for example of class $\mathcal{C}^2$. Given that the process $B^F(\cdot, T)$ is the solution of the stochastic differential equation (12), a classical result (the Feynman-Kac formula) implies that

$$V_t^F = \pi_{\sigma_f}(t, B^F(t, T)), \tag{13}$$

[5] See Geman, El Karoui and Rochet (1995).

where the function π_{σ_f} solves the following parabolic partial differential equation:

$$\begin{cases} \frac{\partial \pi_{\sigma_f}}{\partial t}(t,x) + \frac{1}{2}x^2(\sigma_f^*(t,T^O) - \sigma_f^*(t,T))^2 \frac{\partial^2 \pi_{\sigma_f}}{\partial x^2}(t,x) = 0, \\ \pi_{\sigma_f}(T^O, x) = \phi(x), \end{cases} \tag{14}$$

with $(t,x) \in [0,T^O) \times \mathbb{R}$.

The processes (H_t) and (H_t^O) can now be expressed in terms of the forward bond price $B^F(t,T)$. Indeed, applying Itô's formula to the right-hand side of (13) and using (14), one obtains:

$$dV_t^F = \frac{\partial \pi_{\sigma_f}}{\partial x}(t, B^F(t,T))dB^F(t,T).$$

Thus, in view of (12), we obtain:

$$H_t = \frac{\partial \pi_{\sigma_f}}{\partial x}(t, B^F(t,T)). \tag{15}$$

We then deduce from equations (8) and (9) the value of (H_t^O) in terms of (H_t):

$$H_t^O = V_0^F + \int_0^t H_\theta dB^F(\theta,T) - H_t B^F(t,T). \tag{16}$$

This completely defines the self-financing strategy (H_t^0, H_t).

2 The agent's hedging strategy under model risk

In this section, we analyze the model risk associated with the hedging of a single option within the univariate Markov HJM framework, assuming first that a real term structure model exists. Secondly, we acknowledge the impossible task of defining such a "true" model and thus rely on a benchmark. The pairwise comparison between the "benchmark" and the agent's "erroneous" models that underpins this analysis covers several practical applications. Indeed, risk controllers, independent auditors, or regulators often use a "benchmark" model to gauge the validity of alternative term structure models used by traders, investors, and more generally, financial institutions to manage their derivatives positions. We define the benchmark model as a theoretical model which abstracts from any market errors (such as those associated with noisy price series) and we focus more thoroughly on hedging rather than on pricing errors stemming from the erroneous model used by the agent.

More specifically, the agent uses a specific model belonging to the univariate Markov HJM family and characterized by its volatility function $\bar{\sigma}_f(t,T)$ which is not necessarily equal to the "true"/benchmark volatility function $\sigma_f(t,T)$. Note that given the key role played by the volatility specification in the Heath, Jarrow, Morton framework, $\bar{\sigma}_f(t,T)$ jointly accounts for estimation and misspecification risk.

We denote by (H_t^O, H_t) a self–financing strategy which perfectly replicates the option, whereas $(\bar{H}_t^O, \bar{H}_t)$ denotes the agent's strategy. Similarly, (V_t) denotes the value of the hedging portfolio, whereas $(\bar{V}_t)$ denotes the value of the agent's portfolio. In the absence of model risk, the agent relies on (16) to determine the quantity H_t^O of discount bonds of maturity T^0 needed to achieve a perfect replicating strategy. Since the true/benchmark model differs from the agent's model, it appears (see below) that it is impossible for him to jointly maintain the perfect replication and the self-financing conditions of the replicating strategy. In this study, the choice of $\bar{H}_t^O$ is made in order to preserve the self–financing property. We thus need to compute $\bar{H}_t^O$ in the presence of model risk, ruling out any learning from the trader regarding his trading strategy.

At each date $0 \leq t \leq T$, the agent buys or sells discount bonds of maturity T^O and T in order to duplicate the contingent claim. The quantity of bonds $\bar{H}_t$ of maturity T is determined according to the model he chooses. It is given by the delta of the option in the univariate Markov HJM model with a (wrong) volatility specification $\bar{\sigma}_f(t,T)$. In other words, at each time t, $\bar{H}_t$ satisfies:

$$\bar{H}_t = \frac{\partial \pi_{\bar{\sigma}_f}}{\partial x}\left(t, B^F(t,T)\right) \tag{17}$$

where the function $\pi_{\bar{\sigma}_f}$ solves the following parabolic partial differential equation (similar to equation (14) with $\bar{\sigma}_f$ instead of σ_f):

$$\begin{cases} \dfrac{\partial \pi_{\bar{\sigma}_f}}{\partial t}(t,x) + \dfrac{1}{2}x^2(\bar{\sigma}_f^*(t,T^O) - \bar{\sigma}_f^*(t,T))^2 \dfrac{\partial^2 \pi_{\bar{\sigma}_f}}{\partial x^2}(t,x) = 0 \\ \pi_{\bar{\sigma}_f}(T^O, x) = \phi(x). \end{cases} \tag{18}$$

with $(t,x) \in [0,T^O) \times \mathbb{R}$.

At time 0, the initial value of his "replicating" portfolio is denoted by $\bar{V}_0$. The value f$\bar{V}_t$ satisfies:

$$\bar{V}_t := \bar{H}_t^O B(t,T^O) + \bar{H}_t B(t,T), \tag{19}$$

with $\bar{H}_t$ given by (17). By the change of numeraire, the T^O-forward value $\bar{V}_t^F$ of the "replicating" portfolio is

$$\bar{V}_t^F = \bar{H}_t^O + \bar{H}_t B^F(t,T) = \bar{V}_0^F + \int_0^t \bar{H}_\theta dB^F(\theta,T). \tag{20}$$

Thus, one necessarily has

$$\bar{H}_t^O = \bar{V}_0^F + \int_0^t \bar{H}_\theta dB^F(\theta,T) - \bar{H}_t B^F(t,T). \tag{21}$$

We now prove that $\bar{H}_t^O$ can be expressed without any stochastic integral. This is an important issue in practice since, in such a case, the amounts $\bar{H}_t^O$ and $\bar{H}_t$ can be computed as continuous functionals of the real prices observed before date t. Indeed, applying Itô's formula to $\pi_{\bar{\sigma}_f}(t, B^F(t,T))$ and using (17) and (21), we obtain:

$$\begin{aligned}\textstyle\int_0^t \bar{H}_\theta dB^F(\theta,T) &= \pi_{\bar{\sigma}_f}(t, B^F(t,T)) - \pi_{\bar{\sigma}_f}(0, B^F(0,T)) \\ &\quad - \textstyle\int_0^t \frac{\partial \pi_{\bar{\sigma}_f}}{\partial \theta}(\theta, B^F(\theta,T))d\theta \\ &\quad - \tfrac{1}{2}\textstyle\int_0^t \frac{\partial^2 \pi_{\bar{\sigma}_f}}{\partial x^2}(\theta, B^F(\theta,T))d < B^F(\cdot,T) >_\theta .\end{aligned}$$

Since

$$\begin{aligned} dB^F(t,T) &= B^F(t,T)\sigma_f^*(t,T^O)(\sigma_f^*(t,T^O) - \sigma_f^*(t,T))dt \\ &\quad + B^F(t,T)(\sigma_f^*(t,T^O) - \sigma_f^*(t,T))dW_t. \end{aligned} \tag{22}$$

it follows that

$$d < B^F(\cdot,T) >_\theta = B^F(\theta,T)^2(\sigma_f^*(\theta,T) - \sigma_f^*(\theta,T^O))^2 dt.$$

Using (18) one therefore obtains[6]:

$$\begin{aligned} \bar{H}_t^O &= \bar{V}_0^F - \bar{H}_t B^F(t,T) + \pi_{\bar{\sigma}_f}(t, B^F(t,T)) - \pi_{\bar{\sigma}_f}(0, B^F(0,T)) \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2 \pi_{\bar{\sigma}_f}}{\partial x^2}(\theta, B^F(\theta,T))B^F(\theta,T)^2 \\ &\qquad \left\{(\bar{\sigma}_f^*(\theta,T) - \bar{\sigma}_f^*(\theta,T^O))^2 - (\sigma_f^*(\theta,T) - \sigma_f^*(\theta,T^O))^2\right\} d\theta. \end{aligned} \tag{23}$$

[6] Note that in practice, to compute $\bar{H}_t^O$ one needs to numerically approximate the integrals on the right-hand side of (23), so that a perfect self–financing strategy is impossible to achieve. This effect is only due to the choice of a continuous-time modeling framework. We will come back to this point in section 5.4.

3 The agent's Profit and Loss under model risk

We now characterize the model risk profit and loss function with respect to the true/benchmark term structure model, and provide an analytic expression for its decomposition. An important feature of the analysis is that the forward P&L function can be characterized and evaluated at any given time between dates 0 and T^O. Let us consider the case of an agent who initially sells and hedges the option. At any time t, the agent's profit and loss under model risk is the process $(P\&L_t)$ defined by

$$P\&L_t := \bar{V}_t - V_t.$$

Similarly, his forward profit and loss is defined by

$$P\&L_t^F := \frac{P\&L_t}{B(t,T^O)} = \bar{H}_t^O - H_t^O + (\bar{H}_t - H_t)B^F(t,T). \tag{24}$$

Relying on (15), (17), we can write:

$$\begin{aligned}
P\&L_t^F &= \bar{V}_0^F + \textstyle\int_0^t \bar{H}_\theta dB^F(\theta,T) - V_0^F - \int_0^t H_\theta dB^F(\theta,T) \\
&= \bar{V}_0^F - V_0^F + \textstyle\int_0^t (\bar{H}_\theta - H_\theta) dB^F(\theta,T) \\
&= \bar{V}_0^F - V_0^F + \textstyle\int_0^t \left(\frac{\partial \pi_{\bar{\sigma}_f}}{\partial x}(\theta, B^F(\theta,T)) - \frac{\partial \pi_{\sigma_f}}{\partial x}(\theta, B^F(\theta,T)) \right) dB^F(\theta,T).
\end{aligned} \tag{25}$$

Equation (25) shows the impact of model risk induced delta hedging error on the forward P&L. We want to simplify (25) and rewrite it without stochastic integrals. To this aim, let us examine equation (24).

