ON THE FIRST PASSAGE TIMES OF GENERALIZED ORNSTEIN-UHLENBECK PROCESSES

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Abstract. We study the two-dimensional joint distribution of the first passage time of a constant level by spectrally negative generalized Ornstein-Uhlenbeck processes and their primitive stopped at this first passage time. We show an explicit expression of the Laplace transform of the distribution in terms of new special functions. Finally, we give an application in finance which consists on computing the Laplace transform of the price of a European call option on the maximum on the yield in the generalized Vasicek model. The stable case is studied in more details.

1. Introduction

Let $Z := (Z_t,\ t \geq 0)$ be a spectrally negative Lévy process starting from 0. For any $\lambda > 0$, we define a generalized Ornstein-Uhlenbeck (for short GOU) process $X := (X_t, t \geq 0)$, starting from $x \in \mathbb{R}$, with backward driven Lévy process (for short BDLP) Z as the unique solution to the following stochastic differential equation

$$dX_t = -\lambda X_t \, dt + dZ_t. \tag{1.1}$$

They are a generalization of the classical Ornstein-Uhlenbeck process constructed by simply replacing the driving Brownian motion by a Lévy process.

In this paper we are concerned with the positive random variables T_y and the functional I_t defined by

$$T_y = \inf\{s \geq 0;\ X_s > y\} \qquad \text{and} \qquad I_t = \int_0^t X_s \, ds \tag{1.2}$$

respectively. The Laplace transform of T_y is known from Hadjiev (1985). There is a tremendous bunch of literature regarding the distribution of additive functionals, stopped at certain random times, of diffusions processes, see for instance the book of Borodin and Salminen (2002) for a collection of explicit results. However, the law of such functionals for Markov processes with jumps are unknown besides some special cases (e.g. the exponential functional of some Lévy processes, see Carmona et al. (2001) and the Hilbert transform of Lévy processes see Fitzsimmons and Getoor (1992) and Bertoin (1995)). The explicit form of the joint distribution (T_y, I_{T_y}), when X is the classical Ornstein-Uhlenbeck, is given by Lachal (1993). There, he exploits the fact that the bivariate process $(I_t, X_t, t \geq 0)$ is a Markov process. We shall extend his result by providing the Laplace transform of this two-dimensional distribution in the general case, i.e. when X is a GOU process as defined above.

Key words and phrases. Generalized Ornstein-Uhlenbeck process; stable process; first passage time; martingales; special functions; term structure; path dependent options.

Research of the author was supported by Credit Suisse Group, Swiss Re and UBS AG through RiskLab, Switzerland.

We recall that first passage times problems for Markov processes are closely related to the finding of an appropriate martingale associated to the process. In contrast to Lachal (1993), we shall provide a methodology which allows to build up the martingale used to compute the sought joint Laplace transform. As a second step we shall apply martingales techniques to derive the Laplace-Fourier transform.
GOU processes have found many applications in several fields. Recently, they have been used intensively in finance, for modelling the stochastic volatility of a stock price process (see e.g. Barndorff-Nielsen and Shephard (2001)) and for describing the dynamics of the instantaneous interest rate. The later application, as a generalization of the Vasicek model, deserves a particular attention as these processes belong to the class of one factor affine term structure model. These are well known to be tractable, in the sense that it is easy to fit the entire yield curve by basically solving Riccati equations, see Duffie et al. (2001) for a survey on affine processes. From the expression of the joint Laplace transform of (T_y, I_{T_y}), we can provide an analytical formula for the price of an European call option on maximum of the yield in our framework.
The remainder of the paper is organized as follows. In Section 2, we recall some results about Lévy and GOU processes and mention some well known facts about the first passage times above a constant level. In Section 3, we give an explicit form for the joint Laplace transform (T_y, I_{T_y}) in terms of new special functions. Section 4 is devoted to the special case of stable OU processes, that is when Z is a stable process. In the last section, we apply the previous results to the pricing of path dependent option on yield with a more detailed study of the stable Vasicek case.

2. Preliminaries and Recalls

Throughout the rest of this paper $Z := (Z_t,\, t \geq 0)$ denotes a real-valued spectrally negative Lévy process, that is a process with stationary and independent increments, whose Lévy measure ν^Z charges only the negative real line ($\nu^Z((0,\infty)) = 0$). The Lévy measure being the compensator of the jumps measure of the process, Z has non-positive jumps. Further, for every $z \in \mathbb{R}$, $\mathbb{P}_z$ will denote the law of the Lévy process starting from z, we will write simply $\mathbb{P}$ for $\mathbb{P}_0$. Due to the absence of positive jumps, it is possible to extend analytically the characteristic function of Z to the negative imaginary line. Thus, one characterizes this process by its so-called Laplace exponent $\psi : [0,\infty) \to (-\infty,\infty)$ which is specified by the identity

$$\mathbb{E}\big[\exp(uZ_t)\big] = \exp(t\psi(u)), \qquad t, u \geq 0$$

and has the form

$$\psi(u) = bu + \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{0} (e^{ur} - 1 - ur\chi(r))\nu^Z(dr), \tag{2.1}$$

where $\chi(r) := \mathbb{I}_{\{r>-1\}}$, $b \in \mathbb{R}$, $\sigma \geq 0$ and $\nu^Z(.)$ is the Lévy measure on $(-\infty, 0]$ which satisfies the integrability condition $\int_{-\infty}^{0}(1 \wedge x^2)\,\nu^Z(dx) < \infty$. It is known that ψ is a convex function with $\lim_{u\to\infty}\psi(u) = +\infty$. We recall that for processes with independent increments, the Laplace exponent characterizes completely the law of the process (see e.g. Jacod and shiryaev (2002)). We assume that the process X does not drift to $-\infty$, which is the case when $\psi'(0^+)$ is non negative, see Bertoin (1996, Chapter VII) for a thorough description of these processes.

We introduce the first passage time process $\tau := (\tau_z,\ z \geq 0)$ defined, for a fixed $z \geq 0$, by

$$\tau_z = \inf\{s \geq 0;\ Z_s > z\}.$$

Due to the strong Markov property and the spatial homogeneity of Z, τ is a subordinator, which is a Lévy process with a.s. increasing sample paths. Furthermore, we have

$$\mathbb{P}\left[Z_{\tau_z} = z \text{ for every } z \geq 0\right] = 1.$$

Denoting by ϕ the inverse function of the continuous and increasing function ψ, the Laplace exponent of τ is given by, see Bertoin (1996, Theorem VII.1)

$$\mathbb{E}\left[\exp\left(-u\tau_z\right), \tau_z < \infty\right] = \exp\left(-z\phi(u)\right). \tag{2.2}$$

We now define a GOU process $X :=(X_t,\ t \geq 0)$, with parameter $\lambda > 0$ and a BDLP Z, as the unique solution to the following stochastic differential equation

$$dX_t = -\lambda X_t\, dt + dZ_t, \quad X_0 = x \in \mathbb{R}. \tag{2.3}$$

In terms of Z, this is given by

$$X_t = e^{-\lambda t}\left(x + \int_0^t e^{\lambda s}\, dZ_s\right), \quad t \geq 0. \tag{2.4}$$

