Interest rate model risk: what are we talking about ?

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Abstract

Recently, model risk became an increasingly important concept in financial valuation, but also for risk management models and capital adequacy purposes. It arises as a consequence of incorrect modelling, model identification and specification errors, inadequate estimation procedures, as well as mathematical and statistical properties of financial models applied in imperfect financial markets. This paper provides a definition of model risk, identifies its possible origins, proposes a methodology to analyse and quantify model risk before finally proposing some illustrations of its consequences in the context of the valuation and risk management of interest rate contingent claims.

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1 Introduction

The concept of risk is central to players in capital markets. Risk management is the set of procedures, systems, and persons used to control the potential losses of a financial institution. The explosive increase in interest-rates volatility in the late 1970s and early 1980s has produced a revolution in the art and science of interest rate risk management. For instance, in the U.S., in 1994, interest rates rose by more than 200 basis points; in 1995, there were important non parallel shifts in the yield curve. Complex hedging tools and techniques were developed, dozens of plain vanilla and exotic derivatives instruments were created to provide the ability to create customized financial instruments to meet virtually any financial target exposure.

Recent crises in the derivatives markets have raised the question of interest-rate risk management. It is important for bank managers to recognize the economic value and resultant risks related to interest-rate derivative products, including loans and deposits with embedded options. It is equally important for regulators to measure interest-rate risk correctly. This explains why the Basle Committee on Banking Supervision (1995, 1997) issued directives to help supervisors, shareholders, CFOs and managers in evaluating the interest-rate risk of exchange traded and over the counter derivative activities of banks and securities firms, including off balance sheet items. Under these directives, banks are allowed to choose between using a standardized (building block) approach or their own risk measurement models to calculate their value-at-risk, which will then determine their capital charge. No particular type of model is prescribed, as long as each model captures all the risks run by an institution[1].

Many banks and financial institutions already base their strategic tactical decisions for valuation, market-making, arbitrage or hedging on internal models built by scientists. Extending these models to compute their value-at-risk and resulting capital requirement may seem pretty straightforward. But we all know that any model is by definition an imperfect simplification, a mathematical representation for the purposes of replicating the real world. In

[1] As supervisory authorities are aware of model risk associated with the use of internal models, as a precautionary device, they have imposed adjustment factors: the internal model value-at-risk should be multiplied by an adjustment factor subject to an absolute minimum of 3, and a plus factor - ranging from 0 to 1 - will be added to the multiplication factor if backtesting reveals failures in the internal model. This overfunding solution is nothing else than an insurance or an ad-hoc safety factor against model risk.

some cases, a model will produce results that are sufficiently close to reality to be adopted. But in others, it will not. What will happen in such a situation ? A large number of highly reputable banks and financial institutions have already suffered from extensive losses. For instance, in 1992, J.P. Morgan lost 200 million USD in the mortgage-backed securities market due to an inadequate modelization of the prepayments; in 1987, Merrill Lynch lost 350 million USD in stripped mortgage-backed securities due to an incorrect pricing model; more recently, in March 1997, NatWest Markets announced that mispricing on sterling interest rate options due to improper volatility estimations had cost 90 million GBP and Bank of Tokyo-Mitsubishi had to write off 83 million USD on its U.S. interest rate derivatives book because of the application of an inadequate pricing model which lead to systematic overvaluation of the position.

The problem is not limited to the interest-rate contingent claims market. It also exists for instance in the stock market. In RISK Magazine, the late Fisher Black (1990) commented:

> "I sometimes wonder why people still use the Black and Scholes formula, since it is based on such simple assumptions - unrealistically simple assumptions"

The answer can be found in its 1986 presidential allocution at the American Finance Association, where he said:

> "In the end, a theory is accepted not because it is confirmed by conventional empirical tests, but because researchers persuade one another that the theory is correct and relevant."

Why did we focus on interest rate models rather than on stock models ? First, interest rate models are more complex, as the effective underlying variable - the entire term structure of interest rates - is not observable. Second, there exists a wider set of derivative instruments. Third, interest rates contingent claims have certainly suscitated the most abundant theoretical literature on how to price and hedge from the simplest to the most complex instrument, and the set of models available is prolific in variety and underlying assumptions. Fourth, almost every economic agent is exposed to interest rate risk, even if he does not manage a portfolio of securities.

Despite this, as we will see, the literature on model risk is rather sparse and often focuses on specific pricing or implied volatility fitting issues. We believe there are much more challenging issues to be explored. For instance, is model risk symmetric ? Is it priced in the market ? Is it the source of a larger bid-ask spread ? Does it result in overfunding or underfunding of financial institutions ?

In this paper, we will provide a definition of model risk and examine some of its origins and consequences. The paper is structured as follows: section 2 defines model risk; section 3 reviews the steps of the model building process which are at the origin of model risk; section 4 exposes various examples of model risk influence in areas such as pricing, hedging, or regulatory capital adequacy issues; section 5 concludes.

2 Model risk: some definitions

As postulated by Derman (1996a, 1996b), most financial models fall in one of the following categories:

- **fundamental models**, which are based on a set of hypotheses, postulates and data, together with a means of drawing dynamical inferences from them. They attempt to build a fundamental description of some instruments or phenomenon. Good examples are equilibrium pricing models, which rely on a set of hypotheses to provide a pricing formula or methodology for a financial instrument.

- **phenomenological models**, which are analogies or visualisations that describe, represent, or help understanding a phenomenon which is not directly observable. They are not necessarily true, but provide a useful picture of the reality. Good examples are single-factor interest rate models, which look at reality "as if" everybody was only concerned by the short term interest rate, whose distribution will remain normal or lognormal at any point in time.

- **statistical models**, which generally result from a regression or best-fit between different data sets. They rely on correlation rather than causation and describe tendencies rather than dynamics. They are often a useful way to report information on data and on their trends.

In the following, we will mainly focus on models belonging to the first and second categories, but we could easily extend our framework to include statistical models. In any problem, once a fundamental model has been selected or developed, there are typically three main sources of uncertainty:

- **uncertainty about the model structure**: did we specify the right model ? Even after the most diligent model selection process, we cannot be sure that the true model - if any - has been selected.
- **uncertainty about the estimates of the model parameters**, given the model structure. Did we use the right estimator ?
- **uncertainty about the applying the model in a specific situation**, given the model structure and its parameters estimation. Can we use the model extensively ? Or is it restricted to specific situations, financial assets or markets ?

These three sources of uncertainty constitute what we call model risk. **Model risk results from the inappropriate specification of a theoretical model or the use of an appropriate model but in an inadequate framework or for the wrong purpose.** How can we measure it ? Should we use the dispersion, the worst case loss, a percentile, or an extreme loss value function and minimize it ? There is a strong need for model risk understanding and measurement.