First, we observe that an easy computation similar to the one undertaken in the preceding section implies that a self–financing hedging strategy satisfies

$$H_t^O = -H_t B^F(t,T) + \pi_{\sigma_f}(t, B^F(t,T)). \tag{26}$$

using that

$$V_0^F = \pi_{\sigma_f}(0, B^F(0,T))$$

Thus, combining (23), (24) and (26), one obtains:

$$\begin{aligned}
P\&L_t^F = \;& \bar{V}_0^F - \pi(\sigma_f^*, 0, B^F(0,T)) \\
& + \pi_{\bar{\sigma}_f}(t, B^F(t,T)) - \pi_{\sigma_f}(t, B^F(t,T)) \\
& + \tfrac{1}{2} \textstyle\int_0^t \frac{\partial^2 \pi_{\bar{\sigma}_f}}{\partial x^2}(\theta, B^F(\theta,T)) B^F(\theta,T)^2 \\
& \quad \left\{ (\bar{\sigma}_f^*(\theta,T) - \bar{\sigma}_f^*(\theta,T^O))^2 - (\sigma_f^*(\theta,T) - \sigma_f^*(\theta,T^O))^2 \right\} d\theta.
\end{aligned} \tag{27}$$

Equation (27) defines the value of the forward P&L model risk function at any given date $t \leqslant T^O$. This forward P&L decomposition offers an insightful economic interpretation. Indeed, by looking at the right hand side of equation (27), we observe that the forward P&L consists of three distinct terms:

- the first term represents the initial "pricing" error made by the agent. This term drops out if the agent can sell the option at the price given by his own HJM model, or if the agent calibrates his model to match observed market prices.
- the second term represents the model pricing error at any given date $t \leqslant T^O$ chosen to compute the forward P&L. At the maturity of the option, this term vanishes (see (14) and (18)), since the terminal pay-off of the contingent claim is model-independent.
- the last term represents the cumulative impact of the model error on the hedging strategy up to date t. This hedging error depends on the gamma of the agent (in the "wrong" model) and on the squared difference between the forward volatility functions in the true and the "wrong" models.

The above results are fairly general since they apply to any univariate Markov specification within the class of HJM models. In all cases, we notice that the model risk P&L is sensitive to the gamma function of the agent and to the term structure of the forward rate volatilities as specified in both the benchmark and the agent's economies.

This decomposition of the model risk P&L allows us to understand the impact of model risk on an institution's global P&L, and in particular to emphasize the importance of cumulative hedging errors, even in the absence of a (model based) pricing error. Furthermore, this decomposition of the model risk P&L is an important tool for the risk management of any financial institution that attempts to validate its term structure models, and to stress test them with respect to pre-defined benchmark models specified by the risk controller or by an external regulator. Finally, since the true model is unknown, this decomposition of the P&L suggests that the "natural" control variable that the agent (or regulator) should easily monitor in order to minimize model risk is the gamma of the hedged position.

4 Mathematical analysis of the P&L for specific term structure models

The objective of this section is to show that for some properties of the contingent claims' payoff and for some specifications of the functions σ_f^* and $\bar{\sigma}_f^*$, one can compute the Profit and Loss function or some of its moments analytically.

4.1 The P&L specification for a smooth payoff contingent claim

Let us consider the case of contingent claims with a smooth payoff function ϕ of the class $\mathcal{C}^2(\mathbb{R})$. First, one observes that the probabilistic interpretation of the PDE (18) leads to:

$$\pi_{\bar{\sigma}_f}(t,x) = \overline{\mathbb{E}}^F \phi(\bar{B}_{t,x}^F(T^O,T)) \tag{28}$$

where $\overline{\mathbb{E}}^F$ is the expectation under the agent's forward risk-neutral probability measure $\overline{\mathbb{P}}^F$ defined by substituting $\bar{B}(t,T)$ to $B(t,T)$ in (11) (thus, $\overline{\mathbb{P}}^F$ is the forward risk adjusted measure for the HJM model corresponding to $\bar{\sigma}_f(t,T)$)),

$$\frac{d\overline{\mathbb{P}}^F}{d\mathbb{P}} := \frac{\bar{B}(t,T^O)}{\exp(\int_0^t \bar{r}_\theta d\theta)\bar{B}(0,T^O)}. \tag{29}$$

with an obvious definition of $(\overline{r}_\theta)$ (see (5)). As an example, consider the hedging of a single option with a smooth payoff in the absence of pricing errors. It can be shown that in the case of a smooth payoff contingent claim, the P&L at time T^O can be re-expressed as follows:

$$\begin{aligned} P\&L_{T^O}^F \;=\; & \tfrac{1}{2}\int_0^{T^O} \overline{\mathbb{E}}^F \left[\tfrac{\partial^2\phi}{\partial x^2}(\bar{B}_{t,x}^F(T^O,T)) \right. \\ & \left. \exp\left(2\int_t^{T^O}\int_{T^O}^T \bar{\sigma}_f(\theta,u)du d\bar{W}_s^F - \int_t^{T^O}\left(\int_{T^O}^T \bar{\sigma}_f(\theta,u)du\right)^2 d\theta\right)\right] \\ & \left\{\left(\int_{T^O}^T \bar{\sigma}_f(\theta,u)du\right)^2 - \left(\int_{T^O}^T \sigma_f(\theta,u)du\right)^2\right\} d\theta. \end{aligned} \tag{30}$$

The proof is given in Appendix 2.

Furthermore, if this option has a strictly convex payoff, the model risk P&L at date T^O can be signed explicitly. It signs depends on the difference between the integral of the benchmark forward rate volatility and the integral of the agent's volatility.

At time $t = T^O$, equation (30) reduces to

$$P\&L^F_{T^O} = \frac{1}{2}\int_0^{T^O} \frac{\partial^2 \pi_{\bar{\sigma}_f}}{\partial x^2}(\theta, B^F(\theta,T))B^F(\theta,T)^2 \left\{ \left(\int_{T^O}^T \bar{\sigma}_f(\theta,u)du\right)^2 - \left(\int_{T^O}^T \sigma_f(\theta,u)du\right)^2 \right\} d\theta. \tag{31}$$

if we assume that $\overline{V}_0^F = \pi(\overline{\sigma}_f^*, 0, B^F(0,T)))$. The sign of the forward P&L depends upon both the over/under estimation of the volatility with respect to the "true"/benchmark model and the gamma of the position.

In particular, under a conservative volatility specification:

$$\left(\int_{T^O}^T \sigma_f(\theta,u)du\right)^2 \leq \left(\int_{T^O}^T \bar{\sigma}_f(\theta,u)du\right)^2. \tag{32}$$

the P&L is positive (negative) for a convex (concave) long position. Notice that in order to limit the model risk of a trader, the only sensible strategy is once again to set limits on the gamma of the position.

In order to illustrate the previous result, we now provide the details of the P&L computation in the case of an agent who uses the continuous-time version of the Ho and Lee (1986) model. That is, for some strictly positive constant $\bar{\sigma}$,

$$\begin{aligned} \bar{\sigma}_f(t,T) &= \bar{\sigma} \\ \bar{\sigma}_f^*(t,T) &= \bar{\sigma}(T-t) \\ \bar{\sigma}_f^*(t,T) - \bar{\sigma}_f^*(t,T^O) &= \bar{\sigma}(T-T^O). \end{aligned}$$

One can state using (28):

$$\pi_{\bar{\sigma}_f}(t,x) = \overline{\mathbb{E}}^F \phi\left(x \exp\left(\bar{\sigma}(T-T^O)\sqrt{T^O-t}\,G - \tfrac{1}{2}\bar{\sigma}^2(T-T^O)^2(T^O-t)\right)\right),$$

where G is a Gaussian random variable of zero mean and unit variance.

It can be shown that when the agent hedges a short position in a European call option of strike K and maturity T^O, the P&L at date T^O can be expressed as

$$P\&L^F_{T^O} = \frac{K}{2\sqrt{2\pi}\bar{\sigma}(T-T^O)}\int_0^{T^O} \frac{1}{\sqrt{T^O-\theta}} \exp\left(-\frac{(\log(K)-\log B^F(\theta,T)+\frac{1}{2}\bar{\sigma}^2(T-T^O)^2(T^O-\theta))^2}{2\bar{\sigma}^2(T-T^O)^2(T^O-\theta)}\right) \left(\bar{\sigma}^2(T-T^O)^2 - \left(\int_{T^O}^T \sigma_f(\theta,s)ds\right)^2\right) d\theta. \tag{33}$$

The proof of equation (33) follows in Appendix 3. Notice that

$$\begin{aligned} |P\&L^F_{T^O}| \;\leq\; & \tfrac{K}{2\sqrt{2\pi}\bar{\sigma}(T-T^O)} \\ & \int_0^{T^O} \tfrac{1}{\sqrt{T^O-\theta}} \left| \bar{\sigma}^2(T-T^O)^2 - \left(\int_{T^O}^{T} \sigma_f(\theta,s)ds \right)^2 \right| d\theta. \end{aligned} \tag{34}$$

This is the best possible estimate, since the support of the law of the process $(B^F(\theta,T))$ is such that

$$\exp\left(-\tfrac{(\log(K)-\log B^F(\theta,T)+\frac{1}{2}\bar{\sigma}^2(T-T^O)^2(T^O-\theta))^2}{2\bar{\sigma}^2(T-T^O)^2(T^O-\theta)} \right)$$

can take a value which is as close as desired to 1, providing the path of $(B^F(\theta,T))$ is chosen accordingly.

Consider now a constant $C > 0$ such that:

$$\sup_{0\leq\theta\leq T^O} \left| \bar{\sigma}^2(T-T^O)^2 - \left(\int_{T^O}^{T} \sigma_f(\theta,s)ds \right)^2 \right| < C,\ \mathbb{P}-a.s. \tag{35}$$

Then, one can infer from (33) that, for a written call option, the absolute value of the $P\&L^F_{T^O}$ on the agent's hedging strategy admits the following almost sure upper bound:

$$\left| P\&L^F_{T^O} \right| \leq \tfrac{CKT^O\sqrt{T^O}}{2\sqrt{2\pi}\bar{\sigma}(T-T^O)}. \tag{36}$$

The result can easily be generalized to the writing of a put option with the same exercise price and time to maturity, since both contingent claims have the same gamma. Obviously, in the case of a long position in a European call or put option, the agent's loss in absolute value also becomes bounded by expression (36).

4.2 Mathematical results for the Ho and Lee strategy in a Vasicek environment

In order to further illustrate the general results obtained in Section 3, we now suppose that the agent uses a Ho and Lee model while the true/benchmark term structure in the economy is actually governed by the Vasicek model. These two models are chosen since they represent well-known specific cases of the HJM family of univariate models which can be distinguished by their volatility specification. For the Ho and Lee model, we have

$$\bar{\sigma}_f(t,T) = \sigma_r$$

while for the Vasicek model, we have:

$$\sigma_f(t,T) = \sigma_r \exp\left(-\kappa(T-t)\right)$$

where σ_r denotes the constant volatility of the spot rate[7].

In this simple case, we can show that the P&L at date T^O for the option seller (buyer) is always strictly positive (negative). In order to prove this statement, we first substitute the special form of $\sigma_f(t,T)$ into expression (33), and the following expression for the model risk P&L at maturity date T^O for the seller of a European call (or a put) on a zero coupon bond follows:

$$\begin{aligned} P\&L^F_{T^O} &= \tfrac{K\sigma_r}{2\sqrt{2\pi}(T-T^O)} \int_0^{T^O} \tfrac{1}{\sqrt{T^O-\theta}} \\ &\quad \left((T-T^O)^2 - \tfrac{1}{\kappa^2}\left(e^{-\kappa(T-\theta)} - e^{-\kappa(T^O-\theta)}\right)^2\right) \\ &\quad \exp\left(-\tfrac{(\log(K)-\log B^F(\theta,T)+\frac{1}{2}\sigma_r^2(T-T^O)^2(T^O-\theta))^2}{2\sigma_r^2(T-T^O)^2(T^O-\theta)}\right) d\theta, \end{aligned} \quad (37)$$

where the forward price of the zero coupon bond satisfies:

$$\frac{dB^F(t,T)}{B^F(t,T)} = \frac{\sigma_r}{\kappa}\left(e^{-\kappa(T-t)} - e^{-\kappa(T^O-t)}\right)\left[\frac{\sigma_r}{\kappa}(1-e^{-\kappa(T^O-t)})dt + dW_t\right].$$

It can easily be shown that, for an agent who has initially sold the option, the $P\&L_{T^O}$ is a positive random variable for any level of σ_r and κ. For that purpose, let us introduce the function f defined by

$$f(\theta) = \kappa^2(T-T^O)^2 - \left(e^{-\kappa(T-\theta)} - e^{-\kappa(T^O-\theta)}\right)^2$$

This function is positive. Indeed,

$$\begin{aligned} f(\theta) &= \left[\kappa(T-T^O) - \left(e^{-\kappa(T-\theta)} - e^{-\kappa(T^O-\theta)}\right)\right] \\ &\quad \left[\kappa(T-T^O) + \left(e^{(-\kappa(T-\theta))} - e^{-\kappa(T^O-\theta)}\right)\right]. \end{aligned}$$

Since $T > T^O$, we simply have to study the sign of

$$\left[\kappa(T-T^O) - \left(e^{-\kappa(T-\theta)} - e^{-\kappa(T^O-\theta)}\right)\right]$$

or equivalently, the sign of $p(x) - p(y)$, for $x = \kappa(T-\theta)$ and $y = \kappa(T^O-\theta)$ where $p(z) = z - e^{-z}$. Since $p(\cdot)$ is an increasing function and $x > y$, we conclude that the quantity in brackets is positive.

By symmetry, notice that a long position in the option leads to a negative value of $P\&L^F_{T^O}$ for all levels of σ_r and κ in all states of the world.

[7]In the following analysis of model risk, we suppose that there is no estimation risk (i.e. the estimated value of σ_r is the same for $\bar{\sigma}_f(t,T)$ and $\sigma_f(t,T)$) and the interest rate risk premium is still nil.

5 Numerical illustration for the Ho and Lee strategy in a Vasicek environment

In this section, we extend the previous analysis by conducting numerical simulations in the specific case where the agent uses a Ho and Lee model while the true/benchmark term structure is governed by the Vasicek model. We provide numerical simulations in order to quantify the magnitude of model risk on the forward profit and loss, and to analyze its moments and quantiles as well as the sensitivity of these values with respect to the type of position chosen by the agent under different term structure shapes. For illustrative purposes, we consider the P&L at time T^O, but the analysis can easily be applied at any given date $t \leqslant T^O$, and to any univariate Markov HJM model's pairwise comparison.

5.1 Computation of the P&L at the maturity of the option

We first compute the non-explicit formula for the forward P&L in equation (37) by numerical approximation. The procedure consists in discretizing in time the integral on the right-hand side of (37). We therefore suppose that the agent is indeed able to trade and rebalance his position continuously.

The forward P&L obtained in (37) is a random variable. We are interested in computing a sample of N different realizations at date T^O corresponding to N states of the world. In each state, we simulate a trajectory of the forward price $B^F(t,T)$ between 0 and T^O and we use it to compute the right hand side of (37). Each simulation i, for $i = 1, \ldots, N$ gives a realization $P\&L_{T^O}(i)$ of the forward Profit and Loss which allows us to compute first the approximation $\bar{\mathbb{E}}(P\&L_{T^O})$ of the expectation $\mathbb{E}(P\&L_{T^O})$:

$$\bar{\mathbb{E}}(P\&L^F_{T^O}) = \frac{1}{N}\sum_{i=1}^{N} P\&L^F_{T^O}(i). \tag{38}$$

Second, the variance $Var(P\&L_{T^O})$ is approximated by:

$$\bar{V}ar(P\&L^F_{T^O}) = \frac{1}{N}\sum_{i=1}^{N}(P\&L^F_{T^O}(i))^2 - (\bar{\mathbb{E}}(P\&L^F_{T^O}))^2. \tag{39}$$

Finally, in order to compute value-at-risk (VaR) model risk type indicators, that is, the 1% and 5% quantiles, we invert the empirical cumulative distribution function at the points 0.01 and 0.05 and we will denote by $\overline{Q}(1\%)$ and $\overline{Q}(5\%)$ the corresponding model risk quantile estimates.

For the simulations, we have chosen the following set of parameters: the maturity of the options is respectively equal to1 month, 6 month, and 1 year. The exercise prices are set respectively at $K = 90\%$, 100% and 110% of the initial value of the zero coupon bond. The maturity of the zero coupon bond is 5 years. The initial term structure of interest rates is either flat at a level of 7.5%, or ascending (from 5.5% to 8.3%) or descending (from 8.3% to 5.5%). The speed reversion parameter κ is set to 0.04 in all simulations. The size of the simulations sample is $N = 20'000$.

We have chosen to emphasize the impact of the level of σ_r and the volatility of the short-term interest rate on the characteristics of the $P\&L^F_{TO}$. The latter is intended to provide a "geographical" perspective of model risk with respect to low/high interest rate volatilities countries or regions in which interest rate risk exposures are hedged. The volatility parameter varies between 1% (for which $P\&L^F_{TO} \simeq 0$) and 12%.

5.2 Simulation results for a single short option

The results presented in Table 1 and in Figures 1 to 5 are obtained for a short position on a European call or put option written on a zero coupon bond of nominal 100. Unless otherwise stated, all the model risk $P\&L^F_{TO}$ statistics will be expressed relative to the bond market price in order to maintain a uniform benchmark for the analysis of cash and derivatives positions in terms of the value of the underlying security.

Figure 1 shows that the interest rate volatility level plays a crucial role on the various statistics of the $P\&L^F_{TO}$ distribution. For the base case example of a 6-month at-the-money put option written on a five-year discount bond, we observe that the expected forward P&L at time T^O is increasing (decreasing) almost linearly in σ_r for a short (long) option position, and we notice furthermore that the $P\&L^F_{TO}$ volatility around the central moment is also increasing in the short term rate's volatility. For the maximum level of the volatility ($\sigma_r = 12\%$) displayed, $\bar{\mathbb{E}}(P\&L^F_{TO})$ represents almost 1% of the bond's nominal value.

This value may appear negligible. One should however be aware of the fact that the value and sensitivity of all $P\&L^F_{TO}$ statistics are modified when expressed as a percentage of the underlying put or call market values (see Figure 2). For instance, in the case of the put option examined in Figure 1, the expected $P\&L^F_{TO}$ expressed as a percentage of the put market price is decreasing with respect to the short term rate's volatility. For the minimum level of the volatility ($\sigma_r = 1\%$) displayed, $\bar{\mathbb{E}}(P\&L^F_{TO})$ represents almost 27% of the put market value, which is far from being negligible. This suggests that when analyzing model risk for institutions in terms of their derivatives positions only, the importance of the model risk $P\&L^F_{T0}$ becomes highly significant because of the implied leverage.

Likewise, the 99% VaR of a single option position is highly sensitive to the level of the interest rate's volatility, ranging from 0.15% (for $\sigma_r = 1\%$) to 1.5% of the underlying position's market value (for $\sigma_r = 12\%$). Obviously, the sensitivity of the $P\&L^F_{TO}$ key statistics increases as the maturity of the option lengthens. The positive concavity of model risk with respect to the option's time to maturity increases can easily be inferred from Table 1.