Remark 2.1. *Replacing the Lévy process Z by any semimartingale in the SDE (2.3) will give the same unique solution (2.4).*

From the expression (2.4), it is easy to derive the Laplace exponent of X

$$\mathbb{E}_x\left[\exp\left(uX_t\right)\right] = \exp\left(e^{-\lambda t}xu + \int_0^t \psi(e^{-\lambda r}u)\, dr\right), \quad u \geq 0.$$

From the representation (2.4), we also get, for $t \geq 0$,

$$\begin{aligned}
X_t &= e^{-\lambda t}x + \int_0^t e^{-\lambda(t-s)}\, dZ_s \\
&\overset{(d)}{=} e^{-\lambda t}x + \int_0^t e^{-\lambda s}\, dZ_s \\
&\overset{a.s.}{\longrightarrow} \int_0^\infty e^{-\lambda s}\, dZ_s \quad \text{as } t \to \infty,
\end{aligned}$$

where $\overset{(d)}{=}$ denotes equality in distribution. Consequently, the Laplace transform of the limiting distribution of X, denoted by $\widehat{\rho}^X(u)$, $u \geq 0$, is given by

$$\widehat{\rho}^X(u) = \exp\left(\int_0^\infty \psi(e^{-\lambda r}u)\, dr\right) \tag{2.5}$$

whenever the Lévy measure satisfies the following condition

$$\int_{r<-1} \log|r|\, \nu^Z(dr) < \infty, \tag{2.6}$$

see Sato (1999, Chapter III). The measure ρ^X is selfdecomposable and it is the unique stationary distribution of the process X. We mention that in Carmona et al. (1997), they establish a relationship between the invariant measure of GOU processes and the law of exponential functionals of Lévy processes.

The process of jumps associated to X is defined, for a fixed $t \geq 0$, by

$$\Delta X_t = X_t - X_{t-}.$$

From (2.3), we observe that X and Z have the same process of jumps.
The process X is a Feller process. Its infinitesimal generator $\mathcal{A}$ is an integro-differential operator acting on $C_0(\mathbb{R})$, the space of continuous functions vanishing at infinity. It is defined by

$$\begin{aligned}\mathcal{A}f(x) &= \frac{1}{2}\sigma^2 f''(x) + (b-\lambda x)f'(x) + \\ &\int_{-\infty}^{0} \left(f(x+r) - f(x) - f'(x)r\chi(r)\right)\nu^Z(dr).\end{aligned}$$

To complete the description, from the general theory of processes viewpoint, X is a special semimartingale with predictable characteristics triplet given by

$$\left(bt - \lambda \int_0^t X_s\, ds, \frac{1}{2}\sigma^2 t, \nu^Z(dr)dt\right). \tag{2.7}$$

3. Study of the law of $(T_y, \int_0^{T_y} X_s\, ds)$

Let X be a spectrally negative generalized Ornstein-Uhlenbeck process starting from $x \in \mathbb{R}$. For $y > x$, set

$$T_y = \inf\{s \geq 0;\ X_s > y\}. \tag{3.1}$$

For the remaining of the paper, we impose the following condition on the Lévy measure of Z.

Assumption
Either

$$\sigma > 0 \quad \text{or} \quad b - \int_{-1}^{0} r\nu^Z(dr) > \lambda y. \tag{3.2}$$

Our aim in this section is to characterize the joint law of the couple $(T_y, \int_0^{T_y} X_s ds)$ by computing its double Laplace transform. We shall start with computing the following joint Laplace transform

$$I_{\gamma,\theta}(x,y) := \mathbb{E}_x\left[\exp\left(-\gamma T_y + \theta \int_0^{T_y} X_s\, ds\right)\right]. \tag{3.3}$$

Before stating our main result we shall prepare two results.

Lemma 3.1. *For $\gamma, \theta > 0$ such that $\eta := \gamma - \psi(\frac{\theta}{\lambda}) > 0$, and $y > x$, we have*

$$I_{\gamma,\theta}(x,y) = e^{-\frac{\theta}{\lambda}(y-x)}\, \mathbb{E}_x\left[\exp\left(-\eta\, T_y^{(\frac{\theta}{\lambda})}\right)\right], \tag{3.4}$$

where $T_y^{(\frac{\theta}{\lambda})} = \inf\left\{s \geq 0;\ X_s^{(\frac{\theta}{\lambda})} > y\right\}$, $X^{(\frac{\theta}{\lambda})} := (X_t^{(\frac{\theta}{\lambda})}, t \geq 0)$ being a GOU process with the following predictable characteristics triplet

$$\left(\bar{b}t - \lambda \int_0^t X_s\, ds, \frac{1}{2}\sigma^2 t, e^{\frac{\theta}{\lambda}r}\nu^Z(dr)dt\right), \tag{3.5}$$

where $\bar{b} := b + \frac{\theta}{\lambda}\sigma^2 + \int_{-\infty}^{-1}(e^{\frac{\theta}{\lambda}r} - 1)r\nu^Z(dr)$.

Remark 3.2. *We mention that this lemma can be easily extended to compute the joint law of the couple* $(T_y, \int_0^{T_y} \Lambda(X_t)\, ds)$ *where* X *is any diffusion with jumps of the following form*

$$X_t = \int_0^t \Lambda(X_t)\, dt + Z_t, \quad t \geq 0,$$

where $\Lambda(x)$ *is any locally integrable function on* $\mathbb{R}$ *and* $T_y = \inf\{s \geq 0;\, X_s > y\}$ *such that* y *is regular for itself.*

Proof. Fix $y > x$. Exploiting the fact that X has non-positive jumps, we get

$$\int_0^{T_y} X_s\, ds = \frac{1}{\lambda}\left(Z_{T_y} + x - y\right),$$

which yields

$$I_{\gamma,\theta}(x,y) = e^{-\frac{\theta}{\lambda}(y-x)}\, \mathbb{E}_x\left[\exp\left(-\gamma T_y + \frac{\theta}{\lambda} Z_{T_y}\right)\right].$$

Denote by $\mathcal{F}_t = \sigma(X_s, s \leq t)$ the natural filtration of X up to time t. We now consider the Girsanov's transform $\mathbb{P}^{(\xi)}$ of the probability measure $\mathbb{P}$ which is defined by

$$d\mathbb{P}^{(\xi)}_{|\mathcal{F}_t} = \exp\left(\xi Z_t - t\psi(\xi)\right)\, d\mathbb{P}_{|\mathcal{F}_t},\ t, \xi \geq 0.$$

Under $\mathbb{P}^{(\xi)}$, Z, denoted by $Z^{(\xi)}$, is again a Lévy process with the following Laplace exponent, for $u \geq 0$

$$\begin{aligned}
\psi^{(\xi)}(u) \quad &:= \quad \log\left(\mathbb{E}\left[\exp\left(u Z_1^{(\xi)}\right)\right]\right) \\
&= \quad \log\left(\mathbb{E}\left[\exp\left((u+\xi)Z_1\right)\right]\right) - \psi(\xi) \\
&= \quad \psi(u+\xi) - \psi(\xi) \\
&= \quad b(u+\xi) + \frac{1}{2}\sigma^2(u+\xi)^2 + \int_{-\infty}^0 \left(e^{(u+\xi)r} - 1 - (u+\xi)r\chi(r)\right)\nu^Z(dr) \\
&\quad\quad - \left(b\xi + \frac{1}{2}\sigma^2\xi^2 + \int_{-\infty}^0 (e^{\xi r} - 1 - \xi r\chi(r))\nu^Z(dr)\right) \\
&= \quad \left(b + \sigma^2\xi + \int_{-\infty}^{-1}(e^{\xi r} - 1) r\nu^Z(dr)\right) u + \frac{1}{2}\sigma^2 u^2 \\
&\quad\quad + \int_{-\infty}^0 (e^{ur} - 1 - ur\chi(r)) e^{\xi r}\nu^Z(dr).
\end{aligned}$$

By using the representation (2.7), it is straightforward to deduce the predictable characteristics triplet of the GOU process $X^{(\frac{\theta}{\lambda})}$. We point out that $X^{(\frac{\theta}{\lambda})}$ has again non-positive jumps since the two probability measures are absolutely continuous.

Finally, our relationship is obtained as follows

$$\begin{aligned} I_{\gamma,\theta}(x,y) &= e^{-\frac{\theta}{\lambda}(y-x)}\,\mathbb{E}_x\left[\exp\left(-\gamma T_y+\frac{\theta}{\lambda}Z_{T_y}\right)\right] \\ &= e^{-\frac{\theta}{\lambda}(y-x)}\,\mathbb{E}_x\left[\exp\left(-(\gamma-\psi(\frac{\theta}{\lambda}))T_y+\frac{\theta}{\lambda}Z_{T_y}-\psi(\frac{\theta}{\lambda})T_y\right)\right] \\ &= e^{-\frac{\theta}{\lambda}(y-x)}\,\mathbb{E}_x\left[\exp\left(-(\gamma-\psi(\frac{\theta}{\lambda}))T_y^{(\frac{\theta}{\lambda})}\right)\right]. \end{aligned}$$

□

We now recall the Laplace transform of the random variable T_y see Novikov (2003) or Hadjiev (1985) for a similar proof with more restrictive conditions.

Proposition 3.3. *For $y > x$, and $\gamma > 0$, we have*

$$\mathbb{E}_x\left[\exp\left(-\gamma T_y\right)\right]=\frac{H_{\frac{\gamma}{\lambda}}(x)}{H_{\frac{\gamma}{\lambda}}(y)}, \tag{3.6}$$

where

$$H_\nu(x) = \int_0^\infty \exp\left(xr-\frac{1}{\lambda}\int_1^r \psi(u)\,\frac{du}{u}\right) r^{\nu-1}dr. \tag{3.7}$$

Remark 3.4. *The Laplace transform is analytical on the domain $\{\eta\in\mathbb{C}:\Re(\eta)>0\}$.*

Remark 3.5. *It is in the proof of this lemma that the assumption 3.2 is required to ensure that the function H_ν is well defined, see Novikov (2003).*

See Section 5.2 for a proof of this result in the case of stable BDLP. We are now ready to state the following.

Theorem 3.6. *For $\gamma,\theta > 0$ and $y > x$, we have*

$$I_{\gamma,\theta}(x,y) = e^{-\frac{\theta}{\lambda}(y-x)}\frac{H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(x)}{H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(y)}, \tag{3.8}$$

where

$$H_{\nu,\beta}(x) = \int_0^\infty \exp\left(xr-\frac{1}{\lambda}\int_0^r \psi(u+\beta)\,\frac{du}{u}\right) r^{\nu-1}dr. \tag{3.9}$$

Proof. Combining the results of the two previous lemma, we obtain, with the obvious notation

$$I_{\gamma,\theta}(x,y) = e^{-\frac{\theta}{\lambda}(y-x)}\frac{H^{(\frac{\theta}{\lambda})}_{\frac{\eta}{\lambda}}(x)}{H^{(\frac{\theta}{\lambda})}_{\frac{\eta}{\lambda}}(y)}.$$

By remarking that

$$\begin{aligned} H^{(\frac{\theta}{\lambda})}_{\frac{\eta}{\lambda}}(x) &= \int_0^\infty \exp\left(xr-\frac{1}{\lambda}\int_1^r \psi^{(\frac{\theta}{\lambda})}(u)\,\frac{du}{u}\right) r^{\frac{\eta}{\lambda}-1}dr \\ &= \int_0^\infty \exp\left(xr-\frac{1}{\lambda}\int_1^r \psi(u+\frac{\theta}{\lambda})\,\frac{du}{u}\right) r^{\frac{\gamma}{\lambda}-1}dr \\ &= \exp\left(\frac{1}{\lambda}\int_0^1 \psi(u+\frac{\theta}{\lambda})\,\frac{du}{u}\right) H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(x), \end{aligned}$$

we obtain

$$\frac{H^{(\frac{\theta}{\lambda})}_{\frac{\eta}{\lambda}}(x)}{H^{(\frac{\theta}{\lambda})}_{\frac{\eta}{\lambda}}(y)} = \frac{H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(x)}{H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(y)}. \tag{3.10}$$

By using the convexity of ψ and the fact that $\lim_{u\to\infty}\psi(u) = +\infty$, we have for a fixed $\theta > 0$ and a large u, $\psi(u+\theta) \geq \psi(u)$. Moreover, under the condition (3.2), Novikov (2003) shows that $\lim_{u\to\infty} u^{-1}\int_0^u \psi(r)r^{-1}dr = +\infty$. Therefore, by following a line of reasoning similar to the proof of Novikov (2003, Theorem 2) one completes the proof. □

In Section 5.2, the special case with stable BDLP is studied in details. In what follows, we provide the Laplace-Fourier transform of the joint distribution. We first show the following lemma.

Lemma 3.7. *The bivariate process* $(I_t, X_t,\, t \geq 0)$ *is a Markov process. Its infinitesimal generator is defined on* $C_0(\mathbb{R}\times\mathbb{R})$ *by*

$$\mathcal{A}^* f(x,y) = \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x}(x,y) + (b-\lambda x)\frac{\partial f}{\partial x}(x,y) + x\frac{\partial f}{\partial y}(x,y) + \tag{3.11}$$

$$\int_{-\infty}^0 \left(f(x+r,y) - f(x,y) - \frac{\partial f}{\partial x}(x,y) r\chi(r)\right)\nu^Z(dr). \tag{3.12}$$

Proof. We start by recalling that although the additive functional I_t is not markovian, the bivariate process $(I_t, X_t,\, t \geq 0)$ is a strong Markov process, see Blumenthal and Getoor (1968). Denoting by X^c the continuous martingale part of X, we have, by using the Itô's formula for semimartingale, see Jacod and shiryaev (2002), for any function $f \in C^2(\mathbb{R}\times\mathbb{R},\mathbb{R})$

$$\begin{aligned} f(X_t, I_t) &= f(x,0) + \int_0^t \frac{\partial f}{\partial x}(X_{s-}, I_s)\,dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x}(X_s, I_s)\,d\langle X^c\rangle_s + \\ &\quad \int_0^t \frac{\partial f}{\partial y}(X_s, I_s)\,dI_s + \sum_{0<s\leq t} f(X_s, I_s) - f(X_{s-}, I_s) - \frac{\partial f}{\partial x}(X_{s-}, I_s)\Delta X_s \\ &= f(X_0, 0) - \lambda\int_0^t \frac{\partial f}{\partial x}(X_s, I_s) X_s\,ds + \int_0^t \frac{\partial f}{\partial x}(X_{s-}, I_s)\,dZ_s + \\ &\quad \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x}(X_s, I_s)\,d\langle Z^c\rangle_s + \int_0^t \frac{\partial f}{\partial y}(X_s, I_s) X_s\,ds + \\ &\quad \sum_{0<s\leq t} f(X_{s-}+\Delta Z_s, I_s) - f(X_{s-}, I_s) - \frac{\partial f}{\partial x}(X_{s-}, I_s)\Delta Z_s. \end{aligned}$$

Finally, by noticing that $d\langle Z^c\rangle_s = \sigma^2 ds$, we obtain $\mathcal{A}^*$. □

Corollary 3.8. *For* $\gamma, \theta, \lambda > 0$ *and* $y > x$, *we have*

$$I_{\gamma,i\theta}(x,y) = \frac{\overline{H}_{\frac{\gamma}{\lambda},\frac{i\theta}{\lambda}}(x)}{\overline{H}_{\frac{\gamma}{\lambda},\frac{i\theta}{\lambda}}(y)}, \tag{3.13}$$

where

$$\overline{H}_{\nu,\beta}(x) = e^{\beta x} H_{\nu,\beta}(x). \tag{3.14}$$

Proof. In order to simplify the notation in the proof we assume that $\sigma = 0$. We consider the process $M :=(M_t,\, t \geq 0)$ defined, for a fixed $t \geq 0$, by

$$M_t \;=\; \exp\left(-\gamma t + i\theta \int_0^t X_s\, ds\right) \overline{H}_{\frac{\gamma}{\lambda}, \frac{i\theta}{\lambda}}(X_t). \tag{3.15}$$

We shall prove that M is a complex martingale. From the integral representation (3.9), it follows that the function $H_{\nu,\beta}(x)$ is analytic in the domain $\Re(\nu) > 0, \Re(\beta) > 0,\ x \in \mathbb{R}$. Set $u(t,x,y) := e^{-\gamma t + i(\theta y + \frac{\theta}{\lambda}x)}$, $g(x) := H_{\frac{\gamma}{\lambda}, \frac{i\theta}{\lambda}}(x)$ and $f(t,x,y) := u(t,x,y)g(x)$. Thanks to the remark following proposition (3.3), we see that g is a solution of the following integro-differential equation

$$\mathcal{A}^{(i\frac{\theta}{\lambda})} g(x) = (\gamma - \psi(i\frac{\theta}{\lambda}))g(x) \tag{3.16}$$

with

$$\begin{aligned}\mathcal{A}^{\xi} f(x) \;&=\; (\bar{b} - \lambda x)f'(x) + \\ &\int_{-\infty}^{0} \left(f(x+r) - f(x) - f'(x) r\chi(r)\right) e^{\xi r} \nu^Z(dr).\end{aligned}$$

where we recall that $\bar{b} := b + \int_{-\infty}^0 (e^{\xi r} - 1) r\chi(r)\nu^Z(dr)$. We observe that

$$\begin{aligned}\frac{\partial f}{\partial y}(t,x,y) \;&=\; i\theta u(t,x,y)g(x) \\ \frac{\partial f}{\partial x}(t,x,y) \;&=\; u(t,x,y)\left(i\frac{\theta}{\lambda} g(x) + g'(x)\right).\end{aligned}$$

Then by applying the change of variables formula for processes with finite variation, we get

$$\begin{aligned} df(t,X_t,I_t) \;&=\; \left(\frac{\partial f}{\partial t}(t,X_t,I_t) - \lambda X_t \frac{\partial f}{\partial x}(t,X_t,I_t) + \frac{\partial f}{\partial y}(t,X_t,I_t)\right)\, dt \\ &\quad + \frac{\partial f}{\partial x}(t,X_{t-},I_t)\, dZ_t + \int_{-\infty}^0 f(x+r,y) - f(x,y) - \frac{\partial f}{\partial x}(x,y) r J(dr)\, dt \\ &= u(t,x,y)\left(\left(b + \int_{-\infty}^0 (e^{\frac{\theta}{\lambda}r} - 1) r\chi(r)\nu(dr) - \lambda X_t\right) g'(x)\right. \\ &\quad \int_{-\infty}^0 (g(x+r) - g(x) - g'(x) r\chi(r)) e^{\frac{\theta}{\lambda}r}\nu(dr) + \\ &\quad \left. + \left(-\gamma + ib\frac{\theta}{\lambda} + \int_{-\infty}^0 \left(e^{\frac{\theta}{\lambda}r} - 1 - i\frac{\theta}{\lambda} r\xi(r)\right)\nu(dr)\right) g(x)\right)\, dt \\ &\quad + N_t, \end{aligned}$$

where $(N_t,\, t \geq 0)$ is a purely discontinuous martingale. Consequently, by using the fact that g is a solution of the equation (3.16), we have shown that $(M_t,\, t \geq 0)$ is also a purely discontinuous martingale with respect to the natural filtration of X.

Next, we derive the following easy estimates, for any $t \geq 0$

$$\mathbb{E}\left[|M_{T_y\wedge t}|\right] \leq \mathbb{E}\left[|H_{\frac{\gamma}{\lambda},\frac{i\theta}{\lambda}}(X_{T_y\wedge t})|\right] \tag{3.17}$$

$$\leq \mathbb{E}\left[|H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(X_{T_y\wedge t})|\right] \tag{3.18}$$

$$\leq \mathbb{E}\left[|H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(y)|\right] < \infty. \tag{3.19}$$

We complete the proof of the corollary by applying the Doob's optional-sampling Theorem at the bounded stopping time $T_y \wedge t$ and the dominated convergence theorem. □

In the sequel, we assume that the exponential moments of the BDLP Z are finite.
Assumption
The Lévy measure of the BDLP Z satisfies the following condition

$$\int_{r<-1} e^{vr}\nu^Z(dr) < \infty \tag{3.20}$$

for every $v \in \mathbb{R}$.

Theorem 3.9. *Under assumption 3.20, we have, for $\gamma, \theta, \lambda > 0$ and $y > x$,*

$$I_{\gamma,-\theta}(x,y) = e^{\frac{\theta}{\lambda}(y-x)}\frac{H_{\frac{\gamma}{\lambda},-\frac{\theta}{\lambda}}(x)}{H_{\frac{\gamma}{\lambda},-\frac{\theta}{\lambda}}(y)}. \tag{3.21}$$

Proof. It is well known that when the Lévy measure of Z satisfies the assumption 3.20, its Laplace exponent is an entire function, see Skorohod (1991). Then we can follow the same route than for the proof of the Theorem 3.6, but using the martingale $(\exp(-\xi Z_t - t\psi(-\xi)),\ t \geq 0)$, for any $\xi > 0$, in the Girsanov transform. □

Remark 3.10. *For a fixed $\delta > 0$, if we assume only that $\int_{r<-1} e^{-\frac{\delta}{\lambda}r}\nu^Z(dr) < \infty$, then $I_{\gamma,-\theta}$ is well defined for any $\theta < \delta$ since the Laplace exponent is analytic in a convex domain.*

4. The Stable case

We investigate in more detailed the stable OU processes, that is the GOU processes with stable BDLP. We recall that a stable process $Z := (Z_t,\ t \geq 0)$ with index $\alpha \in (0,2]$ is a Lévy process which enjoys the following self-similarity property

$$(Z_{kt}, t \geq 0) \overset{(d)}{=} (k^{1/\alpha} Z_t, t \geq 0), \quad \text{for any } k > 0.$$

The characteristic function of Z has the form, assuming $\alpha \in (0,1) \cup (1,2)$

$$\psi(u) = \tilde{c}|u|^{\alpha}(1 - i\beta \operatorname{sgn}(u)\tan(\pi\alpha/2)), \quad u \in (-\infty, +\infty),$$

where $\tilde{c} > 0$ and $\beta \in [-1,1]$ which corresponds to the skewness parameter. It is a purely discontinuous martingale. Moreover, if the stable process has non-positive jumps (i.e. $\beta = -1$), excluding the negative of stable subordinator, its Laplace exponent is given, for $1 < \alpha < 2$, by

$$\psi(u) = cu^{\alpha}, \quad u \geq 0, \tag{4.1}$$

where $c = \tilde{c} \mid \cos(\frac{1}{2}\pi\alpha) \mid^{-1}$ and $\tilde{c} > 0$, see Sato (1999, Example 46.7)). Finally, it is worth noticing that if Z is a stable process with index α we have the following representation for X, for any $t \geq 0$

$$X_t = e^{-\lambda t} \left(x + Z_{\kappa(t)} \right), \tag{4.2}$$

where $\kappa(t) = \frac{e^{\alpha \lambda t} - 1}{\alpha \lambda}$. The semi-group of X is then given by

$$\begin{aligned} P(X_t \in dy \mid X_0 = x) &= (e^{\lambda t} - 1)^{-1/\alpha} e^{\lambda t/\alpha} \\ & \quad p_1 \left((e^{\lambda t} - 1)^{-1/\alpha} (e^{t/\alpha} y - x) \right) dy, \end{aligned}$$

where $p_1(x)$ denotes the density of the semi-group of the stable process Z at time 1. Here we show a proof of the Laplace transform of the first passage time of a constant level by the stable OU process ($\alpha \in (1, 2]$). There exists another proof of this result, see Hadjiev (1985), but here we provide one which is more constructive. Indeed, our proof is based on the first hitting time of the BDLP to a specific curve, see Novikov (1983) for a similar result, and our approach can be extended to self-similar Markov processes with one sided jumps and for which singleton is regular for itself, see Shepp (1967) and Yor (1984) for self-similar diffusions having continuous paths. We refer to Lamperti (1972) for a characterization of self-similar processes in $\mathbb{R}^+$, the so-called semi-stable processes. It is clear that the first passage time of a constant level of these processes inherits the self-similarity property. Consequently, there exists a unique monotone and continuous function φ such that, for $\gamma > 0$

$$\mathbb{E}_x \left[\exp\left(-\gamma T_y \right) \right] = \frac{\varphi\left(\gamma^{1/\alpha} x \right)}{\varphi\left(\gamma^{1/\alpha} y \right)}, \tag{4.3}$$

where $x \lesseqgtr y$ depending on the side of the jumps of X. We recall that in the stable case $\varphi(x) = e^{-c^{-1/\alpha} x}$. In order to emphasize the role played by the scaling property in the proof of the following result we keep the notation φ. We introduce the following positive random variable

$$T^b_{\bar{x}} = \inf \left\{ s \geq 0;\ Z_s > \bar{x}(s + d)^{1/\alpha} \right\}, \qquad (\bar{x} > x) \tag{4.4}$$

which is the first passage time of the process Z above the curve $\bar{x}(t + d)^{\alpha}$.

Theorem 4.1. *If $1 < \alpha < 2$, then the Mellin transform of the random variable $T^d_{\bar{x}}$ is given by, for $m > 0$,*

$$\mathbb{E}_x \left[(T^d_{\bar{x}} + d)^{-m}, T^d_{\bar{x}} < \infty \right] = d^{-m} \frac{H_{m,\alpha}(d^{-1/\alpha} x)}{H_{m,\alpha}(\bar{x})}, \tag{4.5}$$

where

$$H_{m,\alpha}(-x) = \int_0^\infty \varphi\left(-x r^{1/\alpha} \right) e^{-r} r^{m-1}\, dr \tag{4.6}$$

$$= \alpha \sum_{k=0}^\infty \frac{(-1)^k \Gamma(\frac{k}{\alpha} + m)}{c^{k/\alpha} k!} x^k \tag{4.7}$$

Proof. As a consequence of (2.2) and the self-similarity property of Z, the process $(e^{-\gamma t} \varphi(\gamma^{1/\alpha} Z_t),\ t \geq 0)$ is a $\mathcal{F}$-martingale. Then by the Doob's optional-sampling Theorem, we have (using the bounded stopping time $T^d_{\bar{x}} \wedge t$ and then applying the dominated convergence theorem)

$$\mathbb{E}_x \left[e^{-\gamma T^d_{\bar{x}}} \varphi\left(\gamma^{1/\alpha} Z_{T^d_{\bar{x}}} \right) \right] = \varphi\left(\gamma^{1/\alpha} x \right),$$

where by integrating both side of (4.8) by the measure $e^{-d\gamma}\gamma^{m-1}d\gamma$, we have, by Fubini's theorem

$$\mathbb{E}_x\left[\int_0^\infty e^{-\gamma T^d_{\bar{x}}}\varphi\left(\gamma^{1/\alpha}Z_{T^d_{\bar{x}}}\right)e^{-d\gamma}\gamma^{m-1}d\gamma\right]=\int_0^\infty \varphi\left(\gamma^{1/\alpha}x\right)e^{-d\gamma}\gamma^{m-1}d\gamma.$$

Using the fact that Z has non-positive jumps, it follows that $Z_{T^d_{\bar{x}}}=\bar{x}(T^d_{\bar{x}}+d)^{1/\alpha}$. Thus

$$\mathbb{E}_x\left[\int_0^\infty e^{-\gamma(T^d_{\bar{x}}+d)}\varphi\left(\gamma^{1/\alpha}\bar{x}(T^b_{\bar{x}}+b)^{1/\alpha}\right)\gamma^{m-1}d\gamma\right]=d^{-m}H_{m,\alpha}(d^{-1/\alpha}x),$$

where $H_{m,\alpha}(x)=\int_0^\infty \varphi\left(y^{1/\alpha}x\right)e^{-y}y^{m-1}\,dy$. The change of variable $y=\gamma(T^d_{\bar{x}}+1)$ yields

$$\mathbb{E}_x\left[\int_0^\infty e^{-y}\varphi\left(y^{1/\alpha}\bar{x}\right)y^{m-1}(T^d_{\bar{x}}+d)^{-m}dy\right]=d^{-m}H_{m,\alpha}(d^{-1/\alpha}x).$$

Finally, we get the Mellin transform of the random variable $(T^d_{\bar{x}}+d)$

$$\mathbb{E}_x\left[(T^d_{\bar{x}}+d)^{-m}\right]=d^{-m}\frac{H_{m,\alpha}(d^{-1/\alpha}x)}{H_{m,\alpha}(\bar{x})}. \tag{4.8}$$

The proof is completed by observing that

$$H_{m,\alpha}(-x) \quad = \quad \sum_{k=0}^{\infty}\frac{(-1)^k(c^{-1/\alpha}x)^k}{k!}\int_0^\infty e^{-y}y^{m+\frac{k}{\alpha}-1}dy$$

and using $\Gamma(z)=\int_0^\infty e^{-r}r^{z-1}\,dr,\quad \Re(z)>0.$ □

For more information concerning the property of the function H, we refer to Novikov (1983). As a consequence we state the following result concerning the Laplace transform of the first passage time of a constant level x by the stable OU processes, denoted by T_y.

Theorem 4.2. *The Laplace transform of the random variable T_y is given by*

$$\mathbb{E}_x\left[\exp\left(-\gamma T_y\right)\right]=\frac{H_{\frac{\gamma}{\alpha\lambda},\alpha}((\alpha\lambda)^{1/\alpha}x)}{H_{\frac{\gamma}{\alpha\lambda},\alpha}((\alpha\lambda)^{1/\alpha}x)},\quad y>x. \tag{4.9}$$

Proof. Fix $y>x$. We have the following relationship between first passage times

$$\begin{aligned}
T_y &= \inf\{s\geq 0;\ X_s>y\}\\
&= \inf\left\{s\geq 0;\ e^{-\lambda t}\left(x+\int_0^t e^{\lambda s}\,dZ_s\right)>y\right\}\\
&= \inf\left\{s\geq 0;\ e^{-\lambda t}\left(x+Z_{\tau(s)}\right)>y\right\}\\
&= A\left(\inf\left\{s\geq 0;\ x+Z_s>y\left(\alpha\lambda s+1\right)^{1/\alpha}\right\}\right)\\
&= A\left(T^{(\alpha\lambda)^{-1}}_{(\alpha\lambda)^{1/\alpha}y}\right),
\end{aligned}$$

where we have performed the deterministic time change $A(t)=\kappa^{-1}(t)$, with

$$A(t)=\frac{1}{\alpha\lambda}\ln(\alpha\lambda t+1).$$

Therefore,

$$\begin{aligned}\mathbb{E}_x\left[\exp\left(-\gamma T_y\right)\right] &= \mathbb{E}_x\left[(\alpha\lambda T^{(\alpha\lambda)^{-1}}_{(\alpha\lambda)^{1/\alpha}y}+1)^{-\frac{\gamma}{\lambda\alpha}}\right]\\ &= (\alpha\lambda)^{-\frac{\gamma}{\lambda\alpha}}\mathbb{E}_x\left[(T^{(\alpha\lambda)^{-1}}_{(\alpha\lambda)^{1/\alpha}y}+(\alpha\lambda)^{-1})^{-\frac{\gamma}{\lambda\alpha}}\right]\\ &= \frac{H_{\frac{\gamma}{\alpha\lambda},\alpha}((\alpha\lambda)^{1/\alpha}x)}{H_{\frac{\gamma}{\alpha\lambda},\alpha}((\alpha\lambda)^{1/\alpha}y)}.\end{aligned}$$

□

Theorem 4.3. *For $\gamma > 0$, $\theta \in \mathbb{R}$ and $y > x$, we have*

$$I_{\gamma,\theta}(x, y) = e^{-\frac{\theta}{\lambda}(y-x)}\frac{H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(x)}{H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(y)}, \tag{4.10}$$

where

$$H_{\nu,\beta}(x) = \int_0^\infty \exp\left(xr - \frac{c}{\rho}\int_0^r (u+\beta)^\alpha \frac{du}{u}\right) r^{\nu-1}dr. \tag{4.11}$$

Remark 4.4. *When Z is a Brownian with drift b (i.e. $\alpha = 2, c = \frac{1}{2}$), we obtain*

$$I_{\gamma,\theta}(x, y) = e^{\lambda/2(x^2-y^2)-\lambda b(x-y)}\frac{D_{\overline{\nu}}\left(-\sqrt{2\lambda}(x-\frac{b}{\lambda}-\frac{\theta}{\lambda^2})\right)}{D_{\overline{\nu}}\left(-\sqrt{2\lambda}(x-\frac{b}{\lambda}-\frac{\theta}{\lambda^2})\right)}, \tag{4.12}$$

where $\overline{\nu} := \frac{\theta^2}{2\lambda^3}+\frac{b\theta}{\lambda^2}-\frac{\gamma}{\lambda}$ and $D_\nu(x) = \frac{e^{-x^2/2}}{\Gamma(-\nu)}\int_0^\infty \exp\left(-xy-\frac{1}{2}y^2\right)y^{-\nu-1}dy$ denotes the cylinder parabolic function, see e.g. Lebedev (1972). We also mention that by taking $b = 0$ in (4.12), we recover the result of Lachal (1993).

Remark 4.5. *Although the Lévy measure of stable processes does not satisfy the assumption (3.10), it is in this case possible to extend analytically the joint Laplace transform. Indeed, the explicit form of the Laplace exponent allows to use complex analysis technique as contour integration.*

Proof. For the case $\theta \geq 0$, the result follows immediately from Theorem 3.6. For the case $\theta < 0$ we need the following identity, see Gradshteyn and Ryzhik (2000, p. 313),

$$\int_0^y \frac{r^{\nu-1}}{1+\beta r}dr = \frac{y^\nu}{\nu}F(1,\nu;1+\nu;-y\beta), \quad |\arg(1+y\beta)| < \pi,\ \Re(\nu) > 0.$$

where F stands for the Gauss hypergeometric function. Then, we have

$$\begin{aligned}H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(x) &= \int_0^\infty \exp\left(xr - \frac{c}{\lambda}\int_0^r (u-\frac{\theta}{\lambda})^\alpha\frac{du}{u}\right)r^{\frac{\gamma}{\lambda}-1}dr\\ &= \int_0^\infty \exp\left(xr - \frac{c(r+\frac{\theta}{\lambda})^{\alpha+1}}{\theta(\alpha+1)}F(1,\alpha+1;\alpha+2;\frac{\lambda}{\theta}r+1)\right)r^{\frac{\gamma}{\lambda}-1}dr.\end{aligned}$$

which is well defined for any $\theta > 0$.

□

5. Application to finance

In what follows, we apply the results of the previous section to the pricing of lookback options on yields in the generalized Vasicek model. We extend the results of Leblanc and Scaillet (1998) [1] by allowing jumps in the interest rate dynamics.

5.1. **European option on maximum.** We start by giving some definition and introducing some notation. Let $P(t,T)$ denotes the price at date t of a zero-coupon bond of maturity date $T \geq t$, i.e. the price of the asset delivering one monetary unit at time T. In practice, the zero-coupon bond prices are usually translated into an implicit rate of return, $Y(t,T)$, the so-called yield at time t with maturity T. This relationship is done by

$$Y(t,T) = -\frac{1}{T-t}\log(P(t,T)). \tag{5.1}$$

We restrict our models to the affine class of one factor term structure models. In this case, the yield is an affine function of the instantaneous interest rate

$$Y(t,T) = \frac{1}{T-t}\left[A(T-t)r_t + D(T-t)\right], \tag{5.2}$$

where $A(.)$ and $D(.)$ are some deterministic functions. A definition and a complete characterization of regular affine Markov processes is given in Duffie et al. (2001). Essentially there exist two classes of regular affine Markov processes, the generalized Ornstein-Uhlenbeck processes and the continuous state branching processes. Here, we consider that the dynamics of the instantaneous interest rate, with current value $r_0 \in \mathbb{R}^+$, is, under the risk neutral probability measure, a generalized Ornstein-Uhlenbeck process with parameter $\lambda > 0$, namely the generalized Vasicek model

$$r_t = e^{-\lambda t}\left(r_0 + \int_0^t r_s \, dZ_s\right), \quad t \geq 0.$$

In this framework, see Duffie et al. (2001), it is an easy task to derive the current price of the discount bond

$$\begin{aligned} P_{r_0}(0,T) &:= \mathbb{E}_{r_0}\left[\exp\left(-\int_0^T r_s \, ds\right)\right] \\ &= \exp\left(A(T)r_0 + D(T)\right), \end{aligned}$$

where $A(t) = \frac{1}{\lambda}\left(1 - e^{-\lambda t}\right)$ and $D = -\int_0^t \psi(A(s))ds$, where ψ stands for the Laplace exponent of Z.

Now, we consider an European call option on maximum on yields with strike K, and maturity $T^* > 0$. Its terminal and initial value are respectively defined by

$$h(Y_{T^*}) := \left(\sup_{u\in[0,T^*]} Y(u,T) - K\right)^+$$

and

$$C^Y\left(0,T^*,K;r_0,T\right) := \mathbb{E}_{r_0}\left[e^{-\int_0^T r_s \, ds}\left(\sup_{u\in[0,T^*]} Y(u,T) - K\right)^+\right],$$

[1]We mention a mistake in their paper and the accompany paper Leblanc et al (2000), reported also in Göing-Jaeschke and Yor (2003) and in Alili et al. (2003), concerning the expression of the density of the hitting time of the Ornstein-Uhlenbeck process.

which is the expected discounted value of the payoff under the risk neutral probability measure. It is a so-called lookback option since its value depends on the information available up to time T^*. From the affine term structure of the model, we characterize the current price of the option in term of the instantaneous interest rate r

$$C^Y(0,T^*,K;r_0,T) \quad = \quad \frac{A(T)}{T} C^r\left(0,T^*,\bar{K};r_0,T\right),$$

where $\bar{K} = \frac{TK-D(T)}{A(T)}$. The pricing of this path dependent option is clearly a difficult exercise because the distribution of the supremum of the instantaneous interest rate is unknown when its dynamics follows a GOU process. Next, we shall give a closed form expression for the Laplace transform with respect to time of the option price defined, for $\gamma > 0$, by

$$L_\gamma(0,\bar{K};r_0,T) := \int_0^\infty e^{-\gamma T^*} C^r(0,T^*,\bar{K};r_0,T)\, dT^*. \tag{5.3}$$

Proposition 5.1. *We assume that* $\int_{x<-1} e^{-\frac{1}{\lambda}x}\nu^Z(dx) < \infty$. *Then, for* $r_0 \leq \bar{K}$, *we have*

$$L_\gamma(0,\bar{K};r_0,T) = H_{\frac{\gamma}{\lambda},-\frac{1}{\lambda}}(r_0)\int_{\bar{K}}^\infty e^{x/\lambda}\frac{P_\gamma(x)}{H_{\frac{\gamma}{\lambda},-\frac{1}{\lambda}}(x)}\,dx, \tag{5.4}$$

where

$$P_\gamma(x) \quad := \quad \int_0^\infty dT^* \exp\left(-\gamma T^*\right) P_x(0,T^*).$$

Proof. By using the strong Markov property of the instantaneous interest rate process r we obtain

$$\begin{aligned}
L_\gamma(0,\bar{K};r_0,T) &= \mathbb{E}_{r_0}\left[\int_{\bar{K}}^\infty dx \int_0^\infty dT^* \exp(-\gamma T^* - \int_0^{T^*} r_s ds)\, \mathbb{I}_{\left\{\sup_{[0,T^*]} r_u > x\right\}}\right] \\
&= \mathbb{E}_{r_0}\left[\int_{\bar{K}}^\infty dx \int_{T_y}^\infty dT^* \exp\left(-\gamma T^* - \int_0^{T^*} r_s\, ds\right)\right] \\
&= \mathbb{E}_{r_0}\left[\int_{\bar{K}}^\infty dx \int_{T_y}^\infty dT^* \exp\left(-\gamma(T^*-T_y) - \gamma T_y \right.\right. \\
&\qquad \left.\left. - \int_0^{T_y} r_s\, ds - \int_{T_y}^{T^*} r_s\, ds\right)\right] \\
&= \int_{\bar{K}}^\infty dx\, \mathbb{E}_{r_0}\left[\exp\left(-\gamma T_y - \int_0^{T_y} r_s\, ds\right)\right] P_\gamma(x).
\end{aligned}$$

To get the desired expression for the Laplace transform of the option price it remains to compute $\mathbb{E}_{r_0}\left[\exp\left(-\gamma T_y - \int_0^{T_y} r_s\, ds\right)\right]$. From Theorem 3.9 and remark 3.10, choosing $\theta = 1$, we obtain

$$\mathbb{E}_{r_0}\left[\exp\left(-\gamma T_y - \int_0^{T_y} r_s\, ds\right)\right] \quad = \quad e^{\frac{1}{\lambda}(x-r_0)}\frac{H_{\frac{\gamma}{\lambda},-\frac{1}{\lambda}}(r_0)}{H_{\frac{\gamma}{\lambda},-\frac{1}{\lambda}}(x)},$$

which yields finally to (5.4). $\square$

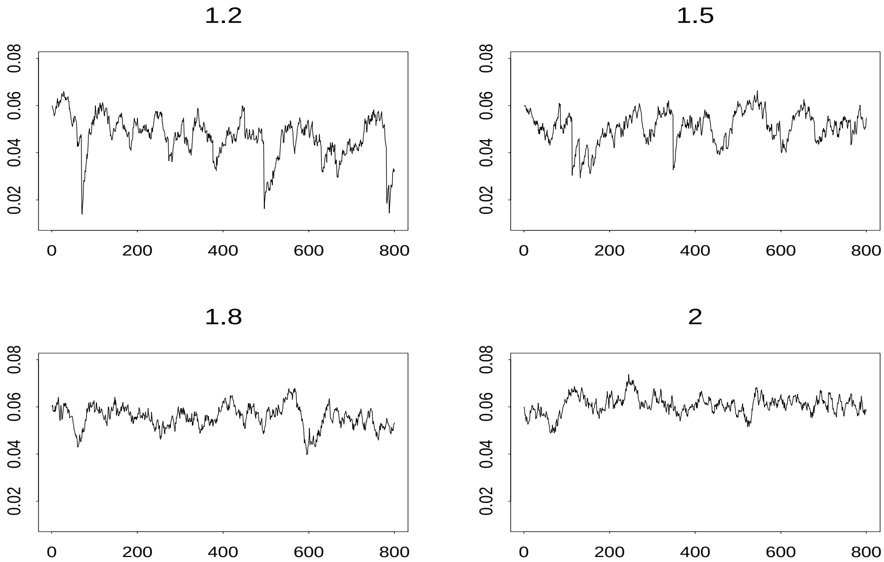

FIGURE 1. Trajectories of generalized Vasicek process for different values of the index of stability $\alpha = 1.2, 1.5, 1.8, 2$ and with $r_0 = 0.09$, $b = 0.01$, $\lambda = 0.1$, $\sigma = 0.002$ and $\delta = 0.0005$; The paths are simulated by using the Euler scheme ($\Delta t = \frac{1}{2}$) and the increments of the α-stable Lévy process are generated from the Chambers-Mallows-Stuck algorithm ($\beta = -1$).

5.2. **The stable BDLP case.** In this part we assume that the dynamics of the instantaneous interest rate r, starting from $r_0 \in \mathbb{R}^+$, is given as the solution to the following stochastic differential equation, for any $\lambda > 0$

$$dr_t = (b - \lambda r_t)\, dt + \sigma\, dB_t + \delta\, dZ_t, \tag{5.5}$$

where B is a standard Brownian motion, Z is a stable Lévy process with index $\alpha \in (1, 2)$, $b \in \mathbb{R}$ is the mean reverting parameter, σ and δ are some positive parameters. We recall that Z is a purely discontinuous martingale and it is considered independent of B. In this case, the Laplace exponent of the BDLP has the following form

$$\psi(u) = bu + \frac{1}{2}\sigma^2 u^2 + c\delta^\alpha u^\alpha, \quad u \geq 0. \tag{5.6}$$

We recall that we restrict ourself to the case where $\alpha \in (1, 2)$ in order that the compensator of the jumps measure of r fulfills the assumption (3.2). We show in Figure 5.2 the effect of the index of stability α on the trajectories of the process. In particular there is evidence of increase of large increments (jumps) with lower value of α.
In this specific case, it is possible to compute explicitly the Laplace transform (5.3) in terms of new special functions.

Proposition 5.2. *For $r_0 \le \bar{K}$, we have*

$$L_\gamma(0,\bar{K};r_0,T) = H_{\overline{\gamma},-\frac{1}{\lambda}}\left(r_0 - \frac{b}{\lambda} + \frac{\sigma^2}{\lambda^2}\right)\int_{\bar{K}}^{\infty} e^{x/\lambda}\frac{P_\gamma(x)}{H_{\overline{\gamma},-\frac{1}{\lambda}}\left(x-\frac{b}{\lambda}+\frac{\sigma^2}{\lambda^2}\right)}\,dx,$$

where $\overline{\gamma} := \frac{\gamma}{\lambda} + \frac{b}{\lambda^2} - \frac{\sigma^2}{\lambda^3}$,

$$H_{\nu,\beta}(x) = \int_0^\infty \exp\left(xy - \frac{\sigma^2}{4\lambda}y^2 - \frac{c\delta^\alpha}{\lambda}\int_1^y (u+\beta)^\alpha \frac{du}{u}\right) y^{\nu-1}dy,$$

$$P_\gamma(x) = \int_0^\infty dT^* \, \exp\left(-\gamma T^*\right)\exp\left(A(T)x + D(T)\right),$$

where

$$A(t) = \frac{1}{\lambda}\left(1-e^{-\lambda t}\right)$$

and

$$D(t) = -\left(\frac{b}{\lambda}+\frac{\sigma^2}{2\lambda^2}\right)t - \left(\frac{b}{\lambda}-\frac{\sigma^2}{\lambda^2}\right)A(t) + \frac{\sigma^2}{4\lambda}(e^{-2\lambda t}-1) - \frac{c\delta^\alpha A(t)}{\alpha+1}F(1,\alpha+1;\alpha+2;\lambda A(t)), \quad (5.7)$$

F denoting the Gauss hypergeometric function.

Proof. First, we compute the price of a zero-coupon bond by using the affine structure of the model. It is given by

$$P_x(0,T) = \exp\left(A(T)x + D(T)\right),$$

where $A(t) = \frac{1}{\lambda}\left(1-e^{-\lambda t}\right)$ and D is given by

$$D(t) = -\int_0^t bA(s) + \frac{1}{2}\sigma^2 A^2(s) + c\delta^\alpha A^\alpha(s)\,ds, \quad D(0) = 0,$$

which yields to (5.7), after we have performed the change of variable $u = -\frac{1}{\lambda}\log(1-s)$ and used the formula 3.194 in Gradshteyn and Ryzhik (2000). Moreover, by using the explicit form of the Laplace exponent (5.6) we derive, after some obvious computation, the desired result. □

Finally we end up this section by investigating the only unpleasant feature of the generalized Vasicek models which is the possibility that interest rates become negative. Here, we focus on the stable Vasicek model, that is we consider that r follows the dynamics (5.5) with $\sigma = 0$. This choice is motivated by the fact that it is a more extreme case compared to the case with Brownian component. From (2.5), we compute the Laplace transform of the limiting distribution of the process r

$$E\left[\exp\left(u\rho^r\right)\right] = \exp\left(c\delta^\alpha u^\alpha \int_0^\infty e^{-\lambda\alpha s}\,ds + bu\int_0^\infty e^{-\lambda s}\,ds\right).$$

$$= \exp\left(\frac{c\delta^\alpha}{\lambda\alpha}u^\alpha + \frac{b}{\lambda}u\right).$$

We recognize the Laplace transform of a α stable random variable with scaling parameter $\frac{\delta^\alpha}{\lambda\alpha}$, skewness parameter $\beta = -1$ and drift parameter $\frac{b}{\lambda}$. In Table 5.2, we show the probability of a negative long term interest rate p^n and the mean value $\bar{r}$ for different values of the index but with the other parameter being constant

($b = 0.01$, $\lambda = 0.1$ and $\delta = 0.00025$). These results are the outcomes of Monte Carlo simulation. We recall that for $\alpha = 2$ the mean value is simply given by the coefficient of the drift term $\frac{b}{\lambda}$, whereas for $1 < \alpha < 2$ the stable random variables without drift are not centered. We observe that the probability of negative interest rate decreases with the index α, but remains very small for moderate values of α. Moreover, the mean value of r stays almost unchanged for the same value of the index and equals the ratio $\frac{b}{\lambda} = 0.1$ which is a realistic level for annual interest rate for instance. It is worthy to mention that it is possible to get both very small values for p^n and reasonable values for long term interest rates $\bar{r}$ for any α by playing with the family of the parameters (λ, b, δ).

TABLE 1

α	$p^n(\approx)$	$\bar{r}$
2	0	1
1.8	1.1×10^{-7}	0.0996
1.5	6.4×10^{-5}	0.099
1.2	0.015	0.086

Acknowledgements This paper is part of my doctoral thesis written under the supervision of Freddy Delbaen. I would like to thank Larbi Alili for useful comments on the manuscript and Monique Jeanblanc for providing Novikov preprint.

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