The academic literature has essentially focused on estimation risk and uncertainty about the model use, but not on the uncertainty about the model structure. Some exceptions are:

- the time series analysis literature - see for instance the collection of papers by Dijkstra (1988) - as well as some econometric problems, where a model is often selected from a large class of models using specific criteria such as the largest R^2, AIC, BIC, MIL, C_P or C_L proposed by Akaike (1973), Mallows (1973), Schwarz (1978) and Rissanen (1978) respectively. These methods propose to select among a collection of parametric models the model which minimizes an empirical loss (typically measured as a squared error or a minus log-likelihood) plus some penalty term which is proportional to the dimension of the model.
- the option pricing literature such as Bakshi, Cao, and Chen (1997) or Bühler, Uhrig-Homburg, Walter and Weber (1998), where prices resulting from the application of different models and different input

parameters estimations are compared to quoted market prices in order to determine which model is the "best" in terms of market calibration.

This sparseness of the literature is rather surprising, as errors arising from uncertainty about the model structure are a-priori likely to be much larger than those arising from estimation errors or misuse of a given model.

3 The steps of the model building process (or how to create model risk)

In this section, we will focus on the model building process (or the model adoption process, if the problem is to select a model among a set of possible candidates) in the particular case of interest rate models. Our problem is the following: we want to develop (or select), estimate and use a model that can explain and fit the term structure of interest rates in order to price or manage a given set of interest rate contingent securities. Our model building process can be decomposed into four steps: identification of the relevant factors, specification of the dynamics for each factor, parameters estimation, and implementation issues.

3.1 Environment characterization and factor(s) identification

The first step in the model building process is the **characterization of the environment** in which we are going to operate. What does the world look like ? Is the market frictionless ? Is it liquid enough ? Is it complete ? Are all prices observable ? Answers to these questions will often result in a set of hypotheses which are fundamental for the model to be developed. But if the model world differs too much from the true world, the resulting model will be useless. Note that, on the other hand, if most economic agents adopt the model, it can become a self fulfilling prophecy.

The next step is the **identification of the factors** that are driving the interest rate term structure. This step involves the identification of both the number of factors and the factors themselves.

Which methodology should be followed ? Up to now, the discussion has been based on the assumption of the existence of a certain number of factors. Nothing has been said about what a factor is (nor on how many of them are needed) ! Basically, two different empirical approaches can be used. On the one hand, the explicit approach assumes that the factors are known and that their returns are observed; using time series analysis, this allows to estimate the factor exposures[2]. On the other hand, the implicit approach is neutral with respect to the nature of the factors and relies purely on statistical methods, such as principal components or cluster analysis, in order to determine a fixed number of unique factors such that the covariance matrix of their returns is diagonal and they maximize the explanation of the variance of the returns on some assets. Of course, the implicit approach is frequently followed by a second step, in which the implicit factors are compared to existing macro-economic or financial variables in order to explicitly identify them.

For instance, most empirical studies using a principal component analysis have decomposed the motion of the interest rate term structure into three independent and non-correlated factors:

- the first one is a **shift** of the term structure, i.e., a parallel movement of all the rates. It usually accounts for up to 80-90 percent of the total variance (the exact number depending on the market and on the period of observation).
- the second one is a **twist**, i.e. a situation in which long rates and short term rates move in opposite directions. It usually accounts for an additional 5-10 percent of the total variance.
- the third one is called a **butterfly** (the intermediate rate moves in the opposite direction of the short and long term rate). Its influence is generally small (1-2 percent of the total variance).

As the first component generally explains a large fraction of the yield curve movements, it may be tempting to reduce the problem to a one-factor model[3], generally chosen as the short term rate. Most early interest rate models (such as Merton (1973), Vasicek (1977), Cox, Ingersoll and Ross

[2] An alternative is to assume that the exposures are known, which then allows to recover cross-sectionally the factor returns for each period.

[3] It must be stressed at this point that this does not necessarily imply that the whole term structure is forced to move in parallel, but simply that one single source of uncertainty is sufficient to explain the movements of the term structure (or the price of a particular interest rate contingent claim).

(1985b), Hull and White (1990, 1993), etc.) are in fact single-factor models. These models are easy to understand, to implement, and to solve. Most of them provide analytical expressions for the prices of simple interest rates contingent claims[4]. But single-factor models suffer from various critiques:

- the long term rate is generally a deterministic function of the short rate.
- the prices of bonds of different maturities are perfectly correlated (or equivalently: there is a perfect correlation between movements in rates of different maturities).
- some securities are sensitive to both the shape and the level of the term structure. Pricing or hedging them will require at least a two factor model.

Furthermore, empirical evidence suggests that multi-factor models do significantly better than single-factor models in explaining the whole shape of the term structure. This explains the early development of two-factor models (see Table 1) which are much more complex than the single-factor ones. As evidenced by Rebonato (1997), by using a multi-factor model, one can often get a better fit of the term structure, but will have to solve partial differential equations in a higher dimension to obtain prices for interest rate-contingent claims.

What is the optimal number of factors to be considered ? The answer generally depends on the interest rate product which is examined, and on the profile (concave, convex, or linear) of its terminal payoff. One factor models are more comprehensible and relevant to a wide range of products or circumstances, but they also have their limits. As an example, a one-factor model is a reasonable assumption to value a Treasury Bill, but much less reasonable for valuing options written on the slope of the yield curve. Securities whose payoffs are primarily dependent on the shape of the yield curve and/or its volatility term structure rather than its overall level will not be modeled well using single-factor approaches. The same remark applies to derivative instruments that marry foreign exchange with term structures of interest rates risk exposures, such as differential swaps for which floating

[4]See Gibson, Lhabitant, and Talay (1997) for an exhaustive survey of existing term structure models specifications.

rates in one currency are used to calculate payments in another currency. Furthermore, for some variables, the uncertainty in their future value is of little importance to the model resulting value, while for others, uncertainty is critical. For instance, interest rates volatility is of little importance for short term stock options, while it is fundamental for interest-rate options. But the answer will also depend on the particular use of the model.

What are the relevant factors ? Here again, there is no clear evidence. As an example, Table 1 lists some of the most common factor specifications that one can find in the literature[5].

It appears that no single technique clearly dominates another when it comes to the joint identification of the number and identity of the relevant factors. Imposing factors by a pre-specification of some macro-economic or financial variables is tempting, but we do not know how many factors are required. Deriving them using a non-parametric technique such as a principal component analysis will generally provide some information about the relevant number of factors, but not about their identity.

Model	Factors
Richard (1978)	real short term rate, expected instantaneous inflation rate
Brennan and Schwartz (1979)	short term rate, long term rate
Schaefer and Schwartz (1987)	short term rate, spread between the short and long term rate
Cox, Ingersoll, Ross (1985b)	short term rate, inflation
Longstaff and Schwartz (1992)	short term rate, short term rate volatility
Schaefer and Schwartz (1984)	long term rate, spread between the short and long term rate
Das and Foresi (1996)	short term rate, mean of the short term rate
Chen (1996)	short term rate, short term rate mean, short term rate volatility

Table 1 : Examples of various two and three factor models

[5]For a detailed discussion on the considerations invoked in making the choice of the number and type of factors and the empirical evidence, see Nelson and Schaefer (1983) or Litterman and Scheinkman (1991).

When selecting a model, one has to verify that all the important parameters and relevant variables have been included. Oversimplification and failure to select the right risk factors may have serious consequences.

3.2 Factor(s) dynamics specification

Once the factors have been determined, **their individual dynamics have to be specified**. Recall that the dynamics specification has distributional assumptions built in.

Should we allow for jumps or restrict ourselves to diffusions ? And in the case of diffusions, should we allow for constant parameters or time-varying ones ? Should we have restrictions placed on the drift coefficient, such as linearity or mean-reversion ? Should we think in discrete or in continuous-time ? What specification of the diffusion term is more suitable, and what are the resulting consequences for the distributional properties of interest rates ? Can we allow for negative nominal interest rate values, if it is with a low probability ? Should we prefer normality over lognormality ? Should the interest rate dynamics be Markovian ? Should we have a linear or a non-linear specification of the drift ? Should we estimate the dynamics using non-parametric techniques rather than impose a parametric diffusion ?

The problem is not simple, since some models are often nested into other models. For instance, let us focus on single-factor diffusions for the short term rate. For instance, let us consider the general Broze, Scaillet, and Zakoian (1994) specification for the dynamics of the short term rate:

$$dr_t = (\alpha + \beta r_t)dt + \sigma_0(r_t^\gamma + \sigma_1)dW_t \quad (1)$$

where W_t is a standard Brownian motion and r_0 is a fixed positive (known) initial value. This model encompasses some of the most common specifications that one can find in the literature (see Table 2). Should we then automatically adopt the most general specification and let the estimation procedure decide on the value of some parameters, or rather specify and justify some restrictions ?

Interest rate model	α	β	σ_0	σ_1	γ
Merton (1973)		0		0	0
Vasicek (1977)				0	0
Cox, Ingersoll, Ross (1985b)				0	$\frac{1}{2}$
Dothan (1978)	0	0		0	1
Geometric Brownian Motion	0			0	1
Brennan and Schwartz (1980)				0	1
Cox, Ingersoll and Ross (1980)	0	0		0	$\frac{3}{2}$
Constant elasticity of variance	0			0	
Chan, Karolyi, Longstaff, and Sanders (1992)				0	
Broze, Scaillet, Zakoian (1994)					

Table 2 : **Parameters for various interest rate models**

Of course, assumptions about the dynamics of the short term rate can be verified on past data[6]. But on the one hand, this involves falling into estimation procedures before selecting the right model. On the other hand, a misspecified model will not necessarily provide a bad fitting to the data. For instance, duration based models could provide better replicating results than multifactor models in the presence of parallel jumps in the term structure. Models with more parameters will generally give a better fit of the data, but may give worse out-of-sample predictions. Model with time-varying parameters can be use to calibrate exactly the model to current market prices, but the error terms might be reported as unstable parameters and/or non-stationary volatility term structures (Carverhill (1995)).

3.3 Parameters estimation

The final step - which comes only after the two previous steps - is the **estimation procedure**. Most people generally confuse model risk with estimation risk. Whereas estimation risk is an essential part of the process, it remains only a nested part at the almost final spectrum of the model production chain.

[6]Or rejected ! Ait Sahalia (1996) rejects all of the existing linear drift specifications for the dynamics of the short term rate using non parametric tests.

The theory of parameter estimation generally assumes that the true model is known. Once the factor have been selected and their dynamics specified, the model parameters must be estimated using a given set of data. Fitting a time-series model is usually straightforward nowadays using an appropriate computer software. However, in the context of model risk, some important issues should be considered.

- Is the set of data **representative** of what we want to model ? A model may be correct, but the data feeding it may be incorrect. If we lengthen the set of data, we might include some elements that are too old and insignificant; if we shorten it, we might end up with non-representative data. Of course, one can always go towards high frequency data, but is it really appropriate to solve a given problem ?

- Is the set of data adequate for **asymptotic and convergence properties** to be fulfilled ? For instance, in the case of the Vasicek (1977) or Cox, Ingersoll and Ross (1985) model, Fournie and Talay (1991) have shown that natural estimators (such as maximum likelihood and generalized method of moments) applied to daily interest rates observations required such a large observation period (110 years, over which model parameters were assumed to be constant) that the corresponding model was totally non-realistic...

- Is the set of data subject to **measurement errors** (for instance, non simultaneous recording of options and underlying quotes, bid-ask bouncing effects or others liquidity effects) ?

- How can we estimate parameters that **may not be observable** ? The factors of our model have to correspond to observable variables in order to be estimated. But in Finance, some of the quantities we are dealing with are pure abstractions. For instance, even if we assume that the volatility of an asset is constant, how can we estimate it ? How about the future volatility ? Some of the variables are directly measurable, while others are human expectations and therefore indirectly measurable.

- What if the result of the estimation procedure is a result that **does not make sense** ? For instance, the Hull and White (1993) extended model

$$dr_t = (\alpha_t + \beta_t r_t)dt + \sigma r_t^{\gamma} dW_t$$

specified with $\gamma \in]0; 0.5[$ does not necessarily provide a unique solution (see Arnold (1973, page 124)). What should you do if the result of your estimation is $\gamma = 0.4$? As another example, Chan, Karolyi, Longstaff, and Sanders (1992a) test empirically the following model

$$dr_t = (\alpha + \beta r_t)dt + \sigma r_t^{\gamma} dW_t$$

They obtain that there is no mean-reversion and that $\gamma = 1.5$, a contradiction with most popular one-factor models.

- Another problem arises with continuous-time financial models: **approximations**. There are numerous sources of approximations when estimating a model. For instance, to be estimated, a continuous-time model must be discretized, that is, must be approximated by a discrete-time model. Or we may not know the explicit underlying transition density, and we must use an approximate likelihood function which may lead to inconsistent estimators (see Going (1997)). If we take the example of the term structure estimation, in a complete market, the required term structure would be directly observable. But in practice, this is not the case: zero-coupon bonds are not available for all maturities and suffer from liquidity and tax effects (see Daves and Ehrhardt (1993), Jordan (1984)), and the term structure must be estimated using coupon bonds. Even in the presence of correct bond data, which methodology should be selected ? In 1990, a survey of software vendors (Mitsubishi (1990)) indicated that 12 out of 13 used linear interpolation to derive yield curves, a methodology that is still recommended by RiskMetrics (1995). But spline techniques are also a recommended technique when smoothness is an issue (Adams and Van Deventer (1994)). Barnhil et al. (1996) have compared four methodologies of estimating the yield curve, namely: linear interpolation along the par-yield curve followed by bootstrap calculation of spot rates, cubic spline interpolation along the par-yield curve followed by bootstrap calculation of spot rates, cubic spline regression estimation of a continuous discount function using all T-Bonds, and the Coleman-Fisher-Ibbotson method of regression estimation of a piecewise constant forward rate function for all T-Bonds. The resulting spot-rates were then fed into a Hull and White extended Vasicek model to compute estimates of European calls on zero-coupon bonds, American calls on coupon bonds, and swaptions. The estimated prices of all the instruments where then compared to the effective market prices based on the known term structure of spot rates. For some of the estimation techniques, it appeared that option pricing errors were between 18% and 80% on average, depending on the estimation procedure.

