Risk Management for Derivatives in Illiquid Markets: A Simulation Study

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Abstract

In this paper we study the hedging of derivatives in illiquid markets. More specifically we consider a model where the implementation of a hedging strategy affects the price of the underlying security. Following earlier work we characterize perfect hedging strategies by a nonlinear version of the Black-Scholes PDE. The core of the paper consists of a simulation study. We present numerical results on the impact of market illiquidity on hedge cost and Greeks of derivatives. We go on and offer a new explanation of the smile pattern of implied volatility related to the lack of market liquidity. Finally we present simulations on the performance of different hedging strategies in illiquid markets.

Key words: Option hedging, volatility, illiquid markets, nonlinear Black-Scholes equation, simulation study
JEL classification: G12, G13
Mathematics Subject Classification (2000) 91B28

1 Introduction

In recent years market liquidity has become an issue of high concern in risk management. In particular, risk managers realized that financial models which are based on the assumption that an investor can trade large amounts of an asset without affecting its price (perfectly liquid markets) may fail miserably in circumstances where market liquidity vanishes. This calls for additional research, extending traditional financial models to markets which are not perfectly liquid.

In the present paper we focus on the risk management for derivative securities via dynamic hedging. We study a model of an illiquid market where the implementation of a dynamic hedging strategy has an impact on the price process of the underlying asset. In practice such a feedback effect could for instance arise, if the volume of trading committed to dynamic hedging strategies is relatively large compared to the overall trading volume on the underlying. Obviously, in this context we cannot rely on results from standard derivative asset analysis, where it is always assumed that option hedgers are small investors. Our paper

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builds on the analysis of Frey (2000). Frey considers a model where the asset price process is driven by some exogenous source of randomness (in his case a standard Brownian motion) *and* by the trading strategy of a representative agent who is hedging derivatives. He obtains a formula for the impact of hedging on market volatility and characterizes perfect hedging strategies by a non-linear version of the standard Black-Scholes partial differential equation (PDE).

Here we complement the analysis of Frey (2000) and carry out an extensive simulation study in order to better understand the implications of market illiquidity for derivative asset analysis. In order to solve the nonlinear PDE numerically we implemented an efficient numerical scheme, which is used to study for a number of different payoffs the properties of hedge cost and greeks in our framework. We go on and offer a new explanation for the famous smile and skew pattern of implied volatility, relating them to properties of market liquidity. Essentially we show that in the context of our model we obtain volatility skews if we assume that market liquidity dries out in a rapidly falling market, a quite plausible assumption in our view. Finally we report results from a simulation study for the tracking error for different hedging strategies in an illiquid market. This quantity measures the difference between the payoff of a derivative and the terminal value of a selffinancing trading strategy designed to replicate this derivative; it is therefore a useful quantity if we want to assess the performance of different hedging strategies in an illiquid market.

The hedging of derivatives in markets which are not perfectly liquid has been the focus of a number of recent studies. Here we mention only the contributions by Jarrow (1994), Frey and Stremme (1997), Frey (1998), Sircar and Papanicolaou (1998), Schönbucher and Wilmott (2000) and Baum (2001). The relation between market liquidity and the smile pattern of implied volatility has previously been addressed in Grossman and Zhou (1996) and Platen and Schweizer (1998). Obviously, our paper builds on these contributions; the relation of our results to earlier work will be discussed in the core of the paper.

The paper is organized as follows. In Section 2 we introduce our basic model. The hedging of options and the relation between market liquidity and volatility skews is discussed in Section 3. Numerical results are presented in Sections 4 and 5.

2 The model

In this Section, we introduce the economic model underlying our analysis. Our exposition follows Frey (2000), and we sometimes refer to this paper for complementary information.

We are working in a stylized financial market with two traded assets: a riskless one (typically a bond or a money market account), called the bond and a risky one (typically a stock or stock index) referred to as the stock. We take the bond as numeraire. Moreover, we assume that the market for the bond is perfectly liquid, meaning that investors can buy or sell arbitrarily large quantities of this security without affecting its price. This reflects the fact that money markets are usually far more liquid than the stock markets. As usual the price of the stock, accounted in units of the numeraire, is modelled as a stochastic process $(S_t)_t$ on some underlying filtered probability space $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_t$.

We assume that there is a group of N agents hedging OTC derivatives on the stock. All these contracts have a common maturity date T and a path-independent payoff. Each of the individual hedgers is relatively small compared to the market and thus acts as a price taker. However, N is large, so that taken as a group the hedgers do have a significant impact on prices. We model this by introducing a representative hedger, who replicates a derivative contract on the stock with maturity date T and payoff $h(S_T)$ using a dynamic trading strategy in stock and bond. The function h represents the aggregated payoff of the contracts which are replicated by the small traders. The trading strategy of the representative hedger is given by a pair $(\alpha_t, \beta_t)_t$ of adapted processes, with α_t and β_t denoting the number of shares respectively the number of bonds in the portfolio at time t. We assume that the representative hedger is a *large trader*, i.e. that the implementation of his hedging strategy has a feedback effect on the price of the stock; this accounts for the fact that taken as a group the hedgers do have an influence on prices. More precisely, in our model the stock price rises (falls) if the representative hedger buys (sells) additional shares of the stock. This is in line with economic intuition on the price impact of large trades; it is also supported by empirical evidence on price impact of large block transactions as given for instance in Holthausen and Leftwich (1987) or Kampovsky and Trautmann (2000).

We now turn to our model for the asset price dynamics in the presence of feedback effects. Following Frey (2000) we refrain from fundamental economic modelling and introduce directly the asset price dynamics which result if the large trader chooses a given stock-trading strategy $(\alpha_t)_t$. In particular, form and size of the price-impact of our hedger's trades are not derived but exogenously imposed. Obviously this simplifies the analysis considerably. Moreover, the primitives of our model are at least in principle observable which facilitates the application of our results.

We need to impose some technical restrictions on the class of stock trading strategies permissible for our trader. Throughout the paper we assume that

A1) The stockholdings $(\alpha_t)_t$ are left-continuous (i.e. $\alpha_t = \lim_{s \overset{<}{\to} t} \alpha_s$), and the right-continuous process α^+ with $\alpha_t^+ = \lim_{s \overset{>}{\to} t} \alpha_s$ is a semimartingale.

A2) The downward-jumps of our strategy are bounded: $\Delta\alpha_t^+ := \alpha_t^+ - \alpha_t > -1/\overline{\rho}$ for some $\overline{\rho} > 0$.

Most of the time we will work with trading strategies which are smooth functions of time and stock-price such as hedging strategies for options in the standard Black-Scholes model. For these strategies the above assumptions are always satisfied.

Our model can be viewed as a perturbation of the standard Black-Scholes model. The size of this perturbation is controlled by a parameter ρ (the market liquidity parameter). In fact, if $\rho = 0$ or if the representative hedger does not trade (i.e. $\alpha_t \equiv 0$), the asset price simply follows a Black-Scholes model with some reference volatility σ. In the sequel we denote the asset price process which results if the liquidity parameter takes a certain value ρ and if the large trader uses a particular trading strategy α by $S_t(\rho, \alpha)$.

