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MULTIVARIATE EXTREMES, AGGREGATION AND DEPENDENCE IN ELLIPTICAL DISTRIBUTIONS*

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ABSTRACT. In this paper we clarify dependence properties of elliptical distributions by deriving general but explicit formulas for the coefficients of upper and lower tail dependence and spectral measures with respect to different norms. We show that an elliptically distributed random vector is regularly varying if and only if the bivariate marginal distributions have tail dependence. Furthermore, the tail dependence coefficients are fully determined by the tail index of the random vector (or equivalently of its components) and the linear correlation coefficient. Whereas Kendall's tau is invariant in the class of elliptical distributions with continuous marginals and a fixed dispersion matrix, we show that this is not true for Spearman's rho. We also show that sums of elliptically distributed random vectors with the same dispersion matrix (up to a positive constant factor) remain elliptical if they are dependent only through their radial parts.

1. Introduction

The class of elliptical distributions provides a rich source of multivariate distributions which share many of the tractable properties of the multivariate normal distribution and enables modelling of multivariate extremes and other forms of non-normal dependences.

In this paper we aim to clarify dependence properties of elliptical distributions. Dependence between the components of a random vector is, of course, related to the shape of the joint distribution. For elliptically distributed random vectors the shape of the distribution is given by the dispersion matrix Σ and the radial random variable R (see Theorem 3.1). The simple structure of elliptical distributