The Minimum Maximum of a Continuous Martingale with Given Initial and Terminal Laws

D. G. Hobson *and* J. L. Pedersen

Let $(M_t)_{0 \le t \le 1}$ be a continuous martingale with initial law $M_0 \sim \mu_0$ and terminal law $M_1 \sim \mu_1$ and let $S = \sup_{0 \le t \le 1} M_t$. In this paper we prove that there exists a greatest lower bound with respect to stochastic ordering of probability measures, on the law of S. We give an explicit construction of this bound. Furthermore a martingale is constructed which attains this minimum by solving a Skorokhod embedding problem. The form of this martingale is motivated by a simple picture. The result is applied to the robust hedging of a forward start digital option.

1. Introduction

Let μ_0 and μ_1 be probability measures on $\mathbb{R}$, let $\mathcal{M} \equiv \mathcal{M}(\mu_0, \mu_1)$ be the space of all martingales $(M_t)_{0 \le t \le 1}$ with initial law μ_0 and terminal law μ_1 and let $\mathcal{M}_C \equiv \mathcal{M}_C(\mu_0, \mu_1)$ be the subspace of $\mathcal{M}$ consisting of the *continuous* martingales. For a martingale $(M_t) \in \mathcal{M}$ let $S \equiv \sup_{0 \le t \le 1} M_t$ and denote the law of S by ν_M. In this article we are interested in the sets $\mathcal{P} \equiv \mathcal{P}(\mu_0, \mu_1) \equiv \{\, \nu_M \mid (M_t) \in \mathcal{M} \,\}$ and $\mathcal{P}_C \equiv \mathcal{P}_C(\mu_0, \mu_1) \equiv \{\, \nu_M \mid (M_t) \in \mathcal{M}_C \,\}$ of possible laws ν. In particular we find a greatest lower bound for $\mathcal{P}_C$. (The problem of finding an upper bound has been studied elsewhere.) Here comparisons of measures are made in the sense of stochastic domination. The fact that (M_t) is a martingale with no jumps imposes quite restrictive conditions on the law of the maximum ν.

Our motivation for studying this problem is twofold. Firstly this work extends results of Perkins which cover the situation when the initial law is a unit mass (see Remark 2.3). Secondly there is an application to mathematical finance and the construction of hedging strategies for exotic options which are robust to model misspecification (see Remark 3.2).

Clearly $\mathcal{M}$ is empty unless the random variables corresponding to the laws μ_i have the same finite mean, and henceforth we will assume without loss of generality that this mean is zero. Moreover a simple application of Jensen's inequality shows that a further necessary condition for the space to be non-empty is that

$$(1.1) \qquad \int_{(x,\infty)} (u - x)\, \mu_0(du) \le \int_{(x,\infty)} (u - x)\, \mu_1(du)$$

for all $x \in \mathbb{R}$. This condition is also sufficient, see for example Strassen [19, Theorem 2] or Meyer [11, Chapter XI]. It follows from the construction in Chacon and Walsh [4] that this is also a necessary and sufficient condition for $\mathcal{M}_C$ to be non-empty. Henceforth we assume that (1.1) holds.

2000 *Mathematics Subject Classification.* Primary 60G44, 60E15. Secondary 60J65.

Key words and phrases. Continuous martingale, maximum process, stochastic domination, greatest lower bound, Brownian motion, Skorokhod embedding, excursion, digital option, robust hedging.

Consider first the problem of determining bounds on $\mathcal{P}(\delta_0,\mu_1)$ where δ_0 is the unit mass at 0. This problem is a special case of a problem first considered in Blackwell and Dubins [2] and Dubins and Gilat [6]. Let $\preceq$ denote stochastic ordering on probability measures, (so that $\rho \preceq \pi$ if and only if $\rho((-\infty,x)) \geq \pi((-\infty,x))$ for all $x\in\mathbb{R}$), and let ρ^* denote the Hardy transform of a probability measure ρ. Then it follows from [2], [6] and Azéma and Yor [1] that

$$\delta_0 \vee \mu_1 \preceq \nu \preceq \mu_1^* \,. \tag{1.2}$$

Kertz and Rösler [10] have shown that the converse to (1.2) also holds: for any probability measure ρ satisfying $\delta_0 \vee \mu_1 \preceq \rho \preceq \mu_1^*$, there is a martingale with terminal distribution μ_1 whose maximum has law ρ. (See also Rogers [17] for a proof of these results based on excursion theory which will motivate many of our arguments). Thus $\mathcal{P}(\delta_0,\mu_1) \equiv \{\,\nu \,|\, \delta_0 \vee \mu_1 \preceq \nu \preceq \mu_1^*\,\}$. Note that the lower bound is attained by a martingale which consists of a single jump at time 1 where the jump is chosen to have law μ_1.

Now consider $\mathcal{P}_C(\delta_0,\mu_1)$. Then the least upper bound is unchanged since there is a continuous martingale whose maximum has law μ_1^*, as can be seen from the example in Rogers [17]. Moreover Perkins [12] gives an expression for the greatest lower bound, which to be consistent with future notation we shall label $\nu^{\#}(\delta_0,\mu_1)$. This lower bound will arise as a special case of the construction we give below for general initial conditions. See Remark 2.3 for a discussion of the Perkins construction and its relationship to the construction we give. In summary, when the starting measure is a point mass, $\mathcal{P}_C(\delta_0,\mu_1) \subseteq \{\,\nu \,|\, \nu^{\#}(\delta_0,\mu_1) \preceq \nu \preceq \mu_1^*\}$ and both $\nu^{\#}(\delta_0,\mu_1)$ and μ_1^* are elements of $\mathcal{P}_C$.

We are interested in the problem with a general initial condition. As Kertz and Rösler [10, Remark 3.3] observe,

$$\mathcal{P}_C(\mu_0,\mu_1) \subseteq \mathcal{P}(\mu_0,\mu_1) \subseteq \{\,\nu \,|\, \mu_0 \vee \mu_1 \preceq \nu \preceq \mu_1^*\,\} \,.$$

Further Hobson [7] derives a least upper bound $\nu_{0,1}^*$ for both of the sets $\mathcal{P}_C(\mu_0,\mu_1)$ and $\mathcal{P}(\mu_0,\mu_1)$. Since there is a continuous martingale with the correct marginal distributions whose maximum has law $\nu_{0,1}^*$, the least upper bound is attained in each case.

The main result of this article is that there is a greatest lower bound $\nu^{\#} \equiv \nu^{\#}(\mu_0,\mu_1)$ for $\mathcal{P}_C$, and that this bound is attained, i.e. there exists $\nu^{\#}\in\mathcal{P}_C$ such that $\nu^{\#}\preceq\nu$ for all $\nu\in\mathcal{P}_C$. The measure $\nu^{\#}$ is difficult to characterize but we give a simple pictorial representation in Figure 1 below. It turns out that it is simple to show that $\nu^{\#}$ is a lower bound, but comparatively difficult to show that it is attained.

If the continuity restriction is dropped then it is easy to define a lower bound $\nu_{\#}$ for $\mathcal{P}(\mu_0,\mu_1)$ non-constructively via

$$\nu_{\#}((-\infty,x)) \equiv \sup_{M\in\mathcal{M}} (\nu_M(-\infty,x)) \,.$$

However there is a simple example in Hobson [7] to show that for general initial measures this lower bound for $\mathcal{P}$ is not attained. Again any minimal element of $\mathcal{P}$ corresponds to a martingale with a single jump at time 1. These two factors explain why it is more interesting to restrict attention to continuous martingales, a restriction that we now make.

The problem of characterising the greatest lower bound for the maximum of a martingale constrained to have given initial and terminal laws has an application to the pricing of derivative securities in mathematical finance. The derivatives in question are forward start barrier options and lookbacks. This idea has been explored in Hobson [8] and Brown, Hobson and Rogers [3]. It was this derivative pricing problem which provided the original motivation for studying martingale inequalities of the type in this paper.

The remainder of this paper is constructed as follows. In the next section we construct the measure $\nu^{\#}(\mu_0, \mu_1)$, and give some examples. In Section 3 we show that this measure is stochastically dominated by every measure in $\mathcal{P}_C(\mu_0, \mu_1)$. We also briefly outline the connection between this result and a problem in the robust hedging of financial derivatives. Finally, in Sections 4 and 5, we show that $\nu^{\#}$ is an element of $\mathcal{P}_C$ and hence that it is a greatest lower bound. At first reading of these final two sections the reader is invited to think of measures μ_i which are discrete as this frequently simplifies the analysis. Note that even in this case, the law $\nu^{\#}$ is not discrete; see Example 2.

2. The main result

The main result is contained in the next theorem. Let μ_0 and μ_1 be two centered probability measures on $\mathbb{R}$ satisfying the inequality (1.1) (i.e. $\mathcal{M}_C(\mu_0, \mu_1)$ is then non-empty). For $i = 0, 1$ we set

$$c_i(x) = \mathbf{E}(M_i - x)^+ = \int_{(x,\infty)} (u - x)\, \mu_i(du) \tag{2.1}$$

for $x \in \mathbb{R}$ and from (1.1) it follows that $c_1(x) \geq c_0(x)$. Hence the function

$$c(x) = c_1(x) - c_0(x) \tag{2.2}$$

is non-negative. Define

$$\Gamma(x) = \mu_1((-\infty, x)) - \sup_{y<x} \frac{c(x) - c(y)}{x - y} \ . \tag{2.3}$$

Theorem 2.1. Γ *is a left continuous distribution function. Further for any continuous martingale* $(M_t)_{0 \leq t \leq 1} \in \mathcal{M}_C(\mu_0, \mu_1)$ *, and for any* $x \in \mathbb{R}$ *we have that*

$$\mathbf{P}(S < x) \leq \Gamma(x) \ .$$

Moreover there exists a continuous martingale $(M_t^{\#})_{0 \leq t \leq 1} \in \mathcal{M}_C(\mu_0, \mu_1)$ *with maximum* $S^{\#}$ *for which* $\mathbf{P}(S^{\#} < x) = \Gamma(x)$ *for each* $x \in \mathbb{R}$ *.*

Corollary 2.2. *Define the probability measure* $\nu^{\#} = \nu^{\#}(\mu_0, \mu_1)$ *by*

$$\nu^{\#}((-\infty, x)) = \Gamma(x) \ . \tag{2.4}$$

Then $\nu^{\#}$ *is a greatest lower bound for* $\mathcal{P}_C(\mu_0, \mu_1)$ *. In particular, for all* $\nu \in \mathcal{P}_C(\mu_0, \mu_1)$ *we have* $\nu^{\#} \preceq \nu$ *and* $\nu^{\#} \in \mathcal{P}_C(\mu_0, \mu_1)$ *.*

Before we prove the theorem in later sections we will first describe the construction of $(M_t^{\#})$ and look at some examples to make the construction clearer. For this we need some notation. Let F_i be the distribution function associated with μ_i . For $x \in \mathbb{R}$ we define

$$\gamma(x) = \sup_{y<x} \frac{c(x) - c(y)}{x - y} \ . \tag{2.5}$$

The two functions $c_i(x)$ are convex and hence the left-hand derivate of $c(x)$ exists and is given by $c'_-(x) = F_1(x-) - F_0(x-)$. If the supremum in (2.5) is not attained then $\gamma(x) = c'_-(x)$. We define the function $x \mapsto g(x)$ as follows. For $x \in \mathbb{R}$, let $g(x) \leq x$ be the maximal value where the supremum in (2.5) is attained and if the supremum is not attained $g(x) = x$. Note that in the cases $\gamma(x) = c'_-(x)$ then $g(x) = x$. See Figure 1.

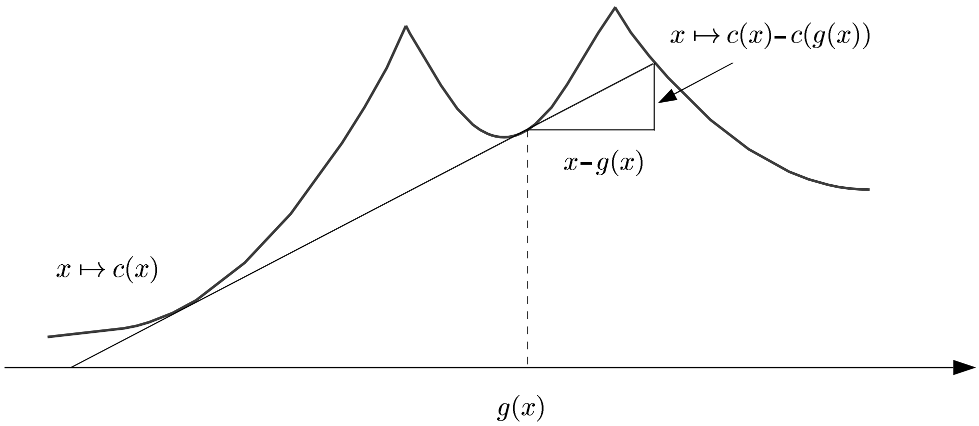

Figure 1. The construction of $\gamma(x)$ and $g(x)$ involves finding a tangent to $c(\cdot)$ which crosses $c(\cdot)$ at x and which is supporting for $c(\cdot)$ to the left of x . $\gamma(x)$ is the slope of the tangent and $g(x)$ is the point supporting the tangent. If there is more than one point supporting the tangent, as in this figure, then $g(x)$ is the largest such point.

With the above notation we can describe the martingale $(M_t^{\#})$. On some suitable sample space define the three elements

- A random variable B_0 with law μ_0 .
- A random variable G with law

$$\mathbf{P}(G \geq s \,|\, B_0 = r) = \exp\left(- \int_{(r,s)} \frac{F_1^c(du)}{F_0(u-) - \Gamma(u)} \right) \prod_{u \in [r,s)} \left(1 - \frac{\Delta F_1(u)}{F_0(u-) - \Gamma(u)} \right)^+$$

  for $s > r$, where F_1^c is the non-atomic part of F_1 . At present we do not exclude the possibility $G = \infty$.
- A Brownian motion $(W_t)_{t \geq 0}$ independent of B_0 and G .

Then $B_t = B_0 + W_t$ is a Brownian motion with initial law μ_0 . Let $S_t = \max_{0 \leq r \leq t} B_r$ and define the stopping times

$$\begin{aligned} \tau_G &= \inf\{\, t > 0 \,:\, S_t \geq G \,\} \\ \tau_g &= \inf\{\, t > 0 \,:\, B_t \leq g(S_t) \,\} \\ \tau &= \tau_G \wedge \tau_g \,. \end{aligned}$$

In later sections we will prove that B_τ has law μ_1 and S_τ has law $\nu^{\#}$. See Figure 2 for a picture of the stopping times. Then $M_t^{\#}$ is a time change of $B_{t \wedge \tau}$ and is given by

$$M_t^{\#} = B_{\frac{t}{1-t} \wedge \tau} \tag{2.6}$$

for $t \leq 1$. This construction involves the use of independent randomisation using the random variable G . For comments on the necessity of such randomisation see Remark 2.3 below. We begin however with some examples of the construction.

Example 1. Let $\mu_0 = \delta_0$ and μ_1 is the uniform distribution on $[-1, 1]$. Then we compute that

$$c(x) = \left(\tfrac{1}{4}(1-x)^2 + x \right) \mathbf{1}_{(-1,0)}(x) + \tfrac{1}{4}(1-x)^2 \, \mathbf{1}_{[0,1)}(x)$$

and $g(x) = x - 2\sqrt{x}$ for $0 < x \leq 1$ and $g(x) = x$ elsewhere (see Figure 3). This example is also studied in Perkins [12].

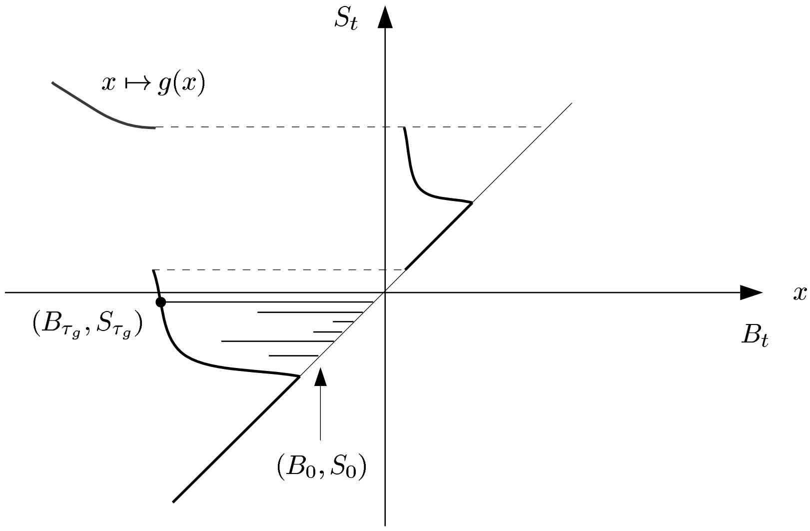

Figure 2. Describing stopping times in (B_t, S_t) plane. The horizontal lines to the left of the line $y = x$ are representations of excursions down from the maximum of Brownian motion.

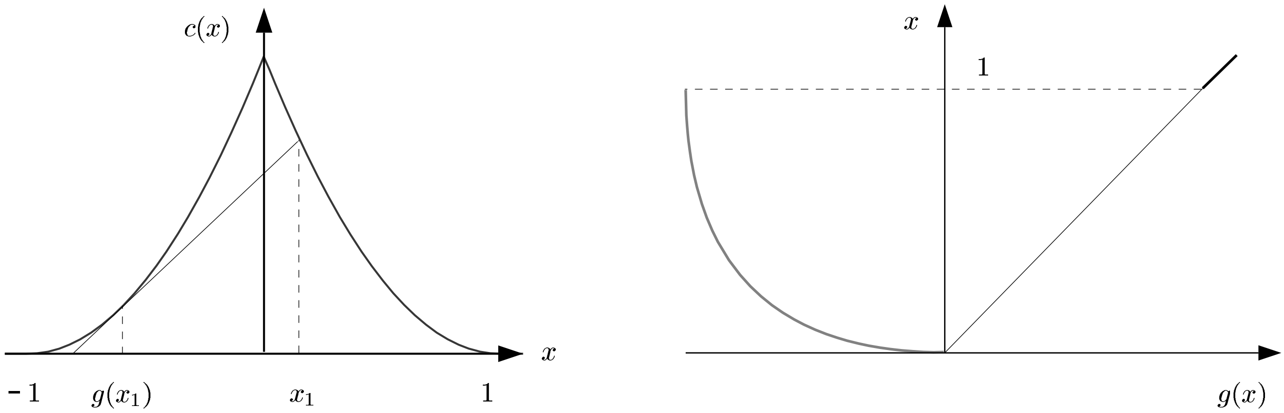

Figure 3. To the left a drawing of $c(x)$ in Example 1. The slope of the tangent at $g(x_1)$ is $\gamma(x_1)$. To the right a drawing of $g(x)$ in Example 1.

Example 2. Let μ_0 be the uniform measure on $\{-1, 1\}$ and let μ_1 have atoms at $-2, 0, 2$ with probability $p, 1-2p, p$ respectively, where $\frac{1}{4} < p < \frac{1}{2}$. Then we compute that

$$\begin{aligned} c(x) =& p(2+x)\,\mathbf{1}_{(-2,-1)}(x) + \big(2p - \tfrac{1}{2} - (\tfrac{1}{2} - p)x\big)\,\mathbf{1}_{[-1,0)}(x) \\ &+ \big(2p - \tfrac{1}{2} + (\tfrac{1}{2} - p)x\big)\,\mathbf{1}_{[0,1)}(x) + p(2-x)\,\mathbf{1}_{[1,2)}(x) \ . \end{aligned}$$

If $\frac{3}{8} \le p < \frac{1}{2}$ then $g(x) = -2$ if $-1 < x \le 2$ and $g(x) = x$ elsewhere. If $\frac{1}{4} < p < \frac{3}{8}$ then

$$g(x) = \begin{cases} -2 & \text{if } \ -1 < x \le 0 \ \text{ and } \ \frac{2}{8p-1} < x \le 2 \\ 0 & \text{if } \ 1 < x \le \frac{2}{8p-1} \\ x & \text{elsewhere} \ . \end{cases}$$

The cases are illustrated in Figure 4 and 5.

Example 3. This is an example to show that the function g can get complicated with even simple expressions for μ_0 and μ_1 . Let μ_0 be the uniform measure on $[-2, -1] \cup [1, 2]$

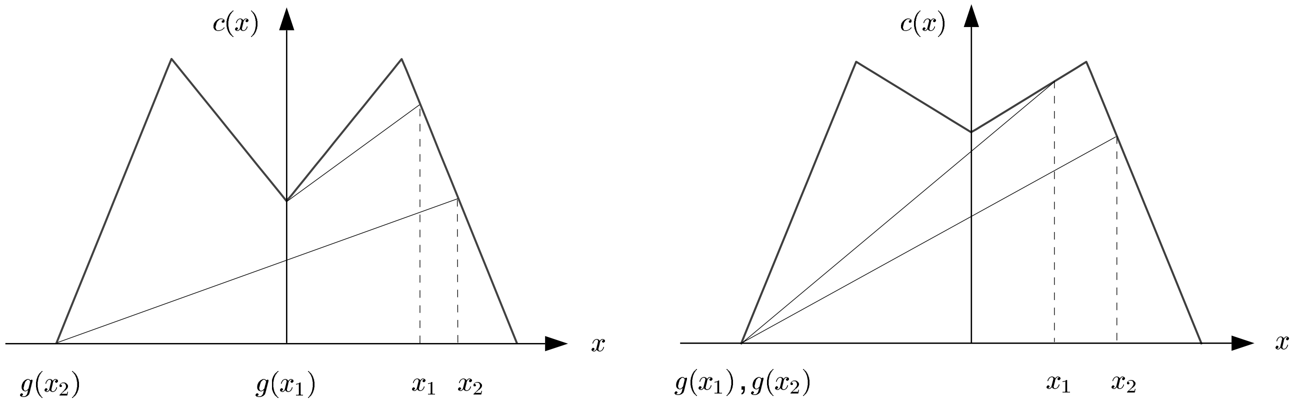

Figure 4. Two drawings of c for different parameter values in Example 2. The left picture is for $p = 1/3$ and the right picture represents $p = 4/10$.

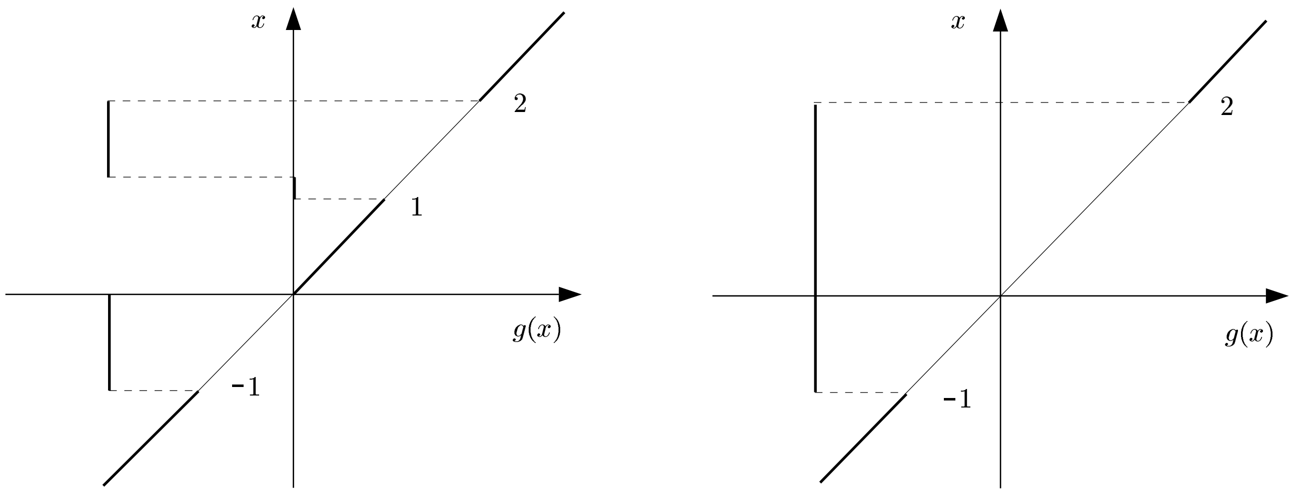

Figure 5. Two drawings of g for different parameter values in Example 2. The left picture is for $p = 1/3$ and the right picture represents $p = 4/10$.

and μ_1 be the uniform measure on $[-3, -2] \cup [-\frac{1}{2}, \frac{1}{2}] \cup [2, 3]$. The functions c and g are illustrated in Figure 6; γ and Γ in Figure 7. Note that for x values in the range of $[\frac{1}{8}, 1]$ we have that $g(x) = x$.

Remark 2.3. Perkins [12] has studied the problem under consideration under the assumption that $\mu_0 = \delta_0$. Although it is defined in a different fashion our function g is exactly $-\gamma_+$ in the notation of [12]. In the Perkins construction the stopping time τ_G is replaced by a stopping time of the form

$$\tau_{\gamma_-} = \inf \{ t > 0 \, : \, B_t \geq \gamma_-(\min_{0 \leq u \leq t} B_u) \}$$

where γ_- is a positive decreasing function. Except when μ_1 has an atom at 0 (i.e. μ_0 and μ_1 have a simultaneous atom) the Perkins construction gives a method of constructing a Skorokhod embedding of the law μ_1 using a stopping time which is adapted to the Brownian motion. The Perkins construction has the property of minimising the law of the maximum of $(B_{t \wedge \tau})_{t \geq 0}$. Furthermore, in the case where $\mu_0 = \delta_0$ this construction has the remarkable additional property that it simultaneously minimises the laws of both $\sup_{0 \leq t \leq \tau} B_t$ and $-\inf_{0 \leq t \leq \tau} B_t$ (Perkins [12, 13]).

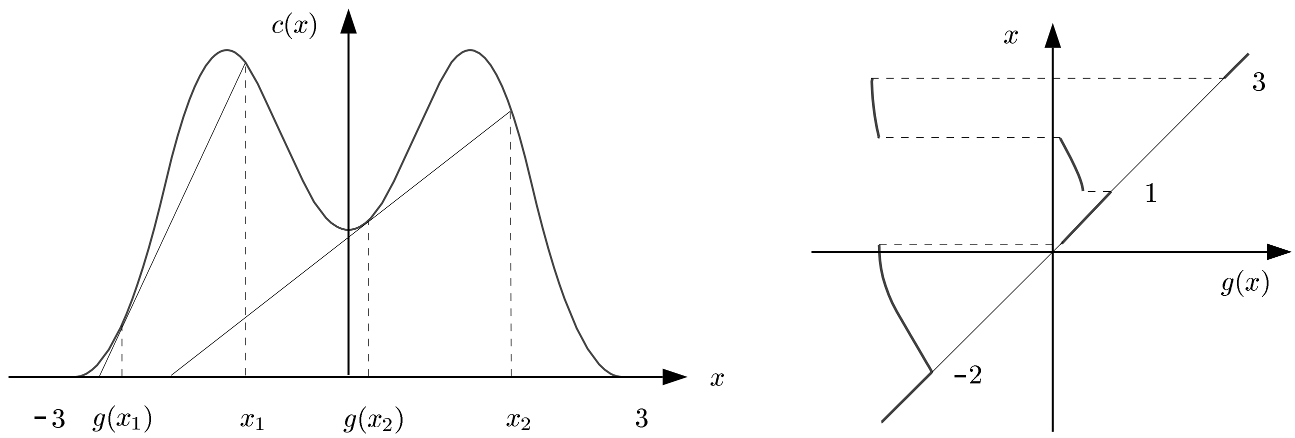

Figure 6. To the left a drawing of $c(x)$ in and to the right a drawing of $g(x)$ in Example 3.

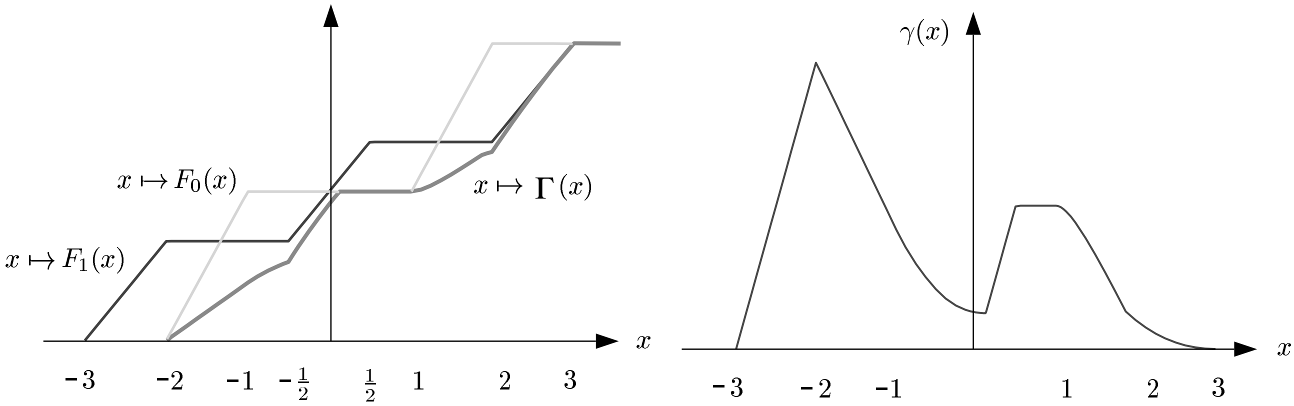

Figure 7. The left diagram represents γ in Example 3. The right hand diagram shows the distribution functions F_0 , F_1 , Γ for this example. Note that $\Gamma(u) \le \min\{F_0(u-)\,, F_1(u-)\}$.

We believe that, using the ideas of Perkins, it should be possible to construct an adapted stopping time for the case $\mu_0 \neq \delta_0$ provided μ_0 and μ_1 have no atoms in common. However, since in general some independent randomisation (represented by the random variable G) is necessary, we have not pursued this direction of research. Moreover, by considering the form of the optimal martingale in Example 2 (with $p = 3/8$), we can see that it is not possible with general starting measures to simultaneously minimise

$$\sup_{0\le t\le 1} M_t \quad \text{and} \quad -\inf_{0\le t\le 1} M_t \ .$$

In his paper Perkins also makes some comments about the problem with general starting measure [12, p220-222]. These comments are predicated on an erroneous claim (3.35) which allows the problem to be reduced to that with $M_0 \sim \delta_0$, but which is in conflict with (1.1). This explains why we reach a different conclusion.

3. The lower bound

The first step in the proof of Theorem 2.1 is to verify that Γ is indeed a lower bound.

Lemma 3.1. *For any* $(M_t) \in \mathcal{M}_C(\mu_0, \mu_1)$ *we have that* $\mathbf{P}(S \geq x) \geq 1 - \Gamma(x)$ *for* $x \in \mathbb{R}$, *where* Γ *is given in* (2.3).

Proof. Let x be fixed. Suppose that $y < x$ then we have the inequality

$$\begin{aligned} \mathbf{1}_{\{S \geq x\}} \geq \mathbf{1}_{\{M_1 \geq x\}} &+ \frac{(M_1 - x)^+}{x - y} - \frac{(M_0 - x)^+}{x - y} - \frac{(M_1 - y)^+}{x - y} + \frac{(M_0 - y)^+}{x - y} \\ &+ \mathbf{1}_{\{y < M_0 < x\}} \frac{M_1 - M_0}{x - y} + \mathbf{1}_{\{S \geq x\}} \mathbf{1}_{\{y < M_0 < x\}} \frac{x - M_1}{x - y} \end{aligned} \tag{3.1}$$

which can be verified on a case by case basis. Since (M_t) is a continuous martingale we have equality in Doob's submartingale inequality and hence

$$\mathbf{E}\left(\frac{x - M_1}{x - y} \, ; \, S \geq x \, , y < M_0 < x \right) = 0 \, .$$

By taking expectation in (3.1) and using martingale property we have that

$$\mathbf{P}(S \geq x) \geq \mathbf{P}(M_1 \geq x) + \frac{c(x) - c(y)}{x - y}$$

for any $y < x$ and the result follows. □

Remark 3.2. The above proof has a financial interpretation in the pricing of a forward start digital option (see [8] and [3] for greater details). Let (M_t) denotes the price process of an asset and suppose for simplicity that there are zero-interest rates and no transaction costs. From the general theory of mathematical finance it follows that the fair price of an European call option with strike x and maturity T is $\mathbf{E}(M_T - x)^+$, where the expectation is taken with respect to the martingale measure. Thus for pricing purpose we may assume that (M_t) is a martingale.

To fit in with previous notation, suppose the current time is -1 and $T = +1$. Suppose we know the call prices at times zero and one for this asset. Then we can derive the laws μ_0 and μ_1 of M_0 and M_1 respectively, under the pricing measure.

Consider the digital option on sale at time -1 which pays one unit if the value of the asset is above the barrier x at any time in the period $[0, 1]$, i.e. the payoff is given by

$$\mathbf{1}_{\{\max_{0 \leq t \leq 1} M_t \geq x\}} \, .$$

If we assume that the price process is continuous then from the above lemma we have that

$$\mathbf{P}(\max_{0 \leq t \leq 1} M_t \geq x) \geq \mathbf{P}(M_1 \geq x) + \sup_{y < x} \frac{[c_1(x) - c_0(x)] - [c_1(y) - c_0(y)]}{x - y}$$

where $c_i(x) = \mathbf{E}(M_i - x)^+$ is the price of a call option with strike x and maturity i .

The inequality (3.1) can be used to motivate a hedging strategy. Initially (at time -1) we fix any $y < x$ and buy a binary option with payoff $\mathbf{1}_{\{M_2 \geq x\}}$, buy $1/(x - y)$ maturity 1 calls with strike x , sell $1/(x - y)$ maturity 0 calls with strike x , sell $1/(x - y)$ maturity 1 calls with strike y , and buy $1/(x - y)$ maturity 0 calls with strike y . This is the static part of the hedge and costs

$$\mu_1([x, \infty)) + \frac{[c_1(x) - c_0(x)] - [c_1(y) - c_0(y)]}{x - y} \, .$$

For the dynamic part of the hedge we proceed as follows. If the underlying at time 0 is lower or equal y or greater or equal x we do nothing. If the underlying at time 0 is between y

and x we buy $1/(x-y)$ units of the underlying and if the underlying reaches the level x we sell $1/(x-y)$ units of the underlying.

From the inequality (3.1) we have that for each y this is a sub-replicative strategy. The cost of the strategy is

$$\mu_1([x,\infty)) + \frac{[c_1(x)-c_0(x)]-[c_1(y)-c_0(y)]}{x-y}$$

which is a lower bound on the price of a digital option. Since $y<x$ is arbitrary the greatest lower bound on the price of a digital option is

$$\mu_1([x,\infty)) + \sup_{y<x} \frac{[c_1(x)-c_0(x)]-[c_1(y)-c_0(y)]}{x-y} . \tag{3.2}$$

If the digital option is offered for sale below this price, then arbitrage profits can be made. Further this analysis is completely independent of the model for the behaviour of the underlying asset. The only assumption that has been made is that the price process is continuous.

Of course the digital option may trade for a price above the bound in (3.2). However the result of Theorem 2.1 is that if the ask price is above the bound (3.2) then it is not possible to create riskless profits unless further assumptions about the dynamics of the price process are made (for instance, that the price process is geometric Brownian motion).

4. **Some preliminary lemmas**

In this section we state some technical results which will be required in the sequel. Some of the proofs are relegated to an appendix, although we try to explain intuitively why they must be true.

Recall the definitions of Γ, γ and g:

$$\Gamma(x) = \mu_1((-\infty,x)) - \gamma(x)$$

$$\gamma(x) = \sup_{y<x} \frac{c(x)-c(y)}{x-y} \tag{4.1}$$

and $g(x)$ is the value of y where the supremum in (4.1) is attained. If the supremum is not attained then we set $g(x)=x$. If the supremum is attained at more than one value of y then we choose the largest (or more precisely the supremum) of the candidate values.

Lemma 4.1. *The function* $x \mapsto \gamma(x)$ *is positive, left-continuous and has no downward jumps.*

Proof. This is a standard piece of analysis given the fact that the left derivative of c exists, is bounded, and indeed equals $F_1(x-)-F_0(x-)$. That it must be true is best seen by drawing a picture, and recalling the intuition that γ represents the gradient of a supporting tangent. See Figure 1. □

Now we prove one of the statements in Theorem 2.1, namely that the candidate law Γ is indeed (a left-continuous version of) a distribution function.

Proposition 4.2. $x \mapsto \Gamma(x)$ *is a left-continuous distribution function, i.e.* Γ *is increasing, left-continuous and satisfies* $\Gamma(-\infty) = 1-\Gamma(+\infty) = 0$. *Further,* $\Gamma(x) \le F_0(x-) \wedge F_1(x-)$ *and* $\Delta\Gamma(x) \le \Delta F_1(x)$.

Proof. From Lemma 4.1 and the representation $\Gamma(x) = F_1(x-) - \gamma(x)$ it follows that Γ is left-continuous and $\Delta\Gamma(x) \le \Delta F_1(x)$. Note further that $\gamma(x) \ge 0 \vee (F_1(x-) - F_0(x-))$ and hence $\Gamma(x) \le F_0(x-) \wedge F_1(x-)$. It is clear that $\gamma(\pm\infty) = 0$ (for example by Dominated Convergence) so to complete the proof we only need to verify that Γ is increasing.

By (2.1) and (2.2) we have the following expression for Γ ,

$$\Gamma(x) = F_0(x-) - \sup_{y<x} \int_{(y,x)} \frac{u-y}{x-y} (\mu_0(du) - \mu_1(du)) \ . \tag{4.2}$$

Fix $y > x$. With the above observations we have the following:
Case 1: $\gamma(y) = c'_-(y)$. Then $\Gamma(y) = F_0(y-) \ge F_0(x-) \ge \Gamma(x)$.
Case 2: $\gamma(y) > c'_-(y)$ *and* $g(y) \le x$. Then from (4.2)

$$\begin{aligned}
\Gamma(y) &= F_0(y-) - \int_{(g(y),y)} \frac{u-g(y)}{y-g(y)} \mu_0(du) + \int_{(g(y),y)} \frac{u-g(y)}{y-g(y)} \mu_1(du) \\
&\ge F_0(y-) - \int_{(g(y),x)} \frac{u-g(y)}{y-g(y)} \big(\mu_0(du) - \mu_1(du)\big) - \int_{[x,y)} \mu_0(du) \\
&= F_0(x-) - \int_{(g(y),x)} \frac{u-g(y)}{y-g(y)} (\mu_0(du) - \mu_1(du)) \\
&\ge F_0(x-) - \sup_{z<x} \int_{(z,x)} \frac{u-z}{x-z} (\mu_0(du) - \mu_1(du)) = \Gamma(x) \ .
\end{aligned}$$

Case 3: $\gamma(y) > c'_-(y)$ *and* $x < g(y)$. From the first line in the previous case

$$\begin{aligned}
\Gamma(y) &\ge F_0(y-) - \int_{(g(y),y)} \frac{u-g(y)}{y-g(y)} \mu_0(du) \\
&\ge F_0(y-) - \int_{(g(y),y)} \mu_0(du) \\
&= F_0(g(y)) \ge F_0(x-) \ge \Gamma(x) \ .
\end{aligned}$$

Hence Γ is increasing. □

The conclusion is that $\nu^\#$ is a probability measure which, by Lemma 3.1 is a lower bound for $\mathcal{P}_C$. We summarise this in a proposition.

Proposition 4.3. *Let $(M_t)_{0\le t\le 1}$ be a continuous martingale with initial law μ_0 and terminal law μ_1 . Let ν be the law of the maximum process S . Then $\nu^\# \preceq \nu$, i.e. for all $\nu \in \mathcal{P}_C(\mu_0, \mu_1)$ we have that $\nu^\# \preceq \nu$.*

It remains to show that $\nu^\# \in \mathcal{P}_C(\mu_0, \mu_1)$. This is the subject of the next section. For the remainder of this section we state further lemmas, beginning with one on the properties of g .

Lemma 4.4. *The function $x \mapsto g(x)$ has the following properties.*
(i) *If $z \ge x$, then either $g(z) \le g(x)$ or $g(z) \ge x$.*
(ii) *If $g(x) < x$, then $c'_-(g(x)) \le \gamma(x) \le c'_+(g(x))$.*
(iii) *If $g(x) = x$ then $F_0(x-) = \Gamma(x)$.*

Proof. These statements are best understood using a picture; recall Figure 1. (ii) follows from interpretation of γ as the gradient of the tangent to c at $g(x)$, and (iii) is true by l'Hôpital's rule. □

It follows from the lemma that the typical behaviour of g is that either $g(x) = x$, or $g(x) < x$ and g is decreasing. In fact if g increases then it must increase to the diagonal.

Lemma 4.5. *We have the following properties.*
(i) *If* $x_n \downarrow x$, *with* $g(x_n) \geq x$ *then* $\Gamma(x+) = F_0(x)$.
(ii) *If* $g(x) < x$ *over an interval* (y,z) , *and if* $g(z-) \equiv \lim_{u\uparrow z} g(u)$, *then* $g(z) = g(z-)$.

Proof. (i): By Lemma 4.4(ii) we have $c'_-(x_n) \leq \gamma(x_n) \leq c'_-(x_n) \vee \{\sup_{y\in[x,x_n]} c'_+(y)\}$ and so $\gamma(x_n) \to c'_+(x) = F_1(x) - F_0(x)$.
(ii): g is decreasing over the interval (y,z) and so $g(z-)$ exists and $g(z-) < z$. Further, Lemma 4.4(i) shows that either $g(z) = z$ or $g(z) \leq g(z-)$. The left-continuity of γ implies

$$\gamma(z) = \gamma(z-) = \lim_{x\uparrow z} \frac{c(x) - c(g(x))}{x - g(x)} = \frac{c(z) - c(g(z-))}{z - g(z-)} \ .$$

Thus $g(z-)$ attains the supremum in (4.1) so, by the definition of g as the maximal solution we have $g(z-) \leq g(z) < z$. This means that the the first case cannot occur and by the second case we must have $g(z) = g(z-)$. □

We set

$$A^+ = \{\, x \in \mathbb{R} \,|\, \Gamma(x+) = F_0(x) \,\} \quad \text{and} \quad A^- = \{\, x \in \mathbb{R} \,|\, \Gamma(x) = F_0(x-) \,\}$$

and let $A = A^+ \cup A^-$. Note that the set A is closed. A will play a special role in the next section where we show that for the optimal martingale, if $M_0 < x \in A$ then necessarily $S \leq x$ also.

Lemma 4.6. *If* $x \notin A^+$ *then* γ *is continuous at* x *and decreasing to the right of* x . *Hence* $\Delta\Gamma(x) = \Delta F_1(x)$.

Proof. If $x \notin A^+$ then by the previous lemma for all y in some interval $(x, x+\delta)$ we have $g(y) < x < y$.

Since $x \notin A^+$, we must have $c'_+(x) < \gamma(x)$, and for y in some smaller interval $(x, x+\delta')$ we have $c(y) < c(x) + (y-x)\gamma(x)$. Then

$$\begin{aligned}
\gamma(y) &\leq \frac{c(x) + (y-x)\,\gamma(x) - c(g(y))}{y - g(y)} \\
&= \frac{c(x) - c(g(y))}{x - g(y)}\,\frac{x - g(y)}{y - g(y)} + \frac{y - x}{y - g(y)}\,\gamma(x) \\
&\leq \frac{x - g(y)}{y - g(y)}\,\gamma(x) + \frac{y - x}{y - g(y)}\,\gamma(x) = \gamma(x) \ .
\end{aligned}$$

Thus γ is decreasing to the right of x .

Right continuity, and hence continuity will follow if $\lim_{y\downarrow x} \gamma(y) \geq \gamma(x)$. Suppose first that $g(x+) < x$. Then

$$\gamma(y) \geq \frac{c(y) - c(g(x))}{y - g(x)} \to \gamma(x) \ .$$

Conversely, if $g(x) = x$ then for $\delta > 0$

$$\gamma(y) \geq \frac{c(y) - c(x-\delta)}{y - (x-\delta)} \to \frac{c(x) - c(x-\delta)}{\delta} \ .$$

As $\delta \downarrow 0$ we recover $\gamma(x+) \geq c'_-(x) = \gamma(x)$. □

Remark 4.7. If $x \notin A^-$ then it is easy to show that γ is decreasing to the left of x .

Lemma 4.8. *We have the following properties.*

(i) *If* I *is an open interval disjoint from* A *then*

$$\int_I \frac{du}{u-g(u)} = -\int_I \frac{d\gamma(u)}{F_0(u-)-\Gamma(u)} \ .$$

(ii) *Further if* $g(x)<x$ *then* γ *satisfies*

$$\gamma(x) = \int_{\{u>x\,,\,g(u)<g(x)\}} \frac{F_0(u-)-\Gamma(u)}{u-g(u)}\, du \ .$$

Proof. γ is decreasing, and so γ is differentiable outside a Lebesgue null set. The main task is to show that γ is absolutely continuous. For full proofs see the appendix. □

We close this section with a couple of lemmas concerning distribution functions, the proofs of which are in the appendix. We denote the distribution function of a measure ϕ by F_ϕ , the atoms by ΔF_ϕ and the non-atomic part of the distribution by F_ϕ^c .

Lemma 4.9. *Let* π,ρ *be two measures on* $\mathbb{R}$ *satisfying* $\pi \preceq \rho$. *Let*

$$J(x) = F_\pi(x-) - F_\rho(x-) \ .$$

If F_π *and* F_ρ *have no simultaneous jumps on the interval* $[u,y)$ *and if* J *is positive over this interval, then*

$$\frac{1}{J(y)} \exp\Big(-\int_u^y \frac{F_\rho^c(dv)}{J(v)}\Big) \prod_{v\in[u,y)} \Big(1-\frac{\Delta F_\rho(v)}{J(v)}\Big)$$
$$= \frac{1}{J(u)} \exp\Big(-\int_u^y \frac{F_\pi^c(dv)}{J(v)}\Big) \Big(\prod_{v\in[u,y)} \Big(1+\frac{\Delta F_\pi(v)}{J(v)}\Big)\Big)^{-1} \ .$$

Lemma 4.10. *Let* π,ρ *and* J *be as above. Fix* $y \in \mathbb{R}$, *and define* $z^\# = \sup_{v<y}\{\, v \mid J(v)=0$ *or* $J(v+)=0\,\}$. *Suppose* $z^\# < y$. *Then*

$$\int_{[z^\#,y)} \frac{F_\pi(du)}{J(u)} \exp\Big(-\int_u^y \frac{F_\pi^c(dv)}{J(v)}\Big) \prod_{v\in[u,y)} \Big(1+\frac{\Delta F_\pi(v)}{J(v)}\Big)^{-1} = 1$$

5. The minimum maximum is attained

In this section we construct a martingale $(M_t^\#)$ which is an element of $\mathcal{M}_C(\mu_0,\mu_1)$ and has the property that it's maximum S has the law $\nu^\#$ from (2.4). Thus, not only is $\nu^\#$ a lower bound for $\mathcal{P}_C(\mu_0,\mu_1)$ but also $\nu^\# \in \mathcal{P}_C(\mu_0,\mu_1)$.

The key idea in the construction of $(M_t^\#)$ is to exhibit the martingale as the solution of a Skorokhod embedding problem (see [7] and [17]). Let $(B_t)_{t\ge 0}$ be a Brownian motion with initial law μ_0 . The problem is to find a stopping time τ satisfying B_τ has the law μ_1 and $\sup_{0\le t\le\tau} B_t$ has the law $\nu^\#$. Then we can define $(M_t^\#)$ as a time change of (B_t) by

$$M_t^\# = B_{\frac{t}{1-t}\wedge\tau} \ . \tag{5.1}$$

$(M_t^\#)$ is a true martingale and not just a local martingale provided that $(B_{t\wedge\tau})_{t\ge 0}$ is uniform integrable.

Before we outline the construction we wish to make one simplifying observation. If μ_0 and μ_1 both contain an atom of size at least m at some point x , then we can, and do, insist that the martingale (M_t) remains constant at x over $[0,1]$ on an appropriate part of the sample space (randomising at time 0 if necessary). This means that our construction we only have to deal with measures μ_0 and μ_1 with no common atoms.

Recall the definition of g from earlier sections and that in Section 2 we defined a Brownian motion (B_t) with initial law μ_0 and a random variable G which depended on (B_t) only through the initial value B_0 :

$$\mathbf{P}(G \geq s \,|\, B_0 = r) = \exp\left(-\int_{(r,s)} \frac{F_1^c(du)}{F_0(u-) - \Gamma(u)} \right) \prod_{u \in [r,s)} \left(1 - \frac{\Delta F_1(u)}{F_0(u-) - \Gamma(u)} \right)^+ . \tag{5.2}$$

Let $S_t = \max_{0 \leq r \leq t} B_r$ and define the stopping times

$$\tau_G = \inf\{\, t > 0 \,:\, S_t \geq G \,\}$$
$$\tau_g = \inf\{\, t > 0 \,:\, B_t \leq g(S_t) \,\} .$$

Set $\tau = \tau_G \wedge \tau_g$.

We have to prove two identities in law, namely that $B_\tau \sim \mu_1$ and $S_\tau \sim \nu^{\#}$. We consider the second identity first, but we begin with a useful lemma. Recall the definitions of the sets A^+, A^- and A before Lemma 4.6.

Lemma 5.1. *We have the following properties.*

(i) *Suppose* $x \in A^-$. *If* $B_0 < x$ *then* $S_\tau < x$.
(ii) *Suppose* $x \in A^+$. *If* $B_0 \leq x$ *then* $S_\tau \leq x$.

Note that both these statements should both be interpreted in an almost sure sense.

Proof. If $B_0 = r < x$ and let H_z denote the first hitting time by (B_t) of level $z > r$.
Case 1: $B_0 = r < x$ *and* (r,x) *is disjoint from* A . If (r,x) is disjoint from A then certainly by Lemma 4.5(i), $g(z) < z$ on $(r,x]$. Suppose $x \in A^-$. We show

$$\int_{(r,x)} \frac{F_1^c(du)}{F_0(u-) - \Gamma(u)} - \sum_{u \in [r,x)} \log \left(1 - \frac{\Delta F_1(u)}{F_0(u-) - \Gamma(u)} \right)^+ = \infty \tag{5.3}$$

so that $\mathbf{P}(G \geq x \,|\, B_0 = r) = 0$ and $\tau_G < H_x$, almost surely. We prove this is the case where F_0 and F_1 have no atoms, but the general case is very similar and just involves additional terms written as sums as well as integrals.

By considering

$$-\infty = \int_{(\cdot\,,x)} d\big[\ln\big(F_0(u) - \Gamma(u) \big) \big] = \int_{(\cdot\,,x)} \frac{dF_0(u)}{F_0(u-) - \Gamma(u)} - \int_{(\cdot\,,x)} \frac{d\Gamma(u)}{F_0(u-) - \Gamma(u)}$$

we deduce that this final integral must be infinite. Also, by Lemma 4.8,

$$\int_{(\cdot\,,x)} \frac{du}{u - g(u)} = -\int_{(\cdot\,,x)} \frac{\gamma(du)}{F_0(u-) - \Gamma(u)}$$

and this first integral is finite so that

$$\int_{(\cdot\,,x)} \frac{dF_1(u)}{F_0(u-) - \Gamma(u)} = \int_{(\cdot\,,x)} \frac{d\Gamma(u)}{F_0(u-) - \Gamma(u)} + \int_{(\cdot\,,x)} \frac{\gamma(du)}{F_0(u-) - \Gamma(u)} = \infty .$$

Now suppose $x \in A \setminus A^-$. Then $\Gamma(x+) = F_0(x)$ and $0 < F_0(x-) - \Gamma(x) \le F_0(x) - \Gamma(x+) + \Delta\Gamma(x) \le \Delta F_1(x)$ by the last part of Proposition 4.2. Then $\mathbf{P}(G > x \,|\, B_0 = r) = 0$ and $S_\tau \le x$ (almost surely).

Case 2: $B_0 = r < x$ *and* (r, x) *not disjoint from* A . We show either that there is an interval $(y, z) \subset (r, x)$ disjoint from A with $z \in A$, or that $S_\tau < x$ for some other reason. In the former situation we can apply the results from the previous case to the interval $(\cdot, z)$ to deduce that $S_\tau < z$ or $S_\tau \le z$ as appropriate.

If there is no interval disjoint from A then either there exists $y \in (A \setminus A^-) \cap (r, x)$ or A^- is dense in (r, x) . If $y \in (A \setminus A^-) \cap (r, x)$ then by the final argument in Case 1 applied at y we have $0 < F_0(y-) - \Gamma(y) \le \Delta F_1(y)$ and $\mathbf{P}(G > x \,|\, B_0 = r) = 0$. If A^- is dense in (r, x) then since A is closed $(r, x) \subseteq A$. Either there exists $y \in (A \setminus A^-) \cap (r, x)$, a case already covered, or $(r, x) \subseteq A^-$. Then for $y, z \in (r, x)$ with $y < z$ we have

$$c(z) - c(y) = \int_y^z c'_-(u)\, du = \int_y^z \gamma(u)\, du \ge (z - y)\, \gamma(z) ,$$

the last equality following since γ is decreasing except on A^+ . Thus

$$\gamma(z) \le \frac{c(z) - c(y)}{z - y} \le \sup_{v<z} \frac{c(z) - c(v)}{z - v} = \gamma(z)$$

and since $g(z)$ is the largest value where this supremum is attained we have $g(z) \ge y$. But y is arbitrary so $g(z) = z$. Finally $\tau \le \tau_g \le H_z$ so $S_\tau \le z < x$.

Case 3: $B_0 = x$ *and* $x \in A^+$. If $\Delta F_1(x) > 0$ then $F_0(x-) - \Gamma(x) \le F_0(x) - \Gamma(x+) + \Delta\Gamma(x) \le \Delta F_1(x)$ and $\mathbf{P}(G > x \,|\, B_0 = r) = 0$ and $S_\tau \le x$ almost surely. Otherwise, F_1 is continuous at x and then $\Delta\Gamma(x) = 0$ so that $F_0(x-) \ge \Gamma(x) = \Gamma(x+) = F_0(x)$. In particular $\Delta F_0(x) = \mathbf{P}(B_0 = x) = 0$. □

Lemma 5.2. *Suppose the open interval* (u, y) *is disjoint from* A *, so that* $g(v) < v$ *over the interval. Then*

$$\mathbf{P}(S_{\tau_g} \ge y \,|\, B_0 = u) = \exp\left(- \int_{(u,y)} \frac{dv}{v - g(v)} \right) .$$

Proof. This result is a standard result from excursion theory, see Rogers [16] or Revuz and Yor [14]. However for completeness, and since excursion ideas are the essential insight in the later proofs we provide a full proof.

By Lévy's Theorem $(S_t, S_t - B_t)$ has the same law as $(L_t, |W_t|)$ for Brownian motion (W_t) with local time (L_t) at 0. (To match initial conditions, if $S_0 = B_0 = u$ we define $W_0 = 0$ and $L_0 = u$). Then, since $\tau_g = \inf\{\, t > 0 \,:\, B_t \le g(S_t) \,\}$ we have that τ_g has the same law as T where $T = \inf\{\, t > 0 \,:\, |W_t| \ge L_t - g(L_t) \,\}$.

If N is the Poisson point process of excursions of $|W_t|$ from 0 then the rate of excursions of height at least h is h^{-1} . We say an excursion at local time l is a success if the maximum modulus of the excursion exceeds $l - g(l)$ and then the number of successes before local time $y > u$ is a Poisson random variable with mean α where

$$\alpha = \int_u^y \frac{dl}{l - g(l)} .$$

In particular, the probability that there have been no successes before the local time reaches y is $e^{-\alpha}$. Finally

$$(S_{\tau_g} \ge y \,|\, B_0 = u) \equiv (L_T \ge y \,|\, L_0 = u)$$

and this last event is the event that the first successes occurs after the local time reaches y . □

Proposition 5.3. *We have that* $S_\tau \sim \nu^\#$.

Proof. For $y \in A$ we have $\mathbf{P}(S_\tau < y) = \mathbf{P}(B_0 < y) = \Gamma(y)$, or $\mathbf{P}(S_\tau \le y) = \mathbf{P}(B_0 \le y) = \Gamma(y+)$.

Otherwise, consider $y \notin A$ and define $z^\# = z^\#(y) = \sup_{z<y}\{z \in A\}$. If $z^\# = y$ then by left continuity $\mathbf{P}(S_\tau < y) = \mathbf{P}(B_0 < y) = \Gamma(y)$. So suppose $z^\#(y) < y$. Then

$$\begin{aligned}
\mathbf{P}(S_\tau \ge y) &= \int_{\mathbb{R}} \mathbf{P}(S_\tau \ge y \,|\, B_0 = u)\, \mu_0(du) \\
&= \mathbf{P}(B_0 \ge y) + \int_{[z^\#,y)} \mathbf{P}(S_{\tau_g} \ge y \,|\, B_0 = u)\, \mathbf{P}(G \ge y \,|\, B_0 = u)\, \mu_0(du) \\
&= \mathbf{P}(B_0 \ge y) + \int_{[z^\#,y)} \mu_0(du)\, \exp\left(- \int_{(u,y)} \frac{dv}{v - g(v)} \right) \\
&\qquad \times \exp\left(- \int_{(u,y)} \frac{F_1^c(dv)}{F_0(v-) - \Gamma(v)} \right) \prod_{v \in [u,y)} \left(1 - \frac{\Delta F_1(v)}{F_0(v-) - \Gamma(v)} \right)^+ \\
&= \mathbf{P}(B_0 \ge y) + \int_{[z^\#,y)} \mu_0(du)\, \exp\left(- \int_{(u,y)} \frac{\Gamma^c(dv)}{F_0(v-) - \Gamma(v)} \right) \\
&\qquad \times \prod_{v \in [u,y)} \left(1 - \frac{\Delta \Gamma(v)}{F_0(v-) - \Gamma(v)} \right)
\end{aligned}$$

where in the last equality we have used Lemma 4.8(i), $\Gamma^c = F_1^c - \gamma^c$ and Lemma 4.6.

If we now apply Lemma 4.9 with $F_\rho(u) = \Gamma(u+)$ and $F_\pi(u) = F_0(u)$ then this becomes

$$\begin{aligned}
\mathbf{P}(S_\tau \ge y) = \mathbf{P}(B_0 \ge y) &+ (F_0(y-) - \Gamma(y)) \\
&\times \int_{[z^\#,y)} \frac{\mu_0(du)}{F_0(u-) - \Gamma(u-)} \exp\left(- \int_{(u,y)} \frac{F_0^c(dv)}{F_0(v-) - \Gamma(v)} \right) \\
&\times \prod_{v \in [u,y)} \left(1 + \frac{\Delta F_0(v)}{F_0(v-) - \Gamma(v)} \right)^{-1} .
\end{aligned}$$

Finally, applying Lemma 4.10, again with $\pi = \mu_0$ and $\rho = \Gamma$ we get that

$$\mathbf{P}(S_\tau \ge y) = \mathbf{P}(B_0 \ge y) + \big(F_0(y-) - \Gamma(y)\big) = 1 - \Gamma(y)$$

and the result follows. □

Proposition 5.4. *For the above construction we have that* $B_\tau \sim \mu_1$.

Proof. From the construction we have that if $B_\tau < S_\tau$, then $B_\tau = g(S_\tau) < S_\tau$. This happens if the Brownian motion has an excursion down below the maximum (at s) which reaches $g(s)$. Results from excursion theory (recall Lemma 5.2) give that this happens at rate $(s - g(s))^{-1}$.

Then

$$\begin{aligned}
\mathbf{P}(B_\tau < y) &= \mathbf{P}(S_\tau < y) + \mathbf{P}(S_\tau \ge y\,,\, B_\tau < y) \\
&= \Gamma(y) + \int_{\{u \ge y\,,\, g(u) < y\}} \mathbf{P}(B_0 < u, S_\tau \ge u)\,\frac{du}{u - g(u)} \\
&= \Gamma(y) + \int_{\{u \ge y\,,\, g(u) < y\}} \frac{F_0(u-) - \Gamma(u)}{u - g(u)}\,du \\
&= \Gamma(y) - \gamma(y) = F_1(y-)\ ,
\end{aligned}$$

where Lemma 4.4(i) guarantees that the sets over which we integrate match up and the last line follows from Lemma 4.8(ii). □

For $(M_t^{\#})$ from (2.6) we have as a corollary of the above proposition the main result of the paper.

Theorem 5.5. $(M_t^{\#})_{0 \le t \le 1} \in \mathcal{M}_C(\mu_0, \mu_1)$ *and for any* $\nu \in \mathcal{P}_C(\mu_0, \mu_1)$ *stochastically dominates* $\nu^{\#}$.

Proof. From Lemma 2.3 in [17] (the condition $\mu_0 \sim \delta_0$ being not necessary) it follows that $(B_{(t/(1-t)) \wedge \tau})_{t \ge 0}$ is uniform integrable, and hence $(M_t^{\#})$ is a martingale, provided that $\lim_{x \uparrow \infty} x\,(1 - \Gamma(x)) = 0$. Since $\lim x\,(1 - F_1(x)) = 0$ by Dominated Convergence, it is sufficient to show that $x\,\gamma(x) \to 0$. This is easily shown by considering the cases $g(x) < x/2$, $x/2 \le g(x) < x$ and $g(x) = x$ separately. We assume $x > 0$.

For $g(x) < x/2$ we have

$$x\,\gamma(x) = x\,\frac{c(x) - c(g(x))}{x - g(x)} < 2c(x)\ ,$$

for $x/2 \le g(x) < x$

$$x\,\gamma(x) \le 2g(x)\,c'_+(g(x))$$

and for $x = g(x)$ we have $x\,\gamma(x) = x\,c'_-(x)$. Since both c and $x\,(1 - F_i(x))$ tend to zero, we are done. □

6. Appendix

Proof. (**Lemma 4.8**). (i): On $I \subset A^c$ we must have $g(v) < v$ and $\Gamma(v) < F_0(v-)$ (by Proposition 4.2 and Lemma 4.4(iii)). Further by Lemma 4.6 and Remark 4.7 γ is continuous and decreasing so that $\gamma(dv)$ must exist. We prove that γ is absolutely continuous on I with Radon-Nikodym derivative

$$\frac{\gamma(dv)}{dv} = -\left(\frac{F_0(v-) - \Gamma(v)}{v - g(v)}\right). \tag{6.1}$$

Suppose $v \in I$ is chosen outside a countable set so that F_0, F_1, Γ and the decreasing function g are all continuous at v . Then, for $y > v$,

$$c(y) - c(v) = \int_v^y c'_-(z)\,dz = \int_v^y \big(F_1(z-) - F_0(z-)\big)\,dz \ge \big(y - v\big)\big(F_1(v) - F_0(y)\big)$$

implies that

$$\begin{aligned}0 \geq \gamma(y)-\gamma(v) &\geq \frac{c(y)-c(g(v))}{y-g(v)}-\frac{c(v)-c(g(v))}{v-g(v)}\\ &\geq \frac{c(v)+(y-v)(F_1(v)-F_0(y))-c(g(v))}{y-g(v)}-\frac{c(v)-c(g(v))}{v-g(v)}\\ &= -\frac{c(v)-c(g(v))}{(y-g(v))(v-g(v))}(y-v)+\frac{F_1(v)-F_0(y)}{y-g(v)}(y-v)\end{aligned}$$

and so

$$\liminf_{y\downarrow v}\frac{\gamma(y)-\gamma(v)}{y-v} \geq -\frac{\gamma(v)}{v-g(v)}+\frac{F_1(v)-F_0(v)}{v-g(v)} = -\left(\frac{F_0(v)-\Gamma(v)}{v-g(v)}\right) .$$

We can obtain the reverse inequality by considering

$$\gamma(y)-\gamma(v) \leq \frac{c(y)-c(g(y))}{y-g(y)}-\frac{c(v)-c(g(y))}{v-g(y)} .$$

Hence γ is differentiable outside a countable set T with derivative given by the right term of (6.1). Since the derivate is integrable over the set $I\setminus T$ it follows from (3.27.5) in [9] that γ is absolutely continuous with the right density.

(ii): Fix x . Let $D(x)=\{\,y>x\,|\,g(y)\leq g(x)\,\}$. Then γ restricted to D is easily seen to be strictly decreasing and onto $[0,\gamma(x))$ and hence has a well defined inverse $q_x(\cdot)$. Let $p_x(\cdot)$ be given by $p_x(u)=g(q_x(u))$. Then

$$p_x(u)\leq g(x)<x<q_x(u)$$

and $\gamma(q_x(u))=u$. See Figure 8.

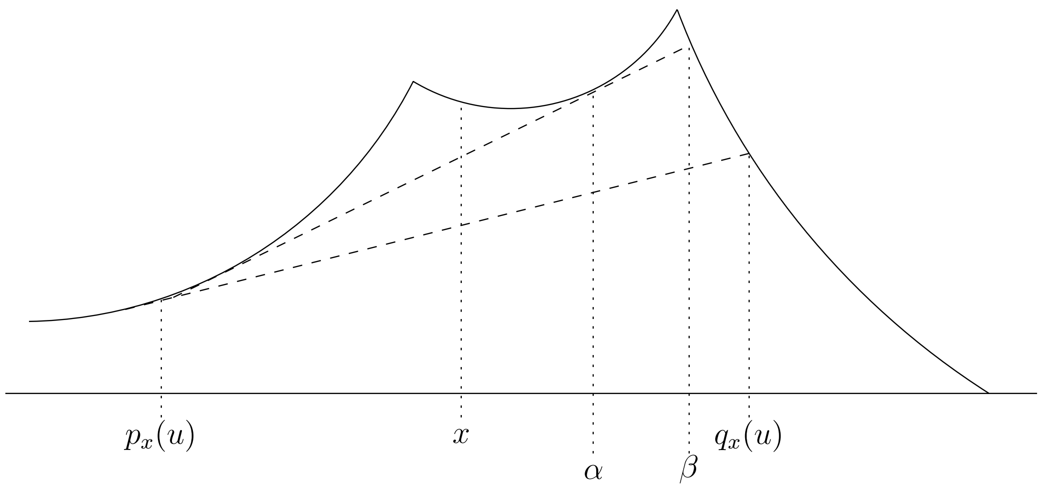

Figure 8. $D(x)$ is the set $\{\,y>x\,|\,g(y)\leq g(x)\,\}$ so that the interval $(\alpha,\beta]$ is disjoint from D . The gradient of the line joining $(p_x(u),c(p_x(u)))$ and $(q_x(u),c(q_x(u)))$ is u .

Then

$$\int_{\{u>x\,,\,g(u)<g(x)\}} \frac{F_0(u-)-\Gamma(u)}{u-g(u)}\,du = \int_{D(x)} d\gamma(u) = \int_{\{\,u\,|\,q_x(u)>x\,\}} d\gamma(q_x(u))$$
$$= \int_0^{\gamma(x)} du = \gamma(x)\ .$$

□

Proof. (**Lemma 4.9**). Denote $J^c(x)=F^c_\pi(x)-F^c_\rho(x)$. Then we have that

$$d[\log J(x)] = \frac{dJ^c(x)}{J(x)} + \log\frac{J(x+)}{J(x)}$$
$$= \frac{dJ^c(x)}{J(x)} + \log\left(1+\frac{\Delta F_\pi(x)}{J(x)}\right) + \log\left(1-\frac{\Delta F_\rho(x)}{J(x)}\right)$$

where, in the last line, we have used the fact that F_π and F_ρ have no common jumps. If we integrate over the set $[y,x)$ we obtain that

$$\log\frac{J(x)}{J(y)} = \int_{(y,x)}\frac{F^c_\pi(du)}{J(u)} - \int_{(y,x)}\frac{F^c_\rho(du)}{J(u)}$$
$$+\sum_{u\in[y,x)}\log\left(1+\frac{\Delta F_\pi(u)}{J(u)}\right) + \sum_{u\in[y,x)}\log\left(1-\frac{\Delta F_\rho(x)}{J(x)}\right)$$

and the result follows easily. □

Proof. (**Lemma 4.10**). Define the function

$$K(x)\equiv K_y(x) = -\int_{[x,y)}\frac{F^c_\pi(dv)}{F_\pi(v-)-F_\rho(v-)} - \sum_{v\in[x,y)}\log\left(1+\frac{\Delta F_\pi(v)}{F_\pi(v-)-F_\rho(v-)}\right)\ .$$

Then we have $K(y)=0$ and

$$K(x)\le -\int_x^y d\big(\log\{F_\pi(v-)-F_\rho(v-)\}\big)$$

where the integral on the right hand side can be taken over either $[x,y)$ or (x,y) . It follows that $K(z^\#)=-\infty$. Then we have

$$d[e^{K(u)}] = e^{K(u)}\,\frac{F^c_\pi(du)}{F_\pi(u-)-F_\rho(u-)} + e^{K(u)}\left(e^{K(u+)-K(u)}-1\right)$$
$$= e^{K(u)}\,\frac{F^c_\pi(du)}{F_\pi(u-)-F_\rho(u-)} + e^{K(u)}\,\frac{\Delta F_\pi(u)}{F_\pi(u-)-F_\rho(u-)}\ .$$

Integrate over the set $[z^\#,y)$ we get that

$$1 = e^{K(y)} - e^{K(z^\#)} = \int_{[z^\#,y)}\frac{F_\pi(du)}{F_\pi(u-)-F_\rho(u-)}\,e^{K(u)}\ .$$

□

References

[1] Azéma, J. *and* Yor, M. (1979). Une solution simple au problème de Skorokhod. *Séminaire de Probabilités* XIII, *Lecture Notes in Math.* **721**, Springer (90-115).
[2] Blackwell, D. *and* Dubins, L.E. (1963). A converse to the dominated convergence theorem. *Illinois J. Math.* **7** (508-514).
[3] Brown, H., Hobson, D.G. *and* Rogers, L.C.G. (2001). Robust hedging of barrier options. *Math. Finance* **11** (285-314).
[4] Chacon, R. *and* Walsh, J.B. (1976). One-dimensional potential embedding. *Séminaire de Probabilités* X, *Lecture Notes in Math.* **511**, Springer (19-23).
[5] Dubins, L.E. (1968) On a theorem of Skorokhod *Ann. Math. Statist.* **39** (2094-2097).
[6] Dubins, L.E. *and* Gilat, D. (1978). On the distribution of the maxima of martingales. *Proc. Amer. Math. Soc.* **68** (337-338).
[7] Hobson, D.G. (1998). The maximum maximum of a martingale. *Séminaire de Probabilités* XXXII, *Lecture Notes in Math.* **1686**, Springer (250-263).
[8] Hobson, D.G. (1998). Robust hedging of the Lookback option. *Finance Stoch.* **2** (329-347).
[9] Hoffmann-Jørgensen, J. (1994). *P*robability with a view toward statistics. Vol. I. Chapman & Hall,
[10] Kertz, R.P. *and* Rösler, U. (1990). Martingales with given maxima and terminal distributions. *Israel J. Math.* **69** (173-192).
[11] Meyer, P.A. (1966). *Probability and Potentials*. Blaidsell.
[12] Perkins, E. (1986). The Cereteli-Davis solution H^1-embedding problem and an optimal embedding in Brownian motion. *Seminar on Stochastic Processes, 1985*, Birkhäuser, (172-223).
[13] Perkins, E. (2000). Personal Communication.
[14] Revuz, D. *and* Yor, M. (1999). *Continuous Martingales and Brownian Motion.* (Third Edition) Springer.
[15] Rogers, L.C.G. (1981). Williams characterisation of the Brownian excursion law: proof and applications. *Séminaire de Probabilités* XV, *Lecture Notes in Math.* **850**, Springer (227-250).
[16] Rogers, L.C.G. (1989). A guide tour through excursions. *Bull. London Math. Soc.* **21** (305-341).
[17] Rogers, L.C.G. (1993). The joint law of the maximum and the terminal value of a martingale. *Probab. Theory Relat. Fields* **95** (451-466).
[18] Rost, H. (1971). The stopping distributions of a Markov process. *Invent. Math.* **14** (1-16).
[19] Strassen, V. (1965). The existence of probability measures with given marginals. *Ann. Math. Statist* **36** (423-439).

David Hobson
Department of Mathematical Sciences
University of Bath
Claverton Down, Bath, BA2 7AY
U.K.
E-mail: dhg@maths.bath.ac.uk http:
http//www.bath.ac.uk/∼masdgh

Jesper Lund Pedersen
Risklab, Department of Mathematics
ETH-Zentrum
CH-8092 Zürich
Switzerland
E-mail: pedersen@math.ethz.ch
http://www.math.ethz.ch/∼pedersen