In the following assumption we describe the dynamics of $S_t(\rho, \alpha)$ by a stochastic differential equation (SDE).

A3) Given a Brownian motion W on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_t)$, two constants $\sigma > 0$ and $\rho \geq 0$ and a continuous function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\rho\lambda(S) \leq \overline{\rho}$ for all $S \geq 0$. Suppose that the large trader uses a stock-trading strategy $(\alpha_t)_t$ satisfying Assumptions A1 and A2. Then the asset price process solves the following stochastic differential equation (SDE)

$$dS_t = \sigma S_{t-} dW_t + \rho\lambda(S_{t-}) S_{t-} d\alpha_t^+ \,, \tag{1}$$

where S_{t-} denotes the left limit $\lim_{s \overset{<}{\to} t} S_t$.

We normalize λ by assuming that $\lambda(S_0) = 1$. Note that $1/(\rho\lambda(S_{t-})S_{t-})$ measures the *depth of the market* at time t, i.e. the size of the change in the large trader's stock position which causes the price to move by one unit of account.

There are basically two different approaches for determining ρ and λ. On the one hand statistical methods could be used. In fact, there are several empirical studies on the price impact of large trades which yield estimates of ρ such as Holthausen and Leftwich (1987) or Kampovsky and Trautmann (2000). On the other hand we might estimate ρ and λ from observed option prices, very much in the spirit of the popular implied volatility models. This approach, which is studied in more detail in Frey and Patie (2001), is based on the idea that option traders have a good feeling for problems in the hedging of derivatives caused by illiquidities in the market for the underlying, so that current derivative prices reflect the market's expectations about future liquidity.

3 Dynamic hedging of derivatives

3.1 Basic concepts revisited

In the sequel we discuss some modifications to basic notions in derivative asset analysis necessary to account for the fact that our hedger is a large investor. Consider a trading strategy $\xi = (\alpha_t, \beta_t)_t$ satisfying Assumptions A1 and A2.

VALUE PROCESS: In defining the value of the large trader's position we have to distinguish between the *paper value* or *mark to market value* and the *liquidation value* of his position. The mark to market value of his portfolio at time t is given by $V_t^M := \alpha_t S_t(\rho, \alpha) + \beta_t$, i.e. we simply value the position using current market prices. The liquidation value of a portfolio corresponds to the funds an investor obtains when actually selling his stockholdings. It is difficult to determine liquidation values exactly as they depend on the liquidation strategy chosen by the large trader. In the present paper we restrict ourselves to mark-to-market values; for an analysis of liquidation values in the context of option hedging in illiquid markets we refer to Schönbucher and Wilmott (2000) and in particular to Baum (2001).

GAINS FROM TRADE AND SELFFINANCING STRATEGIES: As in standard derivative asset analysis the gains from trade from a stock-trading strategy $(\alpha_t)_t$ are given by $G_t := \int_0^t \alpha_s dS_s(\rho, \alpha)$; note however, that in our situation the stock price process S depends on the chosen strategy. We call a strategy selffinancing if $V_t^M = V_0^M + G_t$ for all $0 \le t \le T$. As usual a stock-trading-strategy $(\alpha_t)_t$ satisfying Assumptions A1 and A2 and an initial investment V_0 define a unique selffinancing strategy $(\alpha_t, \beta_t)_t$ in stock and bond. Hence when restricting ourselves to selffinancing strategies we do not have to specify the amount of bonds in the portfolio.

TRACKING ERROR: The tracking error of a selffinancing strategy designed to replicate a derivative measures the the difference between the payoff of the derivative and the value at maturity of the strategy; it is therefore an essential quantity if we want to assess the performance of hedging strategies in markets which are not perfectly liquid. Consider some derivative security with payoff $h(S_T)$ and a selffinancing trading strategy with initial value V_0 and stock-trading-strategy $(\alpha_t)_t$. Then e_T^M, the tracking error with respect to mark to market values of this strategy, is defined by

$$e_T^M := h(S_T(\rho, \alpha)) - V_T^M = h(S_T(\rho, \alpha)) - \left(V_0 + \int_0^T \alpha_s dS_s(\rho, \alpha) \right) . \tag{2}$$

A positive (negative) value of e_T^M obviously indicates that the hedger made a loss (profit) on his hedge.

3.2 Asset price dynamics in the presence of a large trader

In this subsection we determine the dynamics of the asset price if the large trader's stock-trading strategy is given by a smooth function ϕ of time and the current stock price. We make the following regularity assumptions on ϕ.

A4) The function $\phi : [0,T]\times\mathbb{R}^+ \to \mathbb{R}$ is of class $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^+)$. Moreover, $\rho\lambda(S)S\phi_S(t,S) < 1$ for all $(t,S) \in [0,T]\times\mathbb{R}^+$.

We have

Proposition 3.1. *Suppose that the large trader uses a stock-trading-strategy of the form $\alpha_t = \phi(t,S_t)$ for a function ϕ satisfying Assumption A4 and that the stock price process $S_t = S_t(\rho,\alpha)$ follows an Itô process of the form*

$$dS_t = v(t,S_t)S_t dW_t + b(t,S_t)S_t dt \tag{3}$$

for two functions v and b. Then we have under Assumption A3

$$v(t,S) = \frac{\sigma}{1-\rho\lambda(S)S\phi_S(t,S)} \quad \text{and}\,[1] \tag{4}$$
$$b(t,S) = \frac{\rho}{1-\rho\lambda(S)S\phi_S(t,S)}\left(\phi_t(t,S) + \frac{\sigma^2 S^2 \phi_{SS}(t,S)}{2(1-\rho\lambda(S)S\phi_S(t,S))^2}\right).$$

The proof can be found in Frey (2000). Of particular relevance for our further analysis is the feedback effect on the volatility of the stock price process: by the trading-activity of the large investor the constant volatility σ is transformed into the time and price dependent volatility $v(t,S)$ in (4). Note that $v(t,S) > \sigma$ if $\phi_S(t,S) > 0$, i.e. if the trader uses a positive feedback strategy which calls for additional buying if the stock price rises. This property is typical for hedging strategies for derivatives with a convex terminal payoff such as European call or put options; see for instance El Karoui, Jeanblanc-Picqué, and Shreve (1998). On the other hand, if our trader uses a contrarian strategy, i.e. if $\phi_S(t,S) < 0$, we have $v(t,S) < \sigma$, i.e. the volatility is decreased. For a detailed discussion of the relation between dynamic hedging and market volatility we refer the reader to Frey and Stremme (1997) and the references given therein.

Proposition 3.1 also explains why in our context, where the large investor is a representative agent summarizing the trading behaviour of many small agents, it is natural to consider strategies of the form $\alpha_t = \phi(t,S_t)$: if an individual hedger, say hedger n, believes that the stock price follows a diffusion process, he will simply use the standard hedging strategy for a small investor in a diffusion model. It is well known that the corresponding stockholdings are given by some function $\varphi_n(t,S)$ of time and current stock price. If all hedgers believe in a diffusion model their aggregate stockholdings are then given by $\phi(t,S) := \sum_{n=1}^N \varphi_n(t,S)$. Proposition 3.1 now shows that the resulting price process is in fact a diffusion, so that the assumption of a diffusion model made by the hedgers is consistent with the ensuing asset price dynamics.

3.3 Perfect replication of derivatives

We now study, if the representative hedger is able to replicate the payoff of derivative contracts by dynamic trading despite the fact that his hedging activity does have an impact on asset prices. We consider only the simplest case where the aggregated payoff is given by a

[1] The error for the drift term $b(t,S)$ in Frey (2000) has been corrected.

terminal-value claim, i.e. by a derivative with path-independent payoff $h(S_T)$. However, it should become clear how to extend our approach to path-dependent derivatives, whose price in a diffusion model can be characterized by some linear parabolic PDE (under the usual small-investor paradigm).

In the following proposition we obtain a characterization of a perfect hedging strategy for the representative hedger in terms of a nonlinear Black-Scholes equation.

Proposition 3.2. *Assume that there is a solution $u \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^+)$ of the following nonlinear Black-Scholes terminal-value problem*

$$u_t(t,S) + \frac{1}{2}\frac{\sigma^2}{(1-\rho\lambda(S)Su_{SS}(t,S))^2}S^2u_{SS}(t,S) = 0, \quad u(T,S) = h(S)\,, \tag{5}$$

whose space derivative $u_S(t,S) := \frac{\partial}{\partial S}u(t,S)$ satisfies Assumption A4. Then the selffinancing strategy with stock-trading-strategy $\alpha_t = u_S(t,S_t)$ and value process $V_t = u(t,S_t)$, $0 \le t \le T$ is a perfect replication strategy for the derivative with payoff $h(S_T)$, i.e. the tracking-error e_T^M of this strategy is equal to zero.

Proof. If the large trader uses a stock-trading-strategy with $\alpha_t = u_S(t,S_t)$, the asset price volatility equals:

$$\sigma_u(t,S) := \sigma/(1-\rho\lambda(S)Su_{SS}(t,S))\,. \tag{6}$$

Applying Itô's formula to u we get

$$\begin{aligned} h(S_T) = u(T,S_T) = u(0,S_0) &+ \int_0^T u_S(t,S_t)dS_t \\ &+ \int_0^T u_t(t,S_t) + \frac{1}{2}u_{SS}(t,S_t){\sigma_u}^2(t,S_t)S_t^2dt\,, \end{aligned}$$

where S stands for $S(\rho,\alpha)$. Now note that the last integral on the right vanishes because of (5). Hence we have the representation

$$h(S_T(\rho,\alpha)) = V_0 + \int_0^T \alpha_t dS_t(\rho,\alpha)$$

which shows that the tracking-error $e_T^M = 0$. □

Comments

1) A characterization of option-replicating strategies for a large trader in terms of a nonlinear PDE has previously been obtained in a number of papers including Frey (1998), Sircar and Papanicolaou (1998) and Schönbucher and Wilmott (2000). We do not discuss existence and uniqueness of solutions to the terminal value problem (5) in this paper.[2]
2) With reference to Proposition 3.2 we will call the solution $u(t,S)$ hedge-cost of the claim with payoff $h(S_T)$. The pricing of derivatives in illiquid markets is discussed in Frey and Patie (2001).
3) The nonlinear PDE (5) is closely related to the PDE-characterizing of superhedging strategies in the uncertain volatility models of Avellaneda, Levy, and Paras (1995) and Lyons

[2] This very technical issue is dealt with in Frey (1998) in a slightly different context. The results of Frey (1998) guarantee existence (for small ρ) and uniqueness for the terminal value problem (5) in case that $\lambda \equiv 1$;

(1995). To see this relation more clearly we generalize (5) slightly and consider nonlinear PDEs of the form

$$u_t(t,S) + \frac{1}{2}(v(t,S,u_{SS}))^2 S^2 u_{SS}(t,S) = 0 \tag{7}$$

for some function $v(t,S,q)$ which is increasing in q. The key feature of this PDE is of course the dependence of the "volatility" $v(t,S,q)$ on the second derivative of the solution. In the models of Avellaneda et. al. and Lyons we have

$$v(t,S,q) = \underline{\sigma} 1_{\{q \leq 0\}} + \overline{\sigma} 1_{\{q>0\}}\,, \tag{8}$$

where $\underline{\sigma}$ and $\overline{\sigma}$ represent a lower and upper a-priori bound on the otherwise unspecified asset price volatility; in our case we have $v(t,S,q) = \sigma/(1-\rho\lambda(S)Sq)$. Note that in our case $v(t,S,q)$ increases gradually in q for q close to zero, whereas (8) is of "bang-bang-type".
4) The economic relevance of Proposition 3.2 depends crucially on our interpretation of the large trader as representative of many small hedgers. As shown in Baum (2001), if we interprete the large trader as a single large agent who can use his power to move the market strategically, other types of strategies should be considered. Baum showed that by using so-called *moderate* hedging strategies (hedging strategies with continuous trajectories of finite variation) the large trader can reduce his hedge cost and still keep his tracking error arbitrarily small. In our view the practical relevance of this result is however be limited, as the implementation of moderate strategies requires the ability to adjust the portfolio very frequently, an assumption which is particularly problematic in the context of illiquid markets. The discrepancy between our results and those of Baum highlights further the difference between a market with many small hedgers or portfolio insurers and a single large portfolio insurance firm, which was first pointed out by Donaldson and Uhlig (1993).

3.4 Market liquidity and smile patterns of implied volatility

It is well known that on equity markets out-of-the-money put options command much higher implied volatilities than the out-of-the-money call options. Following Rubinstein (1985) this phenomenon is usually termed the smile and skew pattern of implied volatility. There have been a number of approaches seeking to explain this smile pattern of implied volatility. Most studies postulate directly an asset price model with level-dependent or stochastic volatility and/or jumps; see for instance Bakshi, Cao, and Chen (1997) or Embrechts, Frey, and Furrer (2001) for a survey and references.

A number of studies have also explored the relation between volatility smiles and feedback effects from dynamic hedging caused by a lack of market liquidity. Grossman and Zhou (1996) study the impact of portfolio insurers on equilibrium asset prices; in their paper portfolio insurers are utility maximizing agents facing the constraint that their terminal wealth should lie above some exogenous threshold K. Grossman and Zhou find that for K low enough option prices do in fact exhibit a volatility skew in equilibrium. In their view "the volatility smile in the options market is evidence that the options market has priced the equilibrium implications of portfolio insurance". Platen and Schweizer (1998) consider a setup which is very close to our model. However, in order to explain smiles these authors have to choose an implausible parametrization of their model; translated in our context the key assumption of Platen and Schweizer, which ensures that feedback effects from hedging generate smiles, is the assumption of a *negative* liquidity parameter ρ.

In the present paper we offer an explanation of the smile pattern which is directly linked to properties of market illiquidity. More specifically we show that we obtain skew patterns of implied volatilities in our model, if we assume that our liquidity profile $\lambda(S)$ is decreasing and $\lambda(S) >> 1$ for S much smaller than the current stock price S_0 (remember that $\lambda(S_0)$ is normalized to one); if we assume a more symmetric liquidity profile with minimum around S_0 we obtain smiles. These assumptions on λ are in line with market psychology and experience from traders. In fact, it is common wisdom among equity traders that in a falling market liquidity is generally lower (i.e. ρ is higher) than in a rising market; very large moves in either direction tend to decrease liquidity and hence to increase ρ. A level-dependent liquidity profile is a simple way to capture this phenomenon.

We use the following simple parametric model for the liquidity profile λ in our simulations.

$$\lambda(S) = 1 + (S - S_0)^2 (a_1 \mathbf{1}_{\{S \le S_0\}} + a_2 \mathbf{1}_{\{S > S_0\}}) , \tag{9}$$

where usually $a_1 > a_2$. A typical graph for λ is presented in Figure 1. Simulations presented

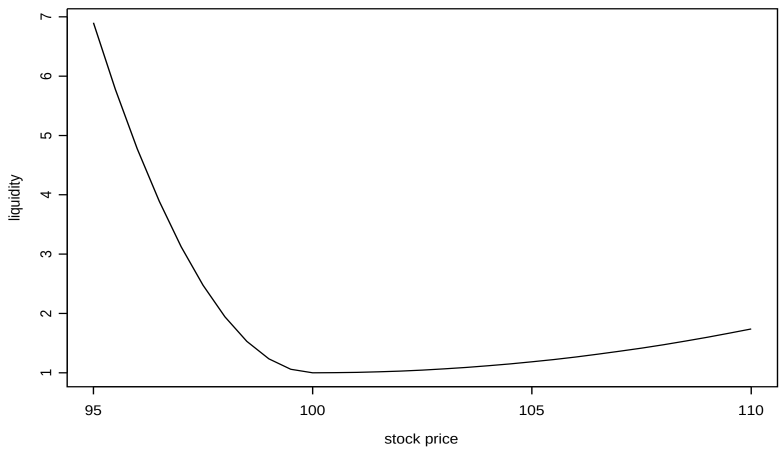

Figure 1: Level-dependent liquidity profile $\lambda(S)$ with $a_1 = 0.236$, $a_2 = 0.0074$ and $S_0 = 100$.

in Section 4.3 show that it is in fact possible to generate smile and skew patterns of implied volatility which resemble closely the patterns observed in real markets by this approach, offering a new and interesting explanation for this phenomenon.

4 Hedge cost and Greeks: numerical results

In our context the impact of market illiquidity on the hedge cost of derivatives is reflected by the nonlinearity of the generalized Black-Scholes equation (5). In order to gain a better understanding of the implications of this nonlinearity for the properties of hedge cost and Greeks we used numerical techniques to compute solutions of (5) for a number of payoffs. We discuss the results in this section; the corresponding pictures can be found in Appendix A.

4.1 The numerical scheme

Obviously the nonlinear PDE (5) cannot be solved explicitly so that we have to resort to techniques from numerical analysis. We chose the implicit scheme, which is an unconditionally stable scheme, for the discretization of the partial derivatives in the PDE. In particular, at each discrete time t we use the unknown value of the solution at the next time step, to approximate the spatial derivatives. In each time step we therefore have to solve a nonlinear system which is done using the Newton method. Details of the methodology are available in Frey and Patie (2001). Further, to avoid problems with volatilities tending to 0 or ∞ we used the following smoothed version of our PDE

$$u_t + \frac{1}{2}\sigma^2 S^2 \max\left\{\alpha_0, \frac{1}{1-\min\{\alpha_1, \lambda(S)Su_{SS}\}}u_{SS}\right\} = 0.$$

We take $\alpha_0 = 0.02$ and $\alpha_1 = 0.85$. In the following, all results are obtained from a grid with 400 time steps and 1000 space steps. For the non linear case ($\rho \neq 0$), the tolerance of the Newton method is set up at 5.10^{-4}. This choice allows the convergence of the Newton algorithm for a wide range of ρ.

4.2 Hedge cost and Greeks

Throughout Section 4.2 we always considered a constant liquidity profile ($\lambda(S) \equiv 1$) and a reference volatility $\sigma = 0.4$. Also, in order to smoothen our solution, we replace the terminal condition with the terminal condition corresponding to the Black-Scholes-price of the derivative with time to maturity one week (typically 8% of the lifetime of the contract); the time to maturity is adjusted accordingly.

European Call

In Figure 2 we present results for a 3-months ($T = 0.25$) option with strike $K = 100$ for different values of the market liquidity ranging from $\rho = 0$ up to $\rho = 0.4$. It is immediately seen that the hedge-cost is increasing in the liquidity parameter ρ, a behaviour we observed for all terminal payoffs considered.[3] The increase is most pronounced for $S \approx K$, which is due to the fact that the increase in volatility caused by the feedback-effect from hedging is most pronounced for those values of the stock.

In Figure 3 we plot the corresponding hedge ratios $u_S(t, S)$. We notice that the hedge ratio is increasing in ρ for $S < K$, and decreasing in ρ for $S > K$, i.e. lower liquidity spreads out the hedge ratio. This behavior is typical for *convex* payoffs like options. We observe (Figure 4) that the gamma $u_{SS}(t, S)$ flattens out as ρ increases, moving the its peak more and more to smaller values of S. Further, the maximum value of gamma is reduced.

Call spread

A call spread is the simplest example of a portfolio with long and short option positions and hence with a terminal payoff which is neither convex nor concave. A call spread can be created by buying one call option with strike K_1 and selling another call with strike $K_2 > K_1$, where it is assumed that both options have the same maturity T; mathematically the payoff is given by $h(S) = (S - K_1)^+ - (S - K_2)^+$. For our simulations we took $K_1 = 100$ and $K_2 = 110$. The hedge cost $u(t, S)$ is graphed in Figure 5. Again, the hedge cost is increasing in ρ. The hedge ratio $u_S(t, S)$ has a bell shape with peak lying between the two strikes (Figure 6). Inspection of Figure 7 shows that the sign of the gamma $u_{SS}(t, S)$ changes as we move from strike K_1 to K_2. The maximum value of the gamma is smaller than for the call, which is due to the fact that the nonlinearities tend to offset each other, since one

[3] Under some technical assumptions it is in fact possible to proof mathematically that solutions of (5) are increasing in ρ using the maximum principle for viscosity solutions of nonlinear parabolic PDE's; see Frey and Patie (2001) for details.

option is bought and one is sold. Finally, defining $\Gamma_t^+ = \sup\{(u_{SS}(t,S))^+ : \ S > 0\}$ and $\Gamma_t^- = \sup\{(u_{SS}(t,S))^- : \ S > 0\}$ we see that Γ_t^+ is decreasing in ρ while Γ_t^- is increasing in ρ. We observed this property for all payoff functions h we considered.

4.3 Generating realistic smile patterns

Next we want to show that by choosing an appropriate parametrization of the liquidity profile $\lambda(S)$ defined in (9) it is possible to generate smile patterns which resemble closely the smile patterns observed in real data. To make this point we proceeded as follows. Define $C(t,S;K,\sigma,\rho,a_1,a_2)$ to be the solution of (5) with terminal condition $h(S) = (S-K)^+$ and parameters (σ,ρ,a_1,a_2). We took from Bakshi, Cao, and Chen (1997) values for average implied volatilities of S&P 500 index options for five different levels of moneyness $\kappa = S_0/K$ ranging from $\kappa_1 = 0.93$ to $\kappa_5 = 1.07$. We took $\sigma = 0.174$, which corresponds to the average historical volatility of the S&P 500, and tried to determine values for ρ, a_1 and a_2 so that the implied volatilities computed by inverting the Black-Scholes formula, using as input the values $C(0,100;100/\kappa_i,0.174,\rho,a_1,a_2)$, $i = 1,\dots,5$ (the hedge cost according to our model), come close to the implied volatilities observed in the market. Figure 8 shows that the skew pattern obtained in this way from our model and the skew pattern observed in the market do indeed resemble each other closely; the parameters describing market liquidity used to produce this plot are $\rho = 0.017$, $a_1 = 0.236$ and $a_2 = 0.007$. Note that a_1 is much bigger than a_2, showing that we have to use a very asymmetric liquidity profile (see also Figure 1) in order to reproduce the properties of quoted equity index option prices. This is however quite plausible: our implied-volatility data comes from the period July to December 1990, and it is quite likely, that after the lessons learned during the stock market crash from October 1987 option traders expected huge liquidity problems in a falling market. Figure 9 finally shows that the implied volatilities estimated from our model remain quite stable as the time to maturity varies.

5 Tracking error simulation

In this part, we study the performance of various hedging strategies of the representative hedger in a market which is not perfectly liquid. We focus on European call options. To measure the performance of different strategies we use several statistics of the distribution of the tracking error such as mean or Value at Risk (a high quantile). Recall from Section 3.1 that the tracking error measures the difference between the payoff of a derivative at maturity and the terminal value of a selffinancing strategy designed to replicate the contract. As it is not possible to compute explicitly the law of the stock price process resulting from a particular trading strategy let alone the tracking error distribution, we must use simulation techniques for our study.

5.1 Euler-Maruyama scheme

Recall that for a given trading strategy $\alpha_t = \phi(t,S_t)$ of the large trader the stock price process follows a diffusion process with drift and volatility given in Proposition 3.1. We approximate the continuous paths of the price diffusion process (3) using the Euler-Maruyama scheme which is described briefly below (see also Kloeden and Platen (1992)). We evaluate the gain process for different hedging strategies and determine the tracking error for each path. To set up the scheme we choose a discretization step $\Delta_t = \frac{T}{n}$ of the time interval $[0,T]$, n being a positive integer, and simulate N trajectories $(i = 1 \dots N)$ of the stock price process using the Euler-Maruyama scheme. We put $S_0^i = S_0$, and for $k = 0 \dots n-1$,

$$S^i_{(k+1)\Delta_t} = S^i_{k\Delta_t} + v(k\Delta_t, S^i_{k\Delta_t}) \left(W^i_{(k+1)\Delta_t} - W^i_{k\Delta_t} \right) + b(k\Delta_t, S^i_{k\Delta_t})\Delta_t,$$

Thus to simulate one trajectory of $\{S^i_t, 0 \le t \le T\}$, one has simply to simulate the family

$$\left(W^i_{\Delta_t}, W^i_{2\Delta_t} - W^i_{\Delta_t}, \dots, W^i_{n\Delta_t} - W^i_{(n-1)\Delta_t} \right)$$

of independent centered Gaussian random variables with variance Δ_t. For each simulated trajectory $\left(S^i_k\right)_{k=0,\dots,n}$ we estimate the tracking error defined in (2) as follows:

$$e_T^{M,i} \approx h(S^i_T) - \left(V_0 + \sum_{k=0}^{n-1} \phi\left(k\Delta_t, S^i_{k\Delta_t}\right) \left(S^i_{(k+1)\Delta_t} - S^i_{k\Delta_t} \right) \right),$$

where $h(S^i_T)$ is the payoff of the derivative at maturity ($h(S) = (S-K)^+$), V_0 is the initial value of the hedge-portfolio, and $\phi\left(k\Delta_t, S^i_{k\Delta_t}\right)$ is the hedge ratio. We recall that a positive value for the tracking error means that the hedger has incurred a loss on his hedge.

5.2 Simulation results

We have studied two issues. First, we looked at the tracking-error distribution assuming that the hedger used the hedging strategy derived in Section 3.3, in the sequel referred to as nonlinear strategy. While a small tracking error is unavoidable as we are working with a discretized version of a continuous-time trading strategy, according to Proposition 3.2 the performance of this strategy and hence the shape of the tracking error distribution should not change significantly if we alter the liquidity parameter ρ. Density plots for the tracking error distribution for $\rho = 0$, $\rho = 0.01$ and $\rho = 0.05$ graphed in Figure 10 show that this is the case, thus vindicating that it is in fact possible to deal with market illiquidities using our nonlinear hedging strategies.

Moreover, we compared the performance of the nonlinear hedging strategy with the standard Black-Scholes hedging strategy, assuming that the market is not perfectly liquid ($\rho = 0.02$). In Figure 11 we plotted density estimates for the tracking error distributions corresponding to both strategies. We see that the density-plot for the Black-Scholes strategy is shifted to the right of the density plot for the nonlinear strategy, i.e. on average we have a larger tracking error. More importantly, the distribution of the tracking error of the Black-Scholes strategy seems to be more dispersed with a heavier right tail, indicating that large losses on our hedges are more likely under the Black-Scholes strategy than under the nonlinear strategy.

To quantify the properties of the distribution of the tracking-error corresponding to different hedge strategies we report the mean, the Value at Risk at a confidence level of 99% ($\mathrm{VaR}_{99\%}$) and the 99% expected shortfall ($\mathrm{ES}_{99\%}$) of the empirical distribution of the tracking-error simulations. These measures are widely used in market risk management; they allow us to assess on the one hand the average performance of each approach and on the other hand the size of extreme losses. As an estimate of $\mathrm{VaR}_{99\%}$, we simply took the $\left[99 \times \frac{N}{100}\right]$ largest value of the simulated values for e^M_T. Mean and the $\mathrm{ES}_{99\%}$ are estimated as follows:

$$\bar{e}^M_T := \frac{1}{N} \sum_{i=1}^{N} e_T^{M,i}, \tag{10}$$

$$\mathrm{ES}_{0.99}\left(e^M_T\right) := \frac{\sum_{i=1}^N e_T^{M,i} \mathbf{1}_{\{e_T^{M,i} > \mathrm{VaR}_{0.99}(e^M_T)\}}}{\sum_{i=1}^N \mathbf{1}_{\{e_T^{M,i} > \mathrm{VaR}_{0.99}(e^M_T)\}}}. \tag{11}$$

Table 1 confirms that the performance of the nonlinear replicating strategy is very good; if we look at the right tail of the tracking error density (rows (3),(4) of table 1) it seems that the performance is slightly better for small values of ρ, which is probably due to numerical effects. Table 2 shows that the tracking error average estimated with the Black-Scholes model (both price and hedge ratio given by this model) in a market which is not perfectly liquid is always positive for $\rho > 0$ and increasing in ρ. This is interesting from a risk management point of view. Indeed, the financial interpretation is that, in presence of market illiquidities, applying a Black-Scholes strategy to replicate derivatives leads to a loss for the derivative's hedger, which increases with the lack of liquidity. This feature is emphasized in the estimates for the risk measures, see rows (3) and (4) of Table 2. An analytical expression for the form of the tracking error, which confirms these observations, is for instance given in Frey (2000).

There are two reasons why a hedging strategy based on the nonlinear PDE performs better than a standard Black-Scholes strategy, namely a different shape of the hedge ratio on the one hand and a higher initial investment $C(0, S_0; \rho) > C(0, S_0; 0) = C^{\mathrm{BS}}(0, S_0)$ on the other. Table 3 shows that the nonlinear strategy performs better than the Black-Scholes strategy with initial investment $V_0 = C(0, S_0; \rho)$ stemming from the nonlinear PDE. This shows that the difference in the initial investment alone is clearly not sufficient to explain the performance difference of the two strategies.

ρ	0	0.01	0.02	0.05
$\overline{e}_T^M$	–0.08	–0.08	–0.08	–0.07
$\mathrm{VaR}_{0.99}\left(e_T^M\right)$	0.67	0.7	0.73	0.83
$\mathrm{ES}_{0.99}\left(e_T^M\right)$	0.84	0.89	0.93	1.07
$\overline{e}_T^M / V_0$	1.8%	1.5%	1.4%	1.%

Table 1: Successively mean, $\mathrm{VaR}_{99\%}$, $\mathrm{ES}_{99\%}$ and average relative value of the tracking error for the *nonlinear hedging strategy* used to replicate an European call option for different value of ρ ($T = 0.5$, $K = 100$, $S_0 = 100$, 5000 simulations with $n = 240$ portfolio rebalancings).

ρ	0	0.01	0.02	0.05
$\overline{e}_T^M$	–0.08	0.24	0.51	2.15
$VaR_{0.99}\left(e_T^M\right)$	0.67	1.44	2.37	26.06
$ES_{0.99}\left(e_T^M\right)$	0.84	1.7	2.88	40.9
$\overline{e}_T^M / V_0$	1.8%	4.2%	8.9%	37.4%

Table 2: Successively mean, $\mathrm{VaR}_{99\%}$, $\mathrm{ES}_{99\%}$ and average relative value of the tracking error for the *Black-Scholes strategy* used to replicate an European call option in a market which is not perfectly liquid ($T = 0.5$, $K = 100$, $S_0 = 100$, 2500 simulations with $n = 240$ portfolio rebalancings).

ρ	0	0.01	0.02	0.05
$\overline{e}_T^M$	–0.08	0.04	0.12	1.15
$\mathrm{VaR}_{0.99}\left(e_T^M\right)$	0.67	1.24	1.98	25.06
$\mathrm{ES}_{0.99}\left(e_T^M\right)$	0.84	0.85	2.49	39.9
$\overline{e}_T^M/V_0$	1.8%	1%	1.9%	17%

Table 3: Successively mean, $\mathrm{VaR}_{99\%}$, $\mathrm{ES}_{99\%}$ and average relative value of the tracking error for the *Black-Scholes strategy, starting with the hedge-cost given by the nonlinear PDE*, used to replicate an European call option in a market which is not perfectly liquid ($T = 0.5$, $K = 100$, $S_0 = 100$, 2500 simulations with $n = 240$ portfolio rebalancings).

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A Figures

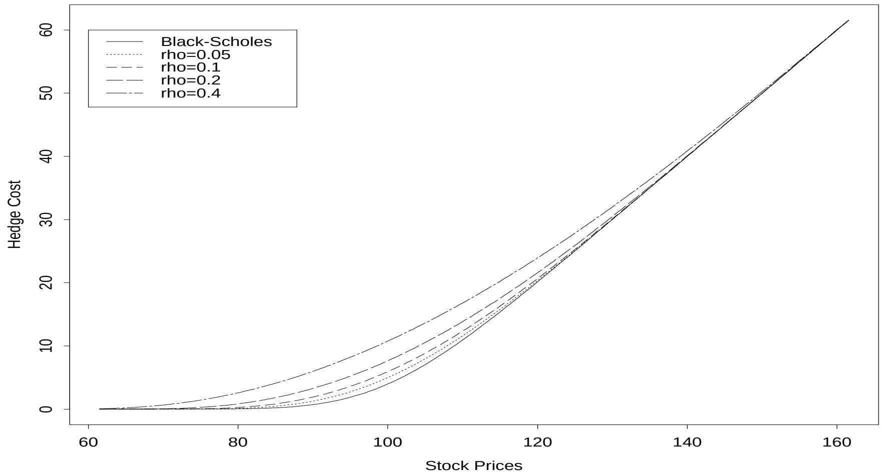

Figure 2: Hedge cost $u(S,T)$ of a European call for various values of ρ (Strike $= 100$, $\sigma = 0.2$, $T - t = 0.25$).

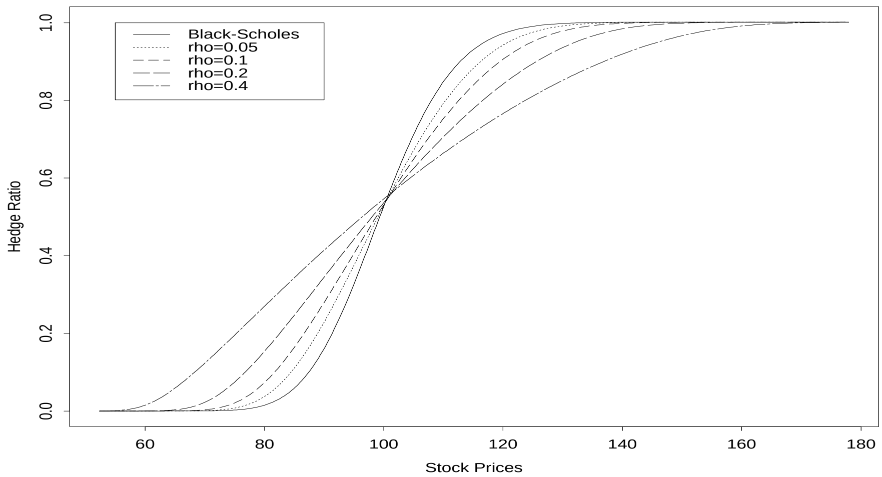

Figure 3: Hedge ratio $u_S(S,T)$ for an European call for various values of ρ (Strike $= 100$, $\sigma = 0.2$, $T - t = 0.25$).

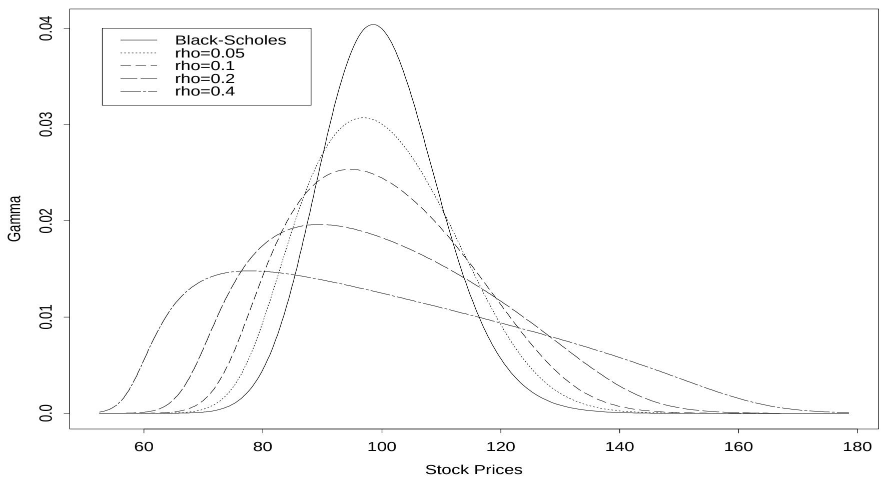

Figure 4: Gamma $u_{SS}(S, T)$ for an European call for various values of ρ (Strike = 100, $\sigma = 0.2$, $T - t = 0.25$).

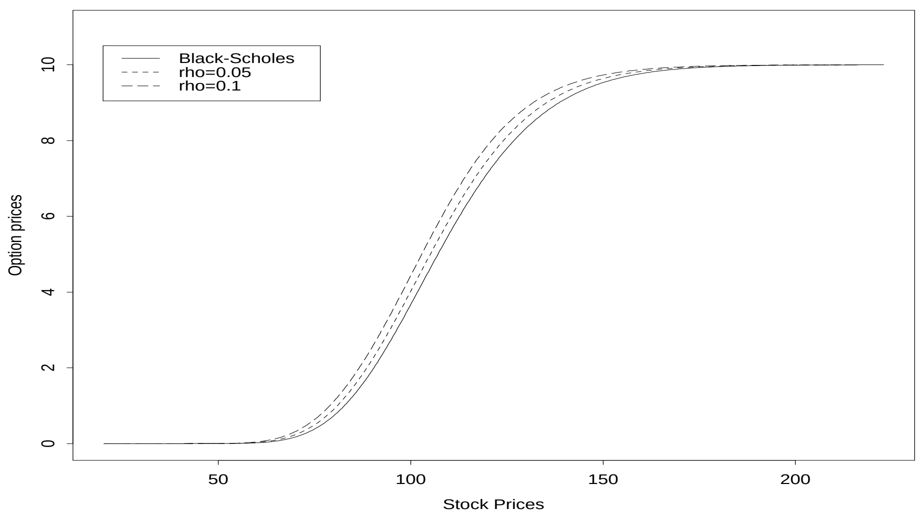

Figure 5: Hedge cost $u(S, T)$ for a call spread for various values of ρ (Strike 1 = 100, Strike 2 = 110, $\sigma = 0.4$, $T - t = 0.25$).

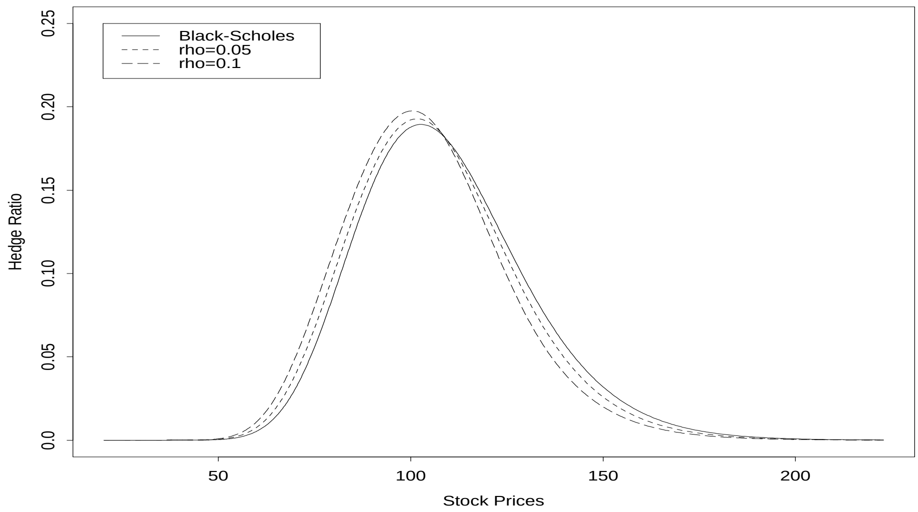

Figure 6: Hedge ratio $u_S(S,T)$ for a call spread for various values of ρ (Strike 1 = 100, Strike 2 = 110, $\sigma = 0.4$, $T - t = 0.25$).

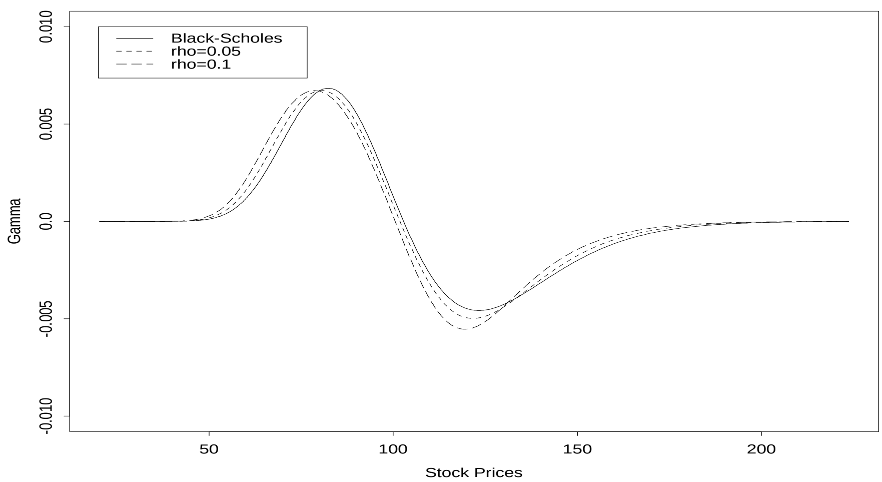

Figure 7: Gamma $u_{SS}(S,T)$ for call spread for various values of ρ (Strike 1 = 100, Strike 2 = 110, $\sigma = 0.4$, $T - t = 0.25$).

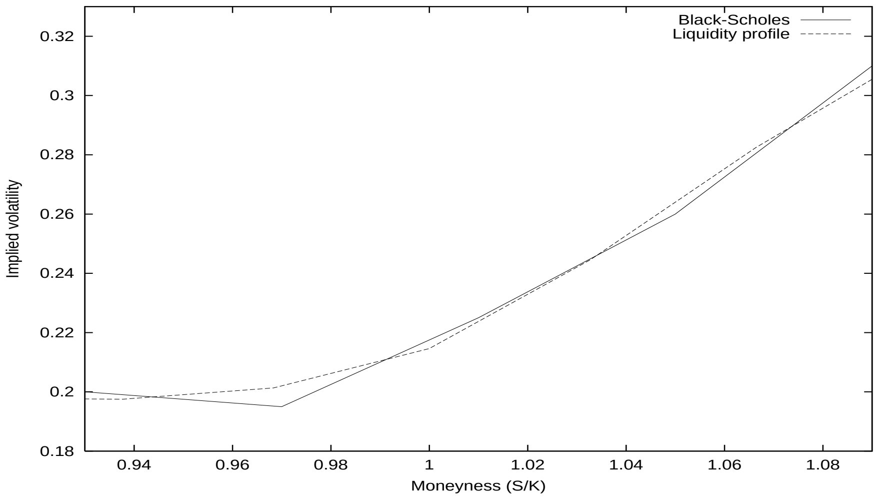

Figure 8: Average implied volatilities for S&P 500 options computed from market prices of traded options using the Black-Scholes model (straight line) compared to implied volatilities computed from solutions of the nonlinear PDE (5) (dotted line). For the nonlinear PDE the following parameter values were used: $\rho = 0.017$, $a_1 = 0.236$, $a_2 = 0.0074$, $\sigma = 17.47\%$.

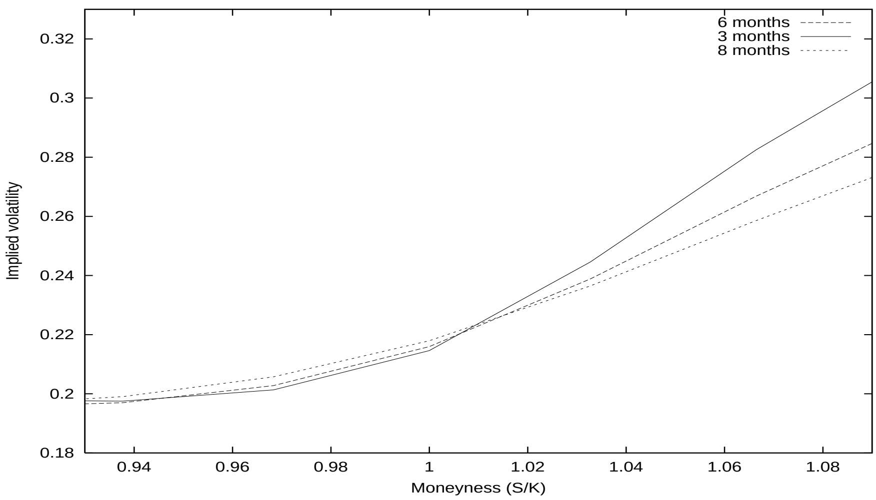

Figure 9: Implied volatilities computed from the nonlinear PDE (5) for different time to maturity (3 months, 6 months, 9 months). The parameters are as in Figure 8.

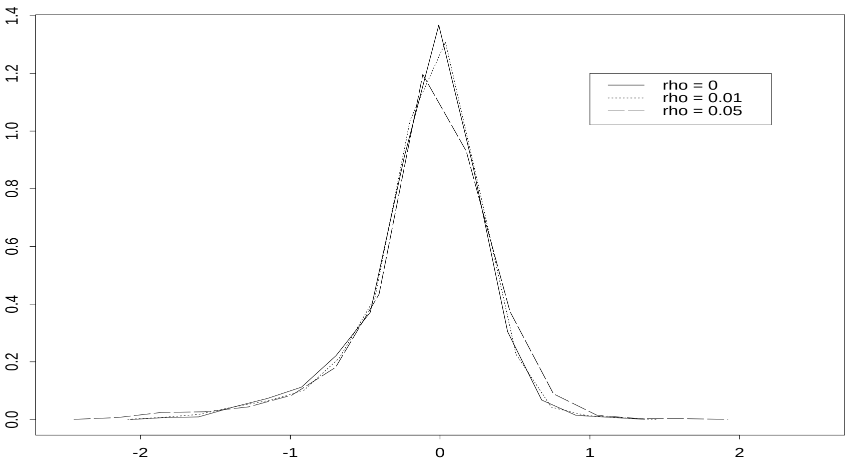

Figure 10: Tracking error density in an illiquid market using the nonlinear strategy for various values of ρ ($N = 5000$, $n = 240$, $T = 0.5$ years).

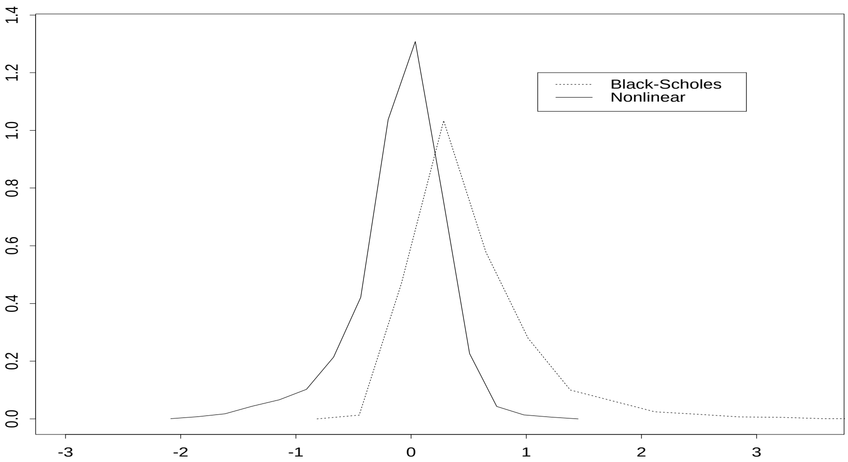

Figure 11: Tracking error density in an illiquid market using the nonlinear strategy (straight line) and the standard Black-Scholes strategy (dotted line) ($\rho = 0.02$, $N = 5000$, $n = 240$, $T = 0.5$ years).