As far as the moneyness of the short put option is concerned, we can observe from Figures 3 and 4 that the out-of-the money option has always the lowest expected $P\&L^F_{TO}$. Furthermore, we see that the impact of model risk on its hedging is lower than for the at and in-the-money put option, especially for low interest rate volatility levels. This is confirmed by Figure 5, which shows that while the volatility of the $P\&L^F_{TO}$ is almost linearly increasing in σ_r for the in and at-the-money put option, it is an increasing concave function of σ_r for the out-of-the-money put. Therefore, the results tend to suggest that in-the-money put or call options are the most sensitive to model risk, especially in highly volatile interest rate environments.

Finally, the simulations also show that, in the specific case of the Ho and Lee (trader's) strategy in a Vasicek (true/benchmark) environment, the shape of the term structure plays a minor role on the at-the-money put option's sensitivity to σ_r. The expected $P\&L^F_{TO}$ and its volatility are nevertheless slightly more pronounced when the term structure of interest rates is downward sloping. Obviously, the sensitivity of model risk $P\&L^F_{TO}$ with respect to the shape of the term structure is exacerbated for in-the-money options, as illustrated in Figure 5.

5.3 Simulation results for option portfolios

The above sensitivity analysis can easily be extended to various option portfolios. In Figure 6, we look at the $P\&L^F_{TO}$ distribution for a six-month long bear spread and notice first that model risk in this case can lead to both gains and losses for the trader. This already illustrates the necessity to study the impact of model risk at the aggregate level. Notice that here, the expected $P\&L^F_{TO}$ is systematically negative, since we are long the more expensive in-the-money put. Furthermore, the volatility and quantiles of the $P\&L^F_{TO}$ are increasing concave functions of σ_r with a negative skewness (which is due to the bearishness of the position). For spread-like positions, model risk is always smaller in absolute value than for a single long position in the contingent claim. This emphasizes the ability of spread positions to provide "natural" hedging against model as well as interest rate risks.

The latter statement does not apply to long straddles. It can be seen in Figure 7 that such a position amplifies the trader's exposure to model risk in an additive way. It is furthermore interesting to notice that while a long straddle position provides limited interest rate risk to its owner, it leads the agent to support considerable, however bounded, model risk if he uses the Ho and Lee hedging model in a (true/benchmark) Vasicek term structure environment.

Thus, a conclusion drawn from this sensitivity analysis is that model risk does not necessarily map into the intended interest rate risk profile of a trader or of an institution. The effects of model risk can be quite harmful in a highly volatile interest rate environment.

This illustrative example has emphasized first of all that model risk is exacerbated in highly volatile interest rate environments, secondly that the model risk exposure of an option position does not necessarily mimic its interest rate exposure, and thus, has to be managed separately. For that purpose, the most cautious strategy (not necessarily the optimal one) rests on the monitoring of the aggregate option position's gamma.

5.4 Simulations of the forward P&L at maturity for a discretely rebalanced self-financing strategy

In practice, most institutions hedge their positions at discrete time intervals, for example at the end of the day. With respect to the continuous hedging problem, discrete rebalancing introduces a discretisation error whose additional influence should be quantified.

In this section, we simulate a discrete time hedging strategy: the term structure still evolves in continuous-time according to the Vasicek model, but the agent acts at discrete times with a self-financing strategy dictated by his perception of the model of the term structure, that is, a Ho and Lee model in this example. The terminal forward P&L of the agent will not correspond to the continuous forward P&L given by equation (33). In most cases, the $P\&L_t^F$ cannot be computed analytically. However, if the interval of discretization is small, we verify numerically that the discrete P&L converges towards the continuously rebalanced strategy's P&L.

Indeed, Figures 8, 9, 10 and 11 obtained for 100 reallocations a day are very similar to the corresponding Figures in the continuous trading case. However, we see that if the agent reallocates his portfolio only once a day, the expected forward P&L is almost unaffected, but its volatility has been exacerbated and can increase by more than a factor of ten. The quantiles can become negative, suggesting that unlike in the continuous case, the discretely rebalanced position model risk P&L does not remain a strictly positive (or negative, in the case of long position) random variable.

The above results suggest that the discreteness in portfolio reallocations magnifies model risk, even for rebalancing time intervals which are commonly used by practitioners. The next challenge is then to assess the optimal portfolio reallocation frequency required to manage interest rate and model risks efficiently.

6 Conclusion

In this study, we explicitly analyze the impact of model risk on the hedging of single and aggregate discount bond option positions within a unified univariate Markov HJM environment. We have seen that the P&L due to model risk has a fairly intuitive economic interpretation and that it essentially depends on the magnitude of the position's gamma and the squared differences between the forward rate volatility curves in the benchmark and the trader's or institution's environment. Simulations were provided to highlight the fact that model risk is highly sensitive to the current level of interest rate volatility, to the type of positions held by the trader (simple versus combined, long or short, in, at, or out-of-the-money), and that it also increases with the time to maturity of the position being hedged. Such a comparative statics exercise should obviously be extended to other univariate Markov interest rate specifications, especially if the institution actively pursues the goal of minimizing interest rate model risk on its trading and/or banking book positions. Given that the "true" model of the term structure is unknown, while avoiding volatility gaming strategies, our results suggest that the independent risk control function should place limits on the position's gamma in order to minimize - or manage - the model risk exposure of a financial institution.

There are several ways in which the above study could be extended. First, we only considered univariate Markov specifications of the HJM term structure models. It would be interesting to examine the consequences on the P&L of a situation in which the trader misjudges the number as well as the specification of the factors driving the evolution of the term structure. Secondly, we ignored market disruptions and their impact on the resulting discontinuous evolution of the term structure. In particular, for emerging markets or countries in which the central bank's monetary policy interventions play an important role, the jump-diffusion component of the short term interest rate dynamics and its misspecification or misestimation should also be examined. Furthermore, issues such as estimation risk, computational risk and discretization issues also belong to the sound risk management of interest rate derivatives' books. We believe that no term structure model is universal and should thus be assessed with respect to the liquidity, the competitiveness and the political stability of the country in which an options' book is being managed. Recent examples in Asia and Russia reinforce the importance of the country economic, political and legal developments on the assessment and magnitude of model risk exposure.

It remains to be seen whether there is a non-diversifiable component to model risk that has to be priced at equilibrium by risk-averse agents. In the affirmative case, it would be interesting to study the hedging demand for model risk by risk-averse agents and to propose concrete solutions to allow for efficient model risk management within financial institutions. Finally, in light of the absence of a true model of the term structure of interest rates, it seems natural to extend this study in order to characterize "optimal" model risk management strategies for agents (institutions) with different risk profiles. This question will hopefully deserve greater attention within the risk management process of financial institutions in light of the pressure stemming from market competition and regulation.

Appendix

Appendix 1

In view of Definition (8), one applies Itô's formula to the product of V_t and of $1/B(t,T^O)$. Thus, under the probability $\mathbb{P}$,

$$\begin{aligned} dV_t^F &= \frac{1}{B(t,T^O)}dV_t - \frac{V_t}{B(t,T^O)}(r_t dt - \sigma_f^*(t,T^O)dW_t) \\ &+\frac{V_t}{B(t,T^O)}\sigma_f^*(t,T^O)^2dt + \frac{H_t\sigma_f^*(t,T^O)}{B(t,T^O)}d < B(\cdot,T),W. >_t \\ &+\frac{H_t^O\sigma_f^*(t,T^O)}{B(t,T^O)}d < B(\cdot,T^O),W. >_t . \end{aligned}$$

where $< .,. >$ denotes the quadratic variation operator.

Then:

- For the first term on the right hand side, one uses the fact that the portfolio is self–financing,

  $$V_t = V_0 + \int_0^t H_\theta^O dB(\theta,T^O) + \int_0^t H_\theta dB(\theta,T)$$

  which can be written as

  $$\begin{aligned} dV_t &= H_t^O dB(t,T^O) + H_t dB(t,T) \\ &= B(t,T^O)(r_t dt - \sigma_f^*(t,T^O)dW_t) + B(t,T)(r_t dt - \sigma_f^*(t,T)dW_t). \end{aligned}$$

- For the second and the third terms, one uses the fact that

  $$V_t = H_t^O B(t,T^O) + H_t B(t,T)$$

  .

- For the two last terms, one observes that

  $$\begin{aligned} d < B(\cdot,T),W. >_t &= -\sigma_f^*(t,T)B(t,T)dt, d < B(\cdot,T^O),W. >_t \\ &= -\sigma_f^*(t,T^O)B(t,T^O)dt. \end{aligned}$$

  Finally, one obtains that, under $\mathbb{P}$,

  $$\begin{aligned} dV_t^F &= H_t B^F(t,T)\sigma_f^*(t,T^O)(\sigma_f^*(t,T^O) - \sigma_f^*(t,T))dt \\ &+H_t B^F(t,T)(\sigma_f^*(t,T^O) - \sigma_f^*(t,T))dW_t \\ &= H_t dB^F(t,T), \end{aligned}$$

  which ends the proof.

Appendix 2

Under $\overline{\mathbb{P}}^F$, the process $(\bar{B}^F_{t,x}(\cdot,T))$, where the subscript t and x indicate that $\bar{B}^F_{t,x}(\cdot,T)$ started with value x at time t, is the solution of

$$\bar{B}^F_{t,x}(\theta,T) = x + \int_t^\theta \bar{B}^F_{t,x}(s,T)(\bar{\sigma}^*_f(s,T^O) - \bar{\sigma}^*_f(s,T))d\bar{W}^F_s, \tag{40}$$

using

$$d\bar{B}^F(t,T) = \bar{B}^F(t,T)\left(\bar{\sigma}^*_f(t,T^O) - \bar{\sigma}^*_f(t,T)\right)dW^F_t.$$

with

$$\bar{W}^F_t := W_t - \int_0^t \bar{\sigma}^*_f(\theta,T^O)d\theta.$$

For $\theta \geq t$, one obtains

$$\begin{aligned}\bar{B}^F_{t,x}(\theta,T) &= x\exp\Big(\textstyle\int_t^\theta \left(\bar{\sigma}^*_f(s,T^O) - \bar{\sigma}^*_f(s,T)\right)d\bar{W}^F_s \\ &\quad -\tfrac{1}{2}\textstyle\int_t^\theta (\bar{\sigma}^*_f(s,T^O) - \bar{\sigma}^*_f(s,T))^2 d\theta\Big).\end{aligned} \tag{41}$$

Thus, given that ϕ is of class $\mathcal{C}^2(\mathbb{R})$, one obtains

$$\begin{aligned}\tfrac{\partial^2 \pi_{\bar{\sigma}_f}}{\partial x^2}(t,x) &= \overline{\mathbb{E}}^F\Big[\tfrac{\partial^2\phi}{\partial x^2}(\bar{B}^F_{t,x}(T^O,T)) \\ &\quad \exp\Big(2\textstyle\int_t^{T^O}(\bar{\sigma}^*_f(s,T^O) - \bar{\sigma}^*_f(s,T))d\bar{W}^F_s \\ &\quad -\textstyle\int_t^{T^O}(\bar{\sigma}^*_f(s,T^O) - \bar{\sigma}^*_f(s,T))^2 ds\Big)\Big]\end{aligned} \tag{42}$$

Appendix 3

With the definition of $\pi_{\bar{\sigma}_f}(t,x)$, we have

$$\begin{aligned}\tfrac{\partial^2 \pi_{\bar{\sigma}_f}}{\partial x^2}(t,x) &= \overline{\mathbb{E}}^F \tfrac{\partial^2\phi}{\partial x^2}\big(x\exp\big(\bar{\sigma}(T-T^O)\sqrt{T^O-t}G \\ &\quad -\tfrac{1}{2}\bar{\sigma}^2(T-T^O)^2(T^O-t)\big)\big) \\ &\quad \exp\left(2\bar{\sigma}(T-T^O)\sqrt{T^O-t}G - \bar{\sigma}^2(T-T^O)^2(T^O-t)\right)\big] \\ &= \tfrac{1}{\sqrt{2\pi}}\textstyle\int_{-\infty}^\infty \exp\left(-\tfrac{y^2}{2}\right)\tfrac{\partial^2\phi}{\partial x^2} \\ &\quad \left(x\exp\left(\bar{\sigma}(T-T^O)\sqrt{T^O-t}y - \tfrac{1}{2}\bar{\sigma}^2(T-T^O)^2(T^O-t)\right)\right) \\ &\quad \exp\left(2\bar{\sigma}(T-T^O)\sqrt{T^O-t}y - \bar{\sigma}^2(T-T^O)^2(T^O-t)\right)dy,\end{aligned}$$

from which, setting

$$\xi := x\exp\left(\bar{\sigma}(T-T^O)\sqrt{T^O-t}y - \tfrac{1}{2}\bar{\sigma}^2(T-T^O)^2(T^O-t)\right)$$

and

$$z(t,\xi) := \frac{1}{\bar\sigma(T-T^O)\sqrt{T^O-t}}\left(\log(\xi)-\log(x)+\tfrac{1}{2}\bar\sigma^2(T-T^O)^2(T^O-t)\right),$$

we obtain:

$$\frac{\partial^2\pi_{\bar\sigma_f}}{\partial x^2}(t,x) = \frac{1}{x^2\sqrt{2\pi}\bar\sigma(T-T^O)\sqrt{T^O-t}}\int_0^{+\infty}\exp\left(-\frac{z(t,\xi)^2}{2}\right)\xi\frac{\partial^2\phi}{\partial\xi^2}(\xi)d\xi. \qquad (43)$$

Thus, substituting in (27) at time $t = T^O$ and for $\bar V_0^F = \pi(\bar\sigma_f^*, 0, B^F(0,T))$, we obtain:

$$\begin{aligned} P\&L_{T^O} \;=\; & \frac{1}{2\sqrt{2\pi}\bar\sigma(T-T^O)}\int_0^{T^O}\int_0^{+\infty}\xi\frac{\partial^2\phi}{\partial\xi^2}(\xi)\frac{1}{\sqrt{T^O-\theta}} \\ & \exp\left(-\frac{(\log(\xi)-\log B^F(\theta,T)+\frac{1}{2}\bar\sigma^2(T-T^O)^2(T^O-\theta))^2}{2\bar\sigma^2(T-T^O)^2(T^O-\theta)}\right) \\ & \left(\bar\sigma^2(T-T^O)^2-\left(\int_{T^O}^T\sigma_f(\theta,s)ds\right)^2\right)d\xi d\theta. \end{aligned}$$

which leads to the $P\&L_{T^O}$ as defined in equation (33).

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Tables & Figures

	Volatility of short term rate (σ_r)					
	$\sigma_r = 5\%$			$\sigma_r = 10\%$		
Maturity of the option (T^0)	$1M$	$6M$	$1Y$	$1M$	$6M$	$1Y$
$P\&L^F_{T^0}$ Expected value	0.19	0.42	0.51	0.37	0.84	1.06
$P\&L^F_{T^0}$ Volatility	0.07	0.14	0.17	0.13	0.28	0.33
$P\&L^F_{T^0}$ Quantile 1%	0.06	0.13	0.16	0.12	0.27	0.36
$P\&L^F_{T^0}$ Quantile 99%	0.31	0.69	0.83	0.63	1.37	1.68

Table 1: Impact of the option maturity on the $P\&L^F_{T^0}$ statistics in the case of a 6-month short at-the-money put option written on a 5-year zero-coupon bond

The table shows the impact of the option maturity on the $P\&L^F_{T^0}$ statistics. The $P\&L^F_{T^0}$ corresponds to the results from the dynamic hedging of a short at-the-money put on a 5-year zero-coupon bond. The option maturity can be 1 month, 6 month or 1 year. The short term rate volatility can be 5% or 10%. The term structure of interest rates is upward sloping from 5.5% (short term) to 8.3% (15-year and above). All results are expressed as a percentage of the initial bond price.

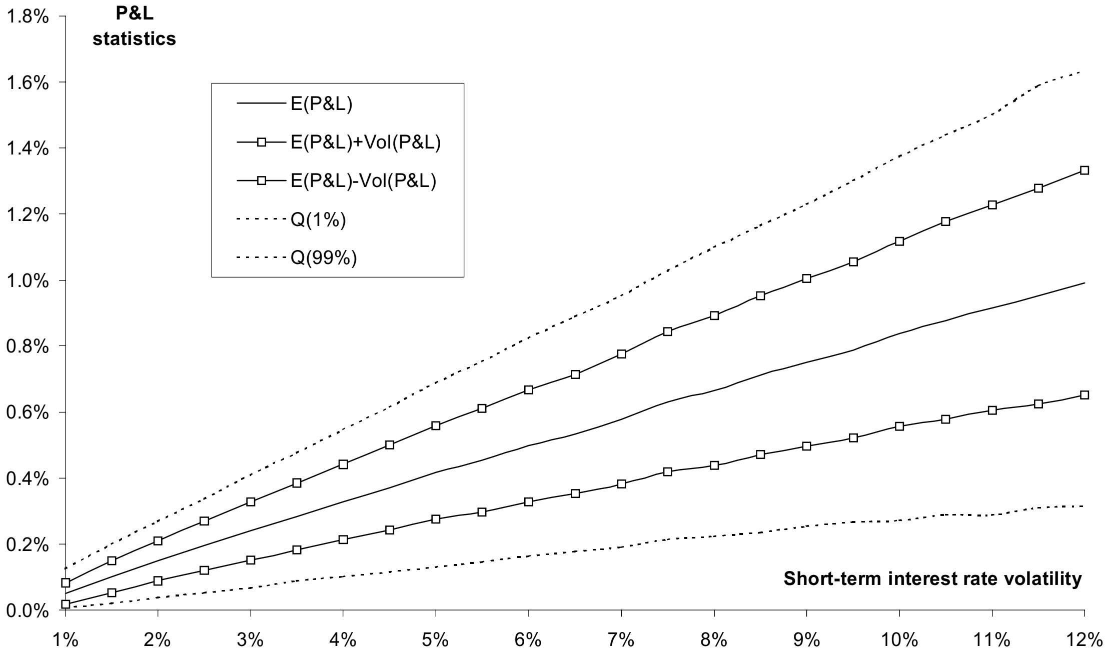

Figure 1: **Statistics for $P\&L^F_{T0}$ in the case of a 6-month short ATM put option on a 5-year zero-coupon bond**

The figure shows the impact of the short term interest rate volatility on the $P\&L^F_{T0}$ statistics. The $P\&L^F_{T0}$ corresponds to the results from the dynamic hedging of a short at-the-money 6-month put on a 5-year zero-coupon bond. The short term rate volatility varies between 1% and 15%. The term structure of interest rates is upward sloping from 5.5% (short term) to 8.3% (15-year and above). The $P\&L^F_{T0}$ statistics are expressed as a percentage of the initial underlying bond price.

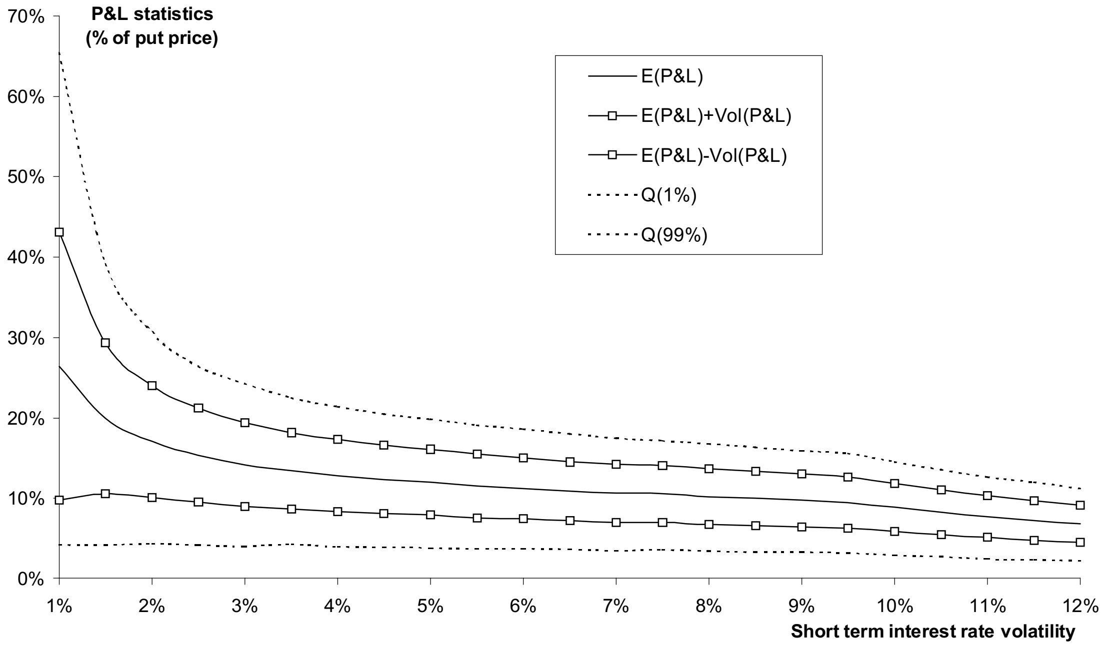

Figure 2: **Statistics for $P\&L^F_{T0}$ in the case of a 6-month short ATM put option on a 5-year zero-coupon bond (as percentage of the initial put price)**

The figure shows the impact of the short term interest rate volatility on the $P\&L^F_{T0}$ statistics. The $P\&L^F_{T0}$ corresponds to the results from the dynamic hedging of a short at-the-money 6-month put on a 5-year zero-coupon bond. The short term rate volatility varies between 1% and 15%. The term structure of interest rates is upward sloping from 5.5% (short term) to 8.3% (15-year and above). The $P\&L^F_{T0}$ statistics are expressed as a percentage of the initial put price.

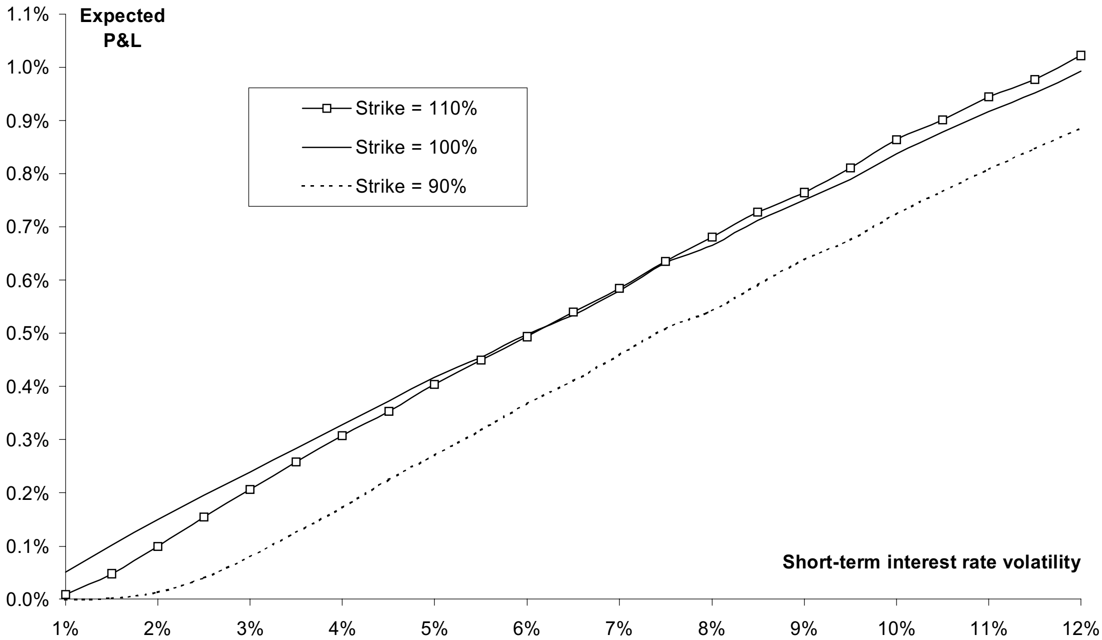

Figure 3: **Exercise price impact on the expected $P\&L_{T0}^{F}$ in the case of a 6-month short put option on a 5-year zero-coupon bond**

The figure shows the impact of the short term interest rate volatility on the expected $P\&L_{T0}^{F}$. The $P\&L_{T0}^{F}$ corresponds to the results from the dynamic hedging of a short at-the-money 6-month put on a 5-year zero-coupon bond. Three exercise prices are considered with respect to the initial bond price: 90%, 100% and 110%. The short term rate volatility varies between 1% and 15%. The term structure of interest rates is upward sloping from 5.5% (short term) to 8.3% (15-year and above). The resulting $P\&L_{T0}^{F}$ is expressed as a percentage of the initial underlying bond price.

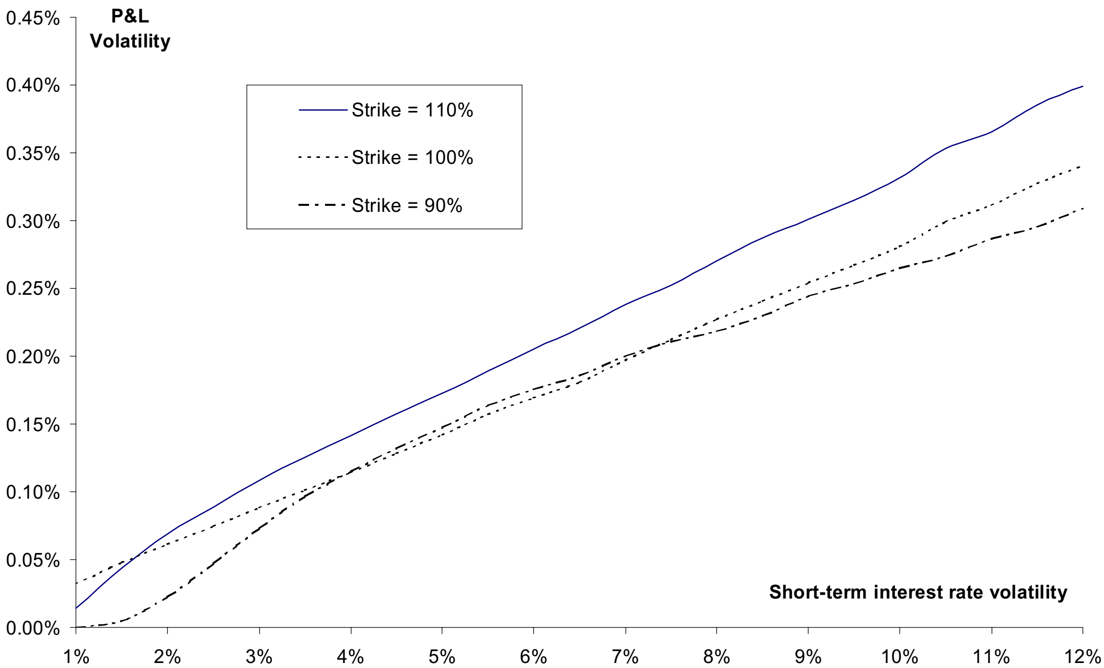

Figure 4: **Exercise price impact on the volatility of the $P\&L^F_{T0}$ in the case of a 6-month short put option on a 5-year zero-coupon bond**

The figure shows the impact of the short term interest rate volatility on the volatility of $P\&L^F_{T0}$. The $P\&L^F_{T0}$ corresponds to the results from the dynamic hedging of a short at-the-money 6-month put on a 5-year zero-coupon bond. Three exercise prices are considered with respect to the initial bond price: 90%, 100% and 110%. The short term rate volatility varies between 1% and 15%. The term structure of interest rates is upward sloping from 5.5% (short term) to 8.3% (15-year and above).

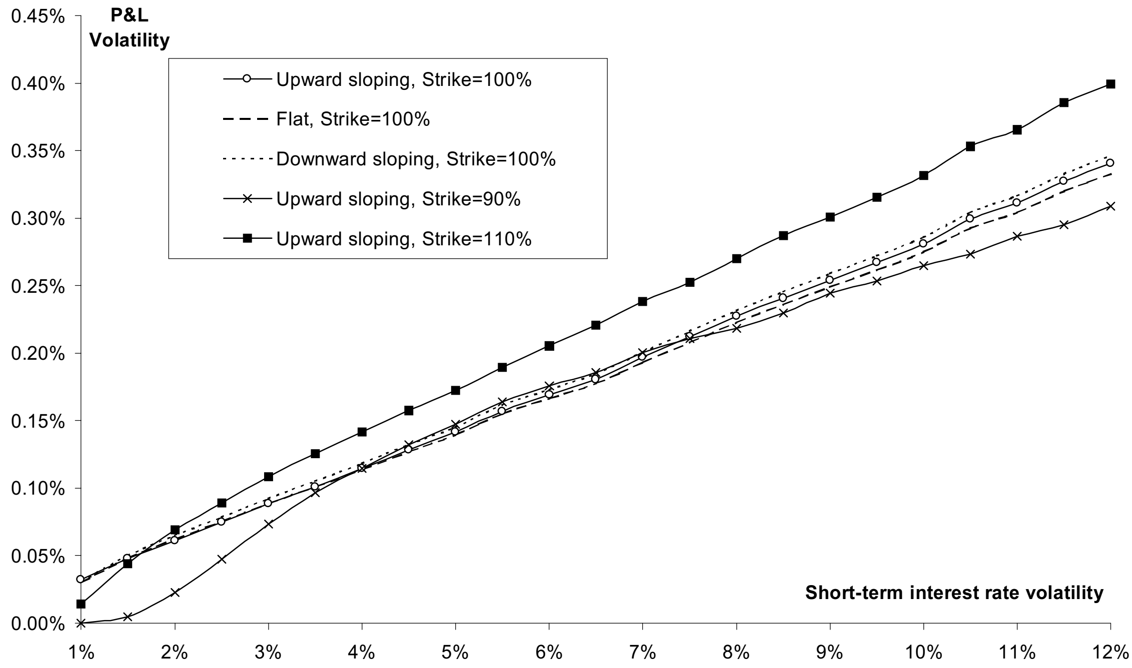

Figure 5: **Impact of the shape of the term structure and of the exercise price on the $P\&L^F_{T0}$ volatility of the in the case of a 6-month short put option written on a 5-year zero-coupon bond**

The figure shows the impact of the short term interest rate volatility on the volatility of the $P\&L^F_{T0}$. The $P\&L^F_{T0}$ corresponds to the results from the dynamic hedging of a short at-the-money 6-month put on a 5-year zero-coupon bond. Different exercise prices are considered with respect to the initial bond price (90%, 100% and 110%), as well as different term structure shapes (upward sloping, downward sloping, flat). The upward sloping term structure of interest rates goes from 5.5% (short term) to 8.3% (15-year and above). The downward sloping term structure of interest rates goes from 8.3% (short term) to 5.5% (15-year and above). The flat term structure is at 7.5%. The short term rate volatility varies between 1% and 15%.

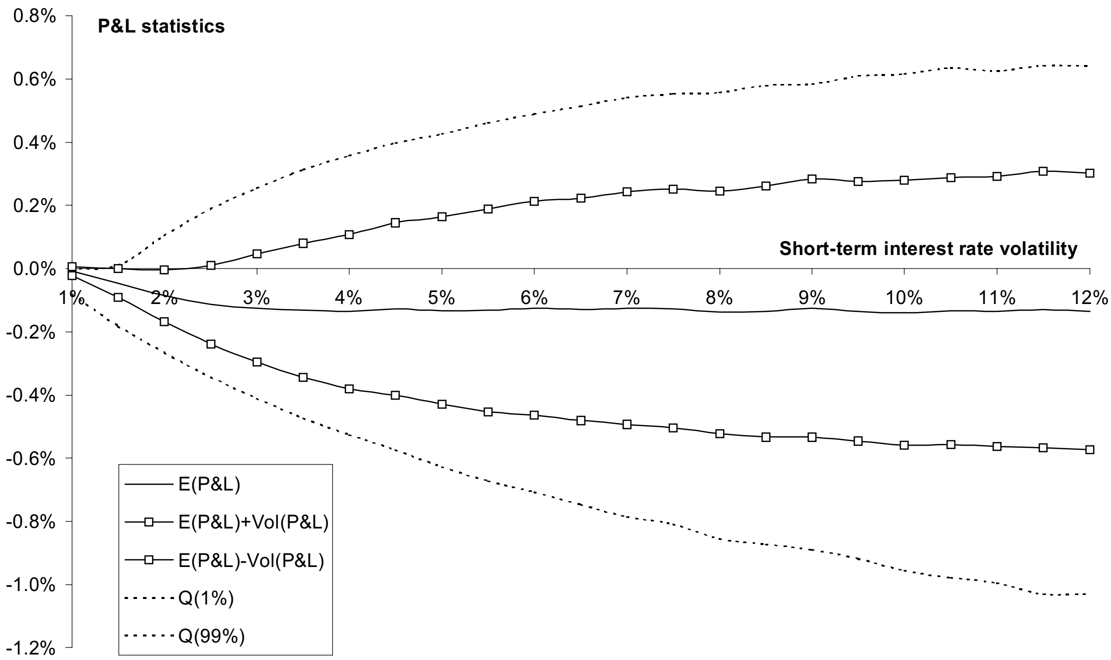

Figure 6: $P\&L_{T0}^{F}$ **statistics in the case of a 6-month long bear-spread on a 5-year zero-coupon bond**

The figure shows the impact of the short term interest rate volatility on the expected value and on the quantiles of the $P\&L_{T0}^{F}$. The $P\&L_{T0}^{F}$ corresponds to the results from the dynamic hedging of a 6-month bear-spread position on a 5-year zero-coupon bond. The spread exercise prices are set at 90% and 100% of the initial bond price. The short term rate volatility varies between 1% and 15%. The term structure of interest rates is upward sloping from 5.5% (short term) to 8.3% (15-year and above). The $P\&L_{T0}^{F}$ statistics are expressed as a percentage of the initial bond price.

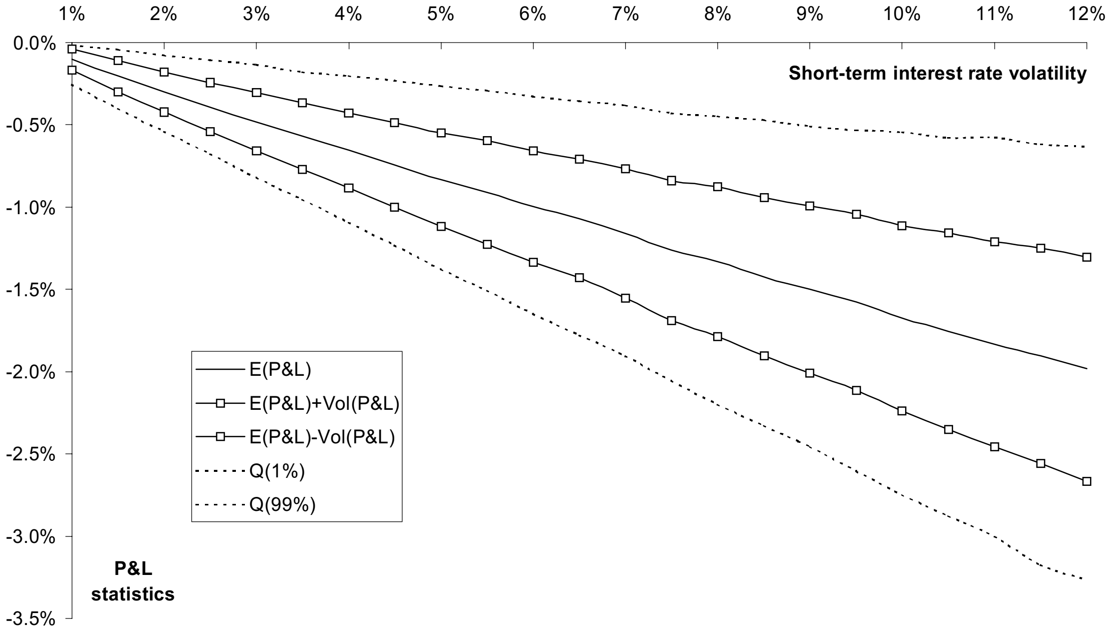

Figure 7: $P\&L^F_{T0}$ **statistics for a 6-month long at-the-money straddle on a 5-year zero-coupon bond**

The figure shows the impact of the short term interest rate volatility on the $P\&L^F_{T0}$ statistics. The $P\&L^F_{T0}$ corresponds to the results from the dynamic hedging of a 6-month long at-the-money straddle position on a 5-year zero-coupon bond. The straddle exercise prices equals the initial bond price. The short term rate volatility varies between 1% and 15%. The term structure of interest rates is upward sloping from 5.5% (short term) to 8.3% (15-year and above). The $P\&L^F_{T0}$ statistics are expressed as a percentage of the initial bond price.

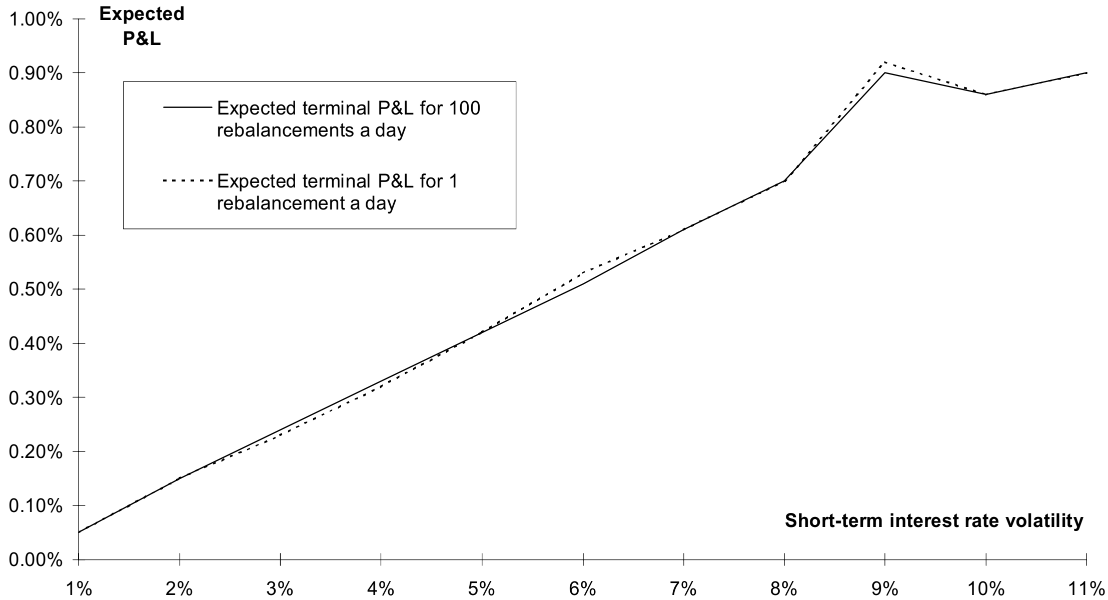

Figure 8: **Expected $P\&L_{T0}^F$ in the case of discrete rebalancing**

The figure shows the impact of the short term rate volatility on the expected $P\&L_{T0}^F$ in the case of a discrete trading strategy. We consider two possible reallocation frequency: 100 times a day, or once a day. The $P\&L_{T0}^F$ corresponds to the results from the dynamic hedging of a short at-the-money 6-month put on a 5-year zero-coupon bond. The short term rate volatility varies between 1% and 11%. The term structure of interest rates is flat at 7.5%. The expected $P\&L_{T0}^F$ is expressed as a percentage of the initial bond price.