- Which **estimation methodology** should we use ? There may exist a large number of econometric techniques to estimate parameters, including non parametric ones[7]. Examples of these are

  - the maximum likelihood (MLE) and its different adaptations, which deals with the probability of having the most likely path between those generated by a model.
  - the generalized method of moments (GMM), which relies upon finding different functions - called "moments" - which should be zero if the model is perfect, and attempt to set them to zero to find correct values of model parameters.
  - filtering techniques, which assume an initial guess and continually improve it as more data become available.

  Which technique is best ? It depends. For instance, compare GMM with MLE. The GMM is reasonably fast, easy to implement and does not require to know the distribution of a noise term, but it does not exploit all the information that we may have regarding a specific model. If we have a complete specification of the joint distribution for interest rates in a multi-factor model, using MLE is more efficient than GMM, but may introduce additional specification errors by specifying arbitrary structures for the measurement errors.

- One should always be cautious with **over-parametrization** or **under-parametrization** of a problem. Calibration can always be achieved using more parameters or by introducing time-varying parameters. But values fluctuating heavily for the estimated parameters can often point to a misspecified or a misestimated model. For instance, Hull and White (1995) themselves wrote:

[7] See for instance Chen and Scott (1993) for MLE, Gibbons and Ramaswamy (1993) or Longstaff and Schwartz (1992) for GMM, or Chen and Scott (1995) for the Kalman filter.

"It is always dangerous to use time-varying model parameters so that the initial volatility curve is fitted exactly. Using all the degrees of freedom in a model to fit the volatility exactly constitutes an **over-parametrization** of the model. It is our opinion that there should be no more than one time varying parameter used in Markov models of the term structure evolution, and this should be used to fit the initial term structure[8]".

3.4 A particular parameter: the market price of risk

A particular parameter in interest rate contingent claim pricing models is the market price of risk. Most valuation models based on the martingale pricing technique require the input of the market price of risk[9]. This parameter is generally not visible in the factor dynamics specification, but appears in the partial differential equation that must be fulfilled by the price of an interest rate contingent claim.

When the underlying variable is a traded asset, such as in the Black and Scholes (1973) framework, the replicating portfolio idea eliminates the need for the market price of risk, as choosing adequate portfolio weights eliminate uncertain returns and thus, risk. But when the underlying variable is not a traded asset, the risk premium has to be specified or estimated from market data. Which methodology is better ? Unfortunately, there is no definite answer. Various specifications can be found in the literature: for instance, Vasicek (1977) exogenously assumes a constant risk premium. Cox, Ingersoll and Ross (1985) show that the endogenous risk premium at equilibrium in their model is $\lambda\sqrt{r(t)}$, a result from their very specific representative investor (which has a logarithmic utility function). But the same risk premium specification is adopted exogenously by Hull and White (1990). But inferring the value of the risk premium from market data is not an easier task. In theory,

[8]This explains why, in practice, the Hull and White (1993) model

$$dr(t) = \kappa(t)[\theta(t) - r(t)]dt + \sigma(t)dW(t)$$

is often implemented with $\kappa(t)$ and $\sigma(t)$ constant and $\theta(t)$ as time-varying. This also explains why when comparing the fit of different models, the BIC criterion is generally prefered to the AIC criterion, so as to penalize adequately the introduction of additional parameters.

[9]Multi-factor models require the input of multiple prices of risk, in fact, one for each factor !

the market price of risk is the same across all derivatives contingent on the same stochastic variable. This should allow one to extract information from one traded security and to use it to value other securities, providing relative valuation as everything becomes dependant on the correct pricing of one initial security. But in practice, the inferred market price of risk may differ across instruments.

As evidenced by Bollen (1997), an incorrect specification of the risk premium can have dramatic consequences (more than 42% of the price) on the valuation of interest rate derivatives. As a consequence, it seems that there is still an important work to be performed in the market price of risk estimation field.

3.5 Model risk and implementation issues

Finally, model risk may also arise even though all of the previous steps were correctly performed. For instance, the model may produce numerically instable or incorrect solutions. As an example, most of the time-invariant models listed in Table 2 suffer from the shortcomings that the short term rate dynamics implies an endogenous term structure, which is not necessarily consistent with the observed one. Furthermore, these models cannot be calibrated to effective yield curves and cannot at the same time fit the initial term structure and a predefined future behavior for the short term rate volatility. As a consequence, practitioners are very reluctant to use them; they often make the parameters time-varying and use this degree of freedom to calibrate exactly the model to current market prices. But in fact, what is called non-stability of the parameters in calibrating the time-invariant model is developed here at time-varying parameters. Model risk is therefore resulting in unstable parameters. But this instability can also result from numerical problems (such as near-singular matrix to be inverted) or from implementation problems: the model may require a large number of iterations to converge (a typical problem in Monte-Carlo simulations or in partial differential equations solving), may require a higher precision for floating point numbers), or may use inappropriate approximations.

Also note that some of the hypotheses of the model may simply not hold in the real world, resulting in a model that performs poorly. For instance, the model assumes that there exists zero-coupon bonds for all required maturities, while in practice, the set of available maturity dates is restricted.

4 Major consequences of model risk

In this section, we will examine the major consequences of model risk in three different domains, namely regarding the pricing, the hedging, and the definition of regulatory capital adequacy rules. When do they arise ? Can we measure them, with or without assuming an objective function ?

4.1 Model risk in pricing

The importance of model risk in pricing should be clear. In the presence of model risk, theoretical prices will diverge from observed ones. If we remain in the framework proposed by Harrisson and Kreps (1979) under which we can compute the price of a contingent claim as the discounted expected value of its future price, the pricing model of an option (say a call option C_t) depends on a pricing function f, on a set of observable parameters Ω_t, and on a set of non-observable parameters θ_t.

$$\widehat{C}_t = f(\Omega_t, \theta_t)$$

But one can add mutually independent zero-mean homoscedastic error terms to the basic model

$$C_t = f(\Omega_t, \theta_t) + \epsilon_t$$

or, as suggested by Jacquier and Jarrow (1995), a multiplicative error specification:

$$C_t = f(\Omega_t, \theta_t) e^{\epsilon_t}$$

with $E(\epsilon_t \,| I_{t,}\theta_t) = 0$ for unbiasedness. This implies that even if we use the true pricing function f, the true parameters Ω_t, and appropriate estimations of the non-observable parameters θ_t, our theoretical prices $\widehat{C}_t$ will differ from the market prices C_t.

How can we distinguish "noise" from model error ? A market error can be the basis of an arbitrage opportunity, whereas a model error cannot. Once we have cleared the observed market prices from these errors, using the true model should provide us with the true price. But in practice, we often have to use the observed price as the true price, as there is no procedure to clear these errors or to define exhaustively the impact of market frictions.

In addition, there still remain some problems regarding the performance of theoretical models for pricing purposes:

- first, the pricing models are often derived under a perfect and complete market paradigm. In practice, they are applied in markets which are incomplete and imperfect. The resulting price is not unique anymore, and one can only derive bounds for the no-arbitrage price.

- second, when comparing model and market prices, one generally uses a quadratic criterion such as the mean and standard deviation of the pricing errors at a given point in time or as the root mean squared error. But such a criterion is only valid if the errors are normally distributed or if the user has a quadratic utility function. The first condition is generally not fulfilled, and the second one is a very specific preference description which has very undesirable properties.

- third, if all traders start using an incorrect model, this model becomes a self-fulfilling prophecy, and comparing theoretical prices to observed ones will result in low average errors. As an example, in the context of stock index options pricing, Chesney, Gibson and Loubergé (1995) show that one can improve artificially the performance of a pricing model by using an implied volatility estimate, while at the same time the basic assumptions of the model are not verified.

4.2 Model risk in hedging (and pricing again !)

The presence of model risk will affect any hedging strategy. As a very simple illustration, let us consider the Black and Scholes (1973) framework: in a complete perfect market, the asset price follows a geometric Brownian motion with constant parameters and constant interest rates[10]:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{2}$$

This defines our true model. We denote by C_t the value at time t of a European call option with maturity T on the asset S_t. By Ito's lemma,

$$dC_t = \left(\frac{\partial C_t}{\partial S_t} \mu S_t + \frac{\partial C_t}{\partial t} + \frac{1}{2} \frac{\partial^2 C_t}{\partial S_t^2} \sigma^2 S_t^2 \right) dt + \frac{\partial C_t}{\partial S_t} \sigma S_t dW_t$$

[10]Working in the Black and Scholes framework leads to an important analytic simplification without any loss of generality. The equivalent derivation in the case of a more general interest rate model can be found in Bossy et al. (1998).

Furthermore, we know that the call price C_t must satisfy the following partial differential equation:

$$\frac{\partial C_t}{\partial S_t} r S_t + \frac{\partial C_t}{\partial t} + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\sigma^2 S_t^2 - rC_t = 0$$

with boundary condition

$$C_T = Max[S_T - K, 0]$$

An investor is short one call option and wants to hedge by creating a replicating portfolio. When hedging in continuous-time using the true model in a frictionless market, a delta hedging strategy should eliminate the option writer's risk completely. At time t, for hedging the short position in the option $(-C_t)$, the investor will hold $\frac{\partial C_t}{\partial S_t}$ units of the underlying asset and $(C_t - \frac{\partial C_t}{\partial S_t} S_t)$ units of cash. The value of his total portfolio Π will be equal to zero if there are no arbitrage opportunities. The portfolio instantaneous variations are defined by

$$d\Pi_t = -dC_t + \frac{\partial C_t}{\partial S_t} dS_t + \left(C_t - \frac{\partial C_t}{\partial S_t} S_t \right) dB_t$$

which can be shown to be equal to

$$d\Pi_t = 0$$

Any other return would give an arbitrage opportunity.

What happens when the hedger uses a misspecified and/or misestimated model ? For simplicity, let us assume that he still uses a single-factor model. By misspecified, we mean that the hedger uses an alternative option pricing model[11]. By misestimated, we mean that the hedger uses the Black and Scholes model, but misestimates the parameters μ and/or σ in equation (2). In both cases, the option pricing model will give a price $\widehat{C_t}$ for the option that differ from the true (market) price C_t and provide an incorrect hedge ratio $\widehat{\frac{\partial C_t}{\partial S_t}}$. Consequently, the hedger's replicating portfolio value will be defined as

$$\Pi_t = -C_t + \widehat{\frac{\partial C_t}{\partial S_t}} S_t + \left(\widehat{C_t} - \widehat{\frac{\partial C_t}{\partial S_t}} S_t \right)$$

[11] For instance, the hedger could use an arithmetic Brownian motion with time-varying parameters, or a mean-reverting diffusion process.

where $\widehat{C_t} - \widehat{\frac{\partial C_t}{\partial S_t}} S_t$ is the amount invested in riskless bonds. Note that Π_t is not necessarily anymore equal to zero. The variation on his portfolio will be

$$\begin{aligned} d\Pi_t &= -dC_t + \widehat{\frac{\partial C_t}{\partial S_t}} dS_t + \left(\widehat{C_t} - \widehat{\frac{\partial C_t}{\partial S_t}} S_t \right) rdt \\ &= -\left(\frac{\partial C_t}{\partial S_t}\mu S_t + \frac{\partial C_t}{\partial t} + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\sigma^2 S_t^2 \right) dt - \frac{\partial C_t}{\partial S_t}\sigma S_t dW_t + \widehat{\frac{\partial C_t}{\partial S_t}} dS_t \\ &\quad + \left(\widehat{C_t} - \widehat{\frac{\partial C_t}{\partial S_t}} S_t \right) rdt \\ &= \left(\left(\widehat{\frac{\partial C_t}{\partial S_t}} - \frac{\partial C_t}{\partial S_t} \right) \mu S_t - \frac{\partial C_t}{\partial t} - \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\sigma^2 S_t^2 + \widehat{C_t} r - \widehat{\frac{\partial C_t}{\partial S_t}} r S_t \right) dt \\ &\quad + \left(\widehat{\frac{\partial C_t}{\partial S_t}} - \frac{\partial C_t}{\partial S_t} \right) \sigma S_t dW_t \end{aligned}$$

Using the partial differential equation that must be satisfied by C_t , the "true" price, we have

$$\begin{aligned} d\Pi_t &= \left(\left(\widehat{\frac{\partial C_t}{\partial S_t}} - \frac{\partial C_t}{\partial S_t} \right) \mu S_t + (\widehat{C_t} - C_t) r + \left(\frac{\partial C_t}{\partial S_t} - \widehat{\frac{\partial C_t}{\partial S_t}} \right) r S_t \right) dt \\ &\quad + \left(\widehat{\frac{\partial C_t}{\partial S_t}} - \frac{\partial C_t}{\partial S_t} \right) \sigma S_t dW_t \end{aligned}$$

that is,

$$\begin{aligned} d\Pi_t &= \left(\left(\widehat{\frac{\partial C_t}{\partial S_t}} - \frac{\partial C_t}{\partial S_t} \right) (\mu - r) S_t \right) dt + (\widehat{C_t} - C_t) rdt \\ &\quad + \left(\widehat{\frac{\partial C_t}{\partial S_t}} - \frac{\partial C_t}{\partial S_t} \right) \sigma S_t dW_t \end{aligned}$$

This equation summarizes the problems of hedging in the presence of model risk. The portfolio instantaneous variation depends on three terms:

- the first one results from a difference between the true delta parameter and the delta given by the model. It also depends on the difference between the drift of the underlying asset and the risk-free rate[12]. Depending on these differences, at maturity, the hedging strategy will

[12] Note that if the hedger uses the Black and Scholes model, but with a misestimated drift coefficient, this first term vanishes as the true delta parameter and the delta given by the model are the same.

create a terminal profit or a terminal loss, and the hedger may end-up with a replicating portfolio that is far from what he should have in order to fulfill his liabilities. For some exotic options, delta hedging can actually even increase the risk of the option writer (see for instance Gallus (1996a)).

- the second one is a consequence of the difference between the true option price and the price given by the model. The initial investment to set up the replicating portfolio is incorrect, and the difference is carried through time at the risk-free rate. As a consequence, the delta hedging strategy may not be self-financing anymore. In other terms, at a given point in time, the hedger may have to borrow and infuse external funds in the strategy in order to keep on implementing the delta-hedge. As the borrowed amount may be larger than the total value of his portfolio, this signifies that delta hedging with model risk can imply bankruptcy.
- the third one results again from a difference between the "true" delta parameter and the delta given by the model. In addition, it depends on a stochastic term, making the hedging strategy result stochastic and path-dependent, and it depends also on σ, the "true" volatility.

To summarize, in the presence of model risk, even though we assume frictionless markets, **the delta hedging strategy is no more replicating or self-financing and even worse, is path-dependent**. The hedger undertakes risk, and should be compensated for it.

How can we account in practice for model risk in hedging ? Rebalancing the hedge more frequently will not help, as there will still be a difference between the true hedging parameters and those given by the model. In some specific cases, a possible solution consists in looking for a super-hedging strategy, i.e. a strategy such that the hedging result is guaranteed whatever the true model[13]. Another solution can be to specify a loss function to be minimized by the hedging strategy[14]. Thus, perfect hedging is transformed into minimum "residual risk" hedging. As a consequence, pricing is not uniquely determined: the risk-neutrality argument cannot be invoked anymore, and

[13]See for instance Lhabitant, Martini, and Reghaï (1998) for options on a zero-coupon bond.

[14]See of instance Bouchaud, Iori, Sornette (1996).

there exists no self-financing strategy for trading a portfolio of the underlying asset and a risk-free bond such that the payoff of the contingent claim equals the value of the self-financing portfolio strategy.

Another important issue in hedging is the aggregation procedure. Using ad-hoc models for each product can provide a better pricing or a better hedging strategy for each individual position. But if those models have distinct idiosyncratic assumptions which are mutually inconsistent, can we simply add them up when examining the aggregated portfolio of various instruments ? Certainly not. Nevertheless, this is widely done in practice, particularly with exotic products.

4.3 Model risk in a capital charge regulatory framework

The regulators seek to ensure that the banks and other financial institutions have sufficient capital to meet large losses within an acceptable margin. Consequently, as we mentioned already, the management of financial institutions must have the ability to identify, monitor and control their global interest rate risk exposure. When an institution's assets and liabilities are contingent on the term structure and its evolution, any change in interest rates may cause a decline in the net economic value of the bank's equity and in its capital-to-asset ratio. Proposition 6 of the Basle Committee Proposal (1997) states:

> "It is essential that banks have interest rate risk measurement systems that capture all material sources of interest rate risk and that assess the effect of interest rates changes in ways which are consistent with the scope of their activities. The assumptions underlying the system should be clearly understood by risk managers and bank management."

This proposition provides banks with a large degree of freedom to choose among a large class of ad-hoc interest rate term structure models. Using their own internal models, banks may calculate their capital requirement as a function of their forecasted ten-days-ahead value-at-risk. The aim is to estimate the potential loss that would not be exceeded with 99% certainty over the next ten trading days.

To ensure that banks use adequate internal models, regulators have introduced the idea of backtesting and multipliers: the market risk capital charge is computed using the bank's own estimate of the value-at-risk, times a multiplier that depends on the number of exceptions[15] over the last 250 days. For instance, in the U.S., the market risk capital charge at time t+1 is defined by[16]

$$MRC_{t+1} = Max\left[VaR_t(10,1); \frac{M_t}{60}\sum_{i=1}^{60} VaR_{t-i}(10,1)\right]$$

where $VaR_t(10,1)$ denotes the value-at-risk on day t using a ten-day holding period and a 99% coverage. As noted by the Basle Committee on Banking Supervision (1996), the multiplier M_t must be at least equal to 3; furthermore, it increases with the magnitude and the number of exceptions, as both are a matter of concern for the regulators. If there are four or fewer exceptions, M_t remains at three. Between five and nine exceptions, M_t increases with the number of exceptions. With ten and more exceptions, M_t is set to 4 and the bank model is deemed to be inaccurate and must be improved. Alternative model-evaluation methods include the binomial distribution and interval forecast evaluation. In the first method, banks report their one-day value-at-risk estimate and their actual portfolio losses; the latter are then modeled as a random variable drawn from an independent binomial distribution with a probability of occurrence specified as one percent; the test consists in computing a likelihood ratio and comparing it to a $\chi^2(1)$ critical value[17]. In the second method, adapted from Christoffersen (1997), the test consists in a conditional or unconditional forecast of the lower one-percent interval of the one-step-ahead return distribution.

The new proposed pre-commitment approach is more flexible: banks choose and report a level of capital that they consider as adequate to back their trading books. This level of capital can be computed by any procedure, including the use of an internal model. But if the cumulative losses of the trading book exceed the chosen capital charge, the bank is penalized - by a way that remains to be specified, for instance by disclosure - by the regulators.

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[15] An exception occurs when the loss exceeds the model calculated value-at-risk.

[16] In fact, there is an additional capital charge for the portfolio idiosyncratic credit risk.

[17] The methodology suffers from various critiques, as evidenced by Kupiec (1995), including poor properties in finite samples and a low power in medium size samples.

Whatever these penalties or value-at-risk adjustments, they result in overfunding and are nothing else than simple "ad-hoc" safety procedures to account for the impact of model risk. A bank might use an inadequate or inappropriate model, but the resulting impact is mitigated by adjusting the capital charge. As a consequence, banks that attempt to use "better quality" models are penalized if model risk analysis is poorly assessed.

In addition, these penalties or value-at-risk adjustments also reduce the moral hazard problem introduced by the freedom to select an internal model. For instance, as evidenced by Aussenegg and Pichler (1997), when estimating the value-at-risk of a bond portfolio, models of the spot rate based on the normality assumption perform very poorly - even when they are extended with time-dependent means and volatilities - while historical simulation performs better. But the value-at-risk based on the normality based models is lower than those implied by the historical simulations based models method. Knowing that its charge in capital will depend on its value-at-risk, will a bank select the most adequate model or the one that gives the lowest value-at-risk ?

Clearly, whatever the way Bank of International Settlements measures and accounts for model risk will create an opportunity for regulatory induced model arbitrage.

4.4 Necessity of a model risk loss function

In all of the above cited cases, the objectives of the model user were clearly different. This shows that **we need to specify a loss function to measure how precise a model proves to be**. The objective will be to select the model that minimizes the value of this loss function for a specific agent or institution.

Of course, the loss function will depend on the specific applications associated with the model. For instance, when pricing, we may select as a loss function such as the root mean squared error, the average error, or the maximum error compared to effectively quoted prices; when hedging, this loss function may depend on the statistical properties of the terminal value of the total position (such as the average terminal profit or loss[18], its variance, etc.) or be defined in terms of intertemporal behavior (for instance in terms

[18] This is often refered to as building a risk neutral strategy "on average", as the hedged portfolio grows at the risk free rate on average for multiple realisations of the underlying, but not necessarily for one given realisation.

of average error over time, maximal loss, first passage time below zero, etc); in regulatory issues, the loss function can be defined in terms of magnitude and number of value-at-risk exceptions, as proposed by Lopez (1998), or any alternative function that captures certain aspects of regulators' concerns (for instance, minimize the systemic risk of large losses).

In addition, such a loss function will often depend on a specific time-horizon which varies with the type of positions considered, the division and/or the responsability levels involved (trading desk versus management), the motivation (private versus regulatory), the asset class (equity, fixed income, derivatives), the activity (trading, pricing, hedging, etc.) the risk aversion, the relative size of the position or the industry (bank versus insurance). It can also differ between a marginal position or the aggregate portfolio, if diversification allows for a model risk reduction. And for a given model and a specific instrument, the loss function will also depend on whether the model user's net position is on the short or on the long side.

This clearly shows that the model risk loss function will depend on each specific application and should be should be decided on an application-by-application basis under the constraints and objectives faced by the financial institution.

5 Conclusion

This paper has shown that the reliance on models to handle interest rate risks carries its own risks, as the use of mathematical models requires simplifications and hypotheses which may cause the models to diverge from reality. Furthermore, developing or selecting a model is always a trade-off between realism and accuracy and computability.

Whatever the model used in interest rate risk management, three key issues should always be addressed:

- Have all important variables and relevant parameters been included in the model ?
- Have all the assumptions about the dynamics of these variable been verified ?
- Are the results from simulation compatible with similar observed market situations ?

Once these points have been verified, it is important to be aware of the presence of the model uncertainty, while still being aware that there is no simple way of overcoming the problem[19].

In fact, what should the properties of a "desirable" and ideal term structure model be? First of all, the model should be applicable in the considered market, parsimonious regarding the number of factors, fast to operate, easy to calibrate and to use. Its results should be easily interpreted and comprehensible by every user (in particular, they should not be counter-intuitive or esoteric; otherwise, the model might be rejected because of lack of understanding, and this will lead the users to a lack of confidence and trust in the model. The model should also be internally consistent and accurate against the market and be arbitrage-free; this is another essential point in building the confidence needed to use the model. Its parameters should be robust and stable from one fitting to another; under normal conditions, unstable parameters are often an indication of a poorly specified model. Finally, the model should be exhaustive across products, and perform equally well under differing economic conditions or strategies.

But all of these features remain "true" for an ideal model. In practice, a "good" model will simply provide a useful applicable approximation for the tasks at hand. Model risk should be assessed with a loss function and a time horizon that are adequate and relevant based on the institution's current objectives; in particular, users of the model (traders, regulators, senior managers, etc.) should be educated with respect to the model limits, and the loss function should be made consistent with the incentives of the model users.

Measuring model risk is challenging, specifically in the domain of interest rates, where there exists a large number of products and incompatible models simultaneously. Model risk should not be considered as a tool to find "the" perfect model, but rather as an instrument and/or a methodology that helps understanding the weaknesses and exploiting the strengths of the alternatives at hand. Progressive dynamic learning has already been proved to be effective in model performance enhancement.

Last, but not least, another essential issue is related to model risk diversification. If model risk cannot be fully diversified, the residual risk should

[19] An alternative to reduce model risk is to allow the possibility of more than one model to be acceptable as sufficiently close approximation to the given data for the required objective. This notion of having more than one model and weighting them is a key element of the Bayesian model averaging, as proposed by West and Harrisson (1989), chapter 12.

be priced by the agents in the market. An important consequence in the banking industry is to determine who bears the costs: the clients, the shareholders, the bondholders, or the government, if there is a systemic model driven failure in the financial market ?

References

- ADAMS K.J., VAN DEVENTER D. (1994), "Fitting yield curves and forward rate curves with maximum smoothness", *Journal of Fixed Income*, vol. 4, June, pp. 52-62
- AIT-SAHALIA Y. (1996), "Testing continuous-time models of the spot interest rate", *Review of Financial Studies*, vol. 9, pp. 385-426
- AKAIKE H. (1973), "Information theory and an extension to the maximum likelihood principle", in *Proceedings of the Second International Symphosium on Information Theory*, P.N.Petrov and F.Csaki (eds.), Akademia Kiado, Budapest, pp. 267-281
- ARNOLD L. (1973), "Stochastic differential equations", John Wiley and Sons, New York
- AUSSENEGG W., PICHLER S. (1997), "Empirical evaluation of simple models to calculate value-at-risk of fixed income instruments", Working paper, Vienna University of Technology
- BAKSHI G., CAO C., CHEN Z (1997), "Empirical performance of alternative option pricing models", Working paper, forthcoming in The Journal of Finance
- BARNHILL T. Jr, JORDAN J., BARNHILL T., MACKEY S. (196), "The effects of term structure estimation on the valuation of interest-rate derivatives", Working paper
- BASLE COMMITTEE ON BANKING SUPERVISION (1995), "Framework for supervisory information about derivatives activities of banks and securities firms"
- BASLE COMMITTEE ON BANKING SUPERVISION (1996), "Supervisory framework for the use of backtesting in conjonction with the internal models approach to market risk capital requirements", manuscript, Bank for Internal Settlements
- BASLE COMMITTEE ON BANKING SUPERVISION (1997), "Principles for the management of interest rate risk", January

- BLACK F., SCHOLES M. (1973), "The pricing of options and corporate liabilities", Journal of Political Economy, vol. 81, pp. 637-659
- BLACK F. (1986), "Noise", Journal of Finance 41, pp. 529-543
- BLACK F. (1990), "Living up to the model", in "From Black Scholes to Black Holes", RISK, pp. 17-22
- BOLLEN N.P.B. (1990), "Derivatives and the market price of risk", The Journal of Futures Markets, vol. 17, no. 7, pp. 839-854
- BOSSY M., GIBSON R., LHABITANT F.S., MARTINI C., PISTRE N., TALAY D. (1998), "Model risk analysis for discount bond options", Risklab Report
- BROZE L., SCAILLET O., ZAKOIAN J.-M. (1995), "Testing for continuous-time models of the short term interest rate", Journal of empirical finance, vol. 2, pp. 199-223
- BOUCHAUD J.P., IORI G., SORNETTE D. (1996), "Real world options", RISK, vol. 9, no. 3, March, pp. 61-65
- BRENNAN M.J., SCHWARTZ E.S. (1979), "A continuous-time approach to the pricing of bonds", Journal of Banking and Finance", vol. 3, pp. 135-155
- BRENNAN M.J., SCHWARTZ E.S. (1980), "Analysing convertible bonds", Journal of Financial and Quantitative Analysis 15, pp.907-929
- BUHLER W., UHRIG-HOMBURG M., WLATER U., WEBER T. (1998), "An empirical comparison of forward and spot rate models for valuing interest rate options", Working paper, forthcoming in The Journal of Finance
- CARVERHILL A. (1995), "A note on the models of Hull and White for pricing options on the term structure", Journal of Fixed Income, September, pp. 89-96
- CHAN K.C., KAROLYI A., LONGSTAFF F., SANDERS A. (1992), "An empirical comparison of alternative models of the short term interest rate", Journal of Finance, vol. 47, pp. 1209-1227

- CHATFIELD Chris (1996), "Model uncertainty and forecast accuracy", Journal of Forecasting, vol. 15, pp. 495-508
- CHEN R., SCOTT L. (1993), "Maximum likelihood estimation of a multi-factor equilibrium model of the term structure of interest rates", Journal of Fixed Income, December, pp. 14-32
- CHEN R., SCOTT L. (1995), "Multi-factor Cox-Ingersoll-Ross models of the term structure: estimates and tests from a Kalman filter model", working paper, Rutgers University and University of Georgia
- CHEN L. (1996), "Stochastic mean and stochastic volatility: a three factor model of the term structure of interest rates and its application to the pricing of interest rate derivatives", Blackwell publishers
- CHESNEY M., GIBSON R., LOUBERGE H. (1995), "Arbitrage trading and index option pricing at the SOFFEX: an empirical study using daily and intradaily data", FinanzMarkt und Portfolio Management, no. 1, pp. 35-61
- CHRISTOFFERSEN P.F. (1997), "Evaluating interval forecasts", Manuscript, Research Departement, International Monetary Fund
- COX J.C., INGERSOLL J.E., ROSS S.A. (1980), "An analysis of variable rate loan contracts", Journal of Finance 35, pp. 389-403
- COX J.C., INGERSOLL J.E., ROSS S.A. (1985), "A theory of the term structure of interest rates", Econometrica, vol. 53, pp. 385-407
- DAS S.R., FORESI S. (1996), "Exact solution for bond and option prices with systematic jump risk", Review of Derivatives Research, vol. 1, pp. 7-24
- DAVES P.R., EHRHARDT M.C. (1993), "Liquidity, reconstitution and the value of U.S. Treasury Strips", Journal of Finance, vol. 48, March, pp. 315-329
- DERMAN E. (1996a), "Valuing models and modeling value: a physicist's perspective on modelling on Wall Street", The Journal of Portfolio Management, Spring, pp. 106-114

- DERMAN E. (1996b), "Model risk", RISK, vol. 9, no 5, May, pp. 34-37
- DIJKSTRA T.K. (1988), "On model uncertainty and its statistical implications", Berlin: Springer Verlag
- DOTHAN U.L. (1978), "On the term structure of interest rates", Journal of Financial Economics, vol. 6, pp. 59-69
- DUARTE Antonio Marcos (1997), "Model risk and risk management", Derivatives Quarterly, Spring, pp. 60-72
- EL KAROUI N., JEANBLANC-PICQUE M. (1994), "Robustness of the Black and Scholes formula", Prepublication no 3, Equipe d'Analyse et de Probabilite, Universite d'Evry
- FOURNIE E., TALAY D. (1991), "Application de la statistique des diffusions à un modèle de taux d'intérêt", Finance, vol. 12, no. 2, pp. 79-111
- GALLUS C. (1996a), "Exploding hedging errors for digital options", Working paper, Deutsche Morgan Grenfell, October
- GALLUS C. (1996b), "Robustness of hedging strategies for European options", Working paper, University of Erlanger
- GIBBONS R.M., RAMASWAMY K. (1993), "A test of the Cox, Ingersoll and Ross model of the term structure", Review of Financial Studies 6, pp. 619-658
- GIBSON R., LHABITANT F.S., TALAY D. (1997), "Modelling the term structure of interest rates: a review of the literature", RiskLab report
- GOING A. (1997), "Estimation in financial models", Risklab report.
- HARRISON M.J., KREPS D.M. (1979), "Martingales and arbitrage in multiperiod securities markets", Journal of Economic Theory, vol. 20, pp. 381-408
- HULL J., WHITE A. (1990), "Pricing interest rate derivative securities", Review of Financial Studies, vol. 3, pp. 573-592

- HULL J., WHITE A. (1993), "One factor interest rate models and the valuation of interest rate derivative securities", Journal of Financial and Quantitative Analysis, vol. 28, no. 2, June, pp. 235-254
- HULL J., WHITE A. (1995), "A note on the models of Hull and White for pricing options on the term structure. Response", Journal of Fixed Income, September, pp. 97-102
- JACQUIER Eric, JARROW Robert (1996), "Model error in contingent claim models dynamic evaluation", Cirano Working Paper 96s-12.
- JAMES J. (1998), "Calibration issues for interest rate models", mimeo, First National Bank of Chicago
- JORDAN J.V. (1984), "Tax effects in term structure estimation", Journal of Finance, vol. 39, June, pp. 393-406
- KUPIEK P. (1995), "Techniques for verifying the accuracy of risk measurement models", Journal of Derivatives, vol. 3, pp. 73-84
- LHABITANT F.S., MARTINI C., REGHAI A. (1998), "Volatility risk for options on a zero-coupon bond", Risklab Report
- LITTERMAN R., SCHEINKMAN J. (1991), "Common factors affecting bond returns", Journal of Fixed Income, vol. 1, pp. 54-62
- LONGSTAFF F., SCHWARTZ E. (1992), "Interest rate volatility and the term structure: a two factor general equilibrium model", Journal of Finance, vol. 47, pp1259-1282
- LOPEZ J.A. (1998), "Methods for evaluating value-at-risk estimates", Federal Reserve Bank of New-York, Working paper
- MALLOWS C.L. (1973), "Some comments on C_p", Technometrics, vol. 15, pp. 661-675
- MERTON R.C. (1973), "Theory of rational option pricing", Bell Journal of Economics and Management Science, vol. 4, pp. 141-183
- MERTON Robert C. (1976), "The impact on option pricing of specification error in the underlying stock price returns", Journal of Finance, vol. 31, no 2, pp. 333-350

- MITSUBISHI (1990), "Mistubishi Finance Risk Directory", London:Risk Magazine
- NELSON C.R., SCHAEFER S. (1983), "The dynamics of the term structure and alternative portfolio immunization strategies", in "Innovations in bond portfolio management", edited by G.G.Kaufman, G.O.Bierwag, JAI Press, Greenwitch, Connecticut
- REBONATO R., COOPER I. (1996), "The limitations of simple two factor interest rate models", The Journal of Financial engineering, vol. 5, no 1, pp. 1-16
- REBONATO R. (1997), "Interest rate option models", New York: John Wiley
- RENAULT Eric (1996), "Econometric models of option pricing errors", Working paper, World Congress of Econometrics Society, Tokyo
- RICHARD S. (1978), "An arbitrage model of the term structure of interest rates", Journal of Financial Economics, vol. 6, pp. 33-57
- RISSANEN J. (1978), "Modeling by shortest data description", Automatica, vol. 14, pp. 465-471
- RISKMETRICS (1995), "Riskmetrics technical document", J.P.Morgan, New-York, May 26
- SCHAEFER S.M., SCHWARTZ E.S. (1984), "A two factor model of the term structure: an approximate analytical solution", Journal of Financial and Quantitative Analysis, vol. 19, no. 4, December, pp. 413-424
- SCHAEFER S.M., SCHWARTZ E.S. (1987), "Time dependant variance and the pricing of bonds", Journal of Finance 42, pp. 1113-1128
- SCHWARZ G. (1978), "Estimating the dimension of a model", Ann. Statist., vol. 6, pp. 461-464
- VASICEK O. (1977), "An equilibrium characterization of the term structure", Journal of Financial Economics, vol. 5, pp. 177-188

- WEST M., HARRISON P.J. (1989), "Bayesian forecasting and dynamic linear models", New-York:Springer Verlag