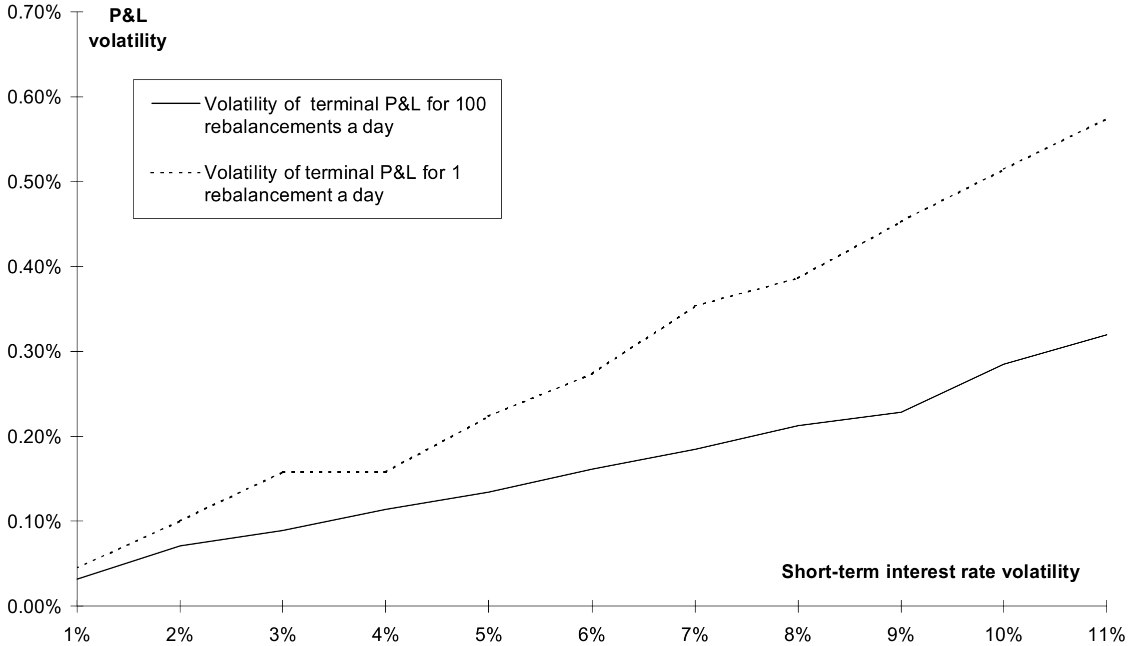

Figure 9: **Volatility of the $P\&L^F_{T0}$ in the case of discrete rebalancing**

The figure shows the impact of the short term rate volatility on the $P\&L^F_{T0}$ volatility in the case of a discrete trading strategy. We consider two possible reallocation frequency: 100 times a day, or once a day. The $P\&L^F_{T0}$ corresponds to the results from the dynamic hedging of a short at-the-money 6-month put on a 5-year zero-coupon bond. The short term rate volatility varies between 1% and 11%. The term structure of interest rates is flat at 7.5%. The $P\&L^F_{T0}$ volatility is expressed as a percentage of the initial bond price.

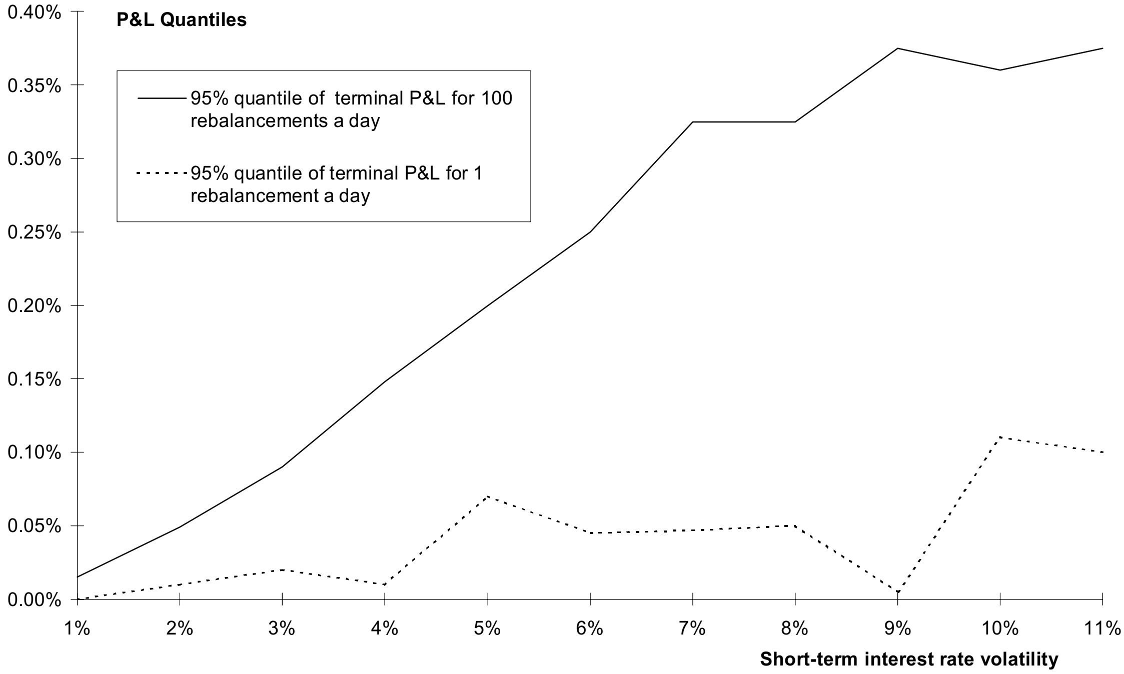

Figure 10: **95% quantiles of the $P\&L^F_{T0}$ in the case of discrete rebalancing**

The figure shows the impact of the short term rate volatility on the $P\&L^F_{T0}$ 95% quantiles in the case of a discrete trading strategy. We consider two possible reallocation frequency: 100 times a day, or once a day. The $P\&L^F_{T0}$ corresponds to the results from the dynamic hedging of a short at-the-money 6-month put on a 5-year zero-coupon bond. The short term rate volatility varies between 1% and 11%. The term structure of interest rates is flat at 7.5%. The quantiles are expressed as a percentage of the initial bond price.

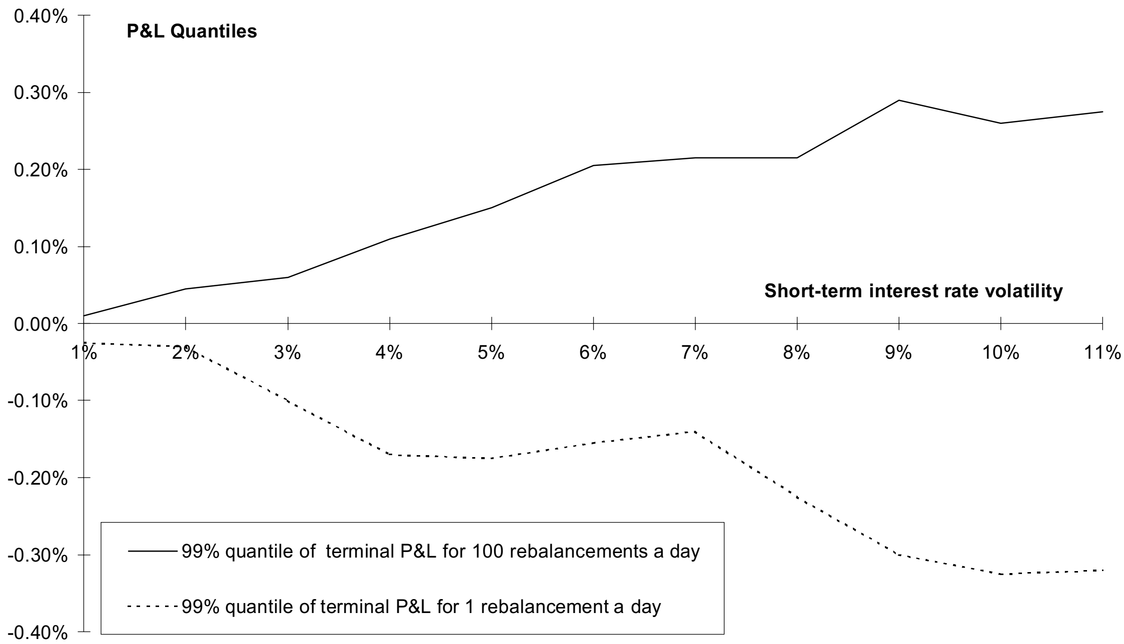

Figure 11: **99% quantiles of the $P\&L^F_{T0}$ in the case of discrete rebalancing**

The figure shows the impact of the short term rate volatility on the $P\&L^F_{T0}$ 99% quantiles in the case of a discrete trading strategy. We consider two possible reallocation frequency: 100 times a day, or once a day. The $P\&L^F_{T0}$ corresponds to the results from the dynamic hedging of a short at-the-money 6-month put on a 5-year zero-coupon bond. The short term rate volatility varies between 1% and 11%. The term structure of interest rates is flat at 7.5%. The quantiles are expressed as a percentage of the initial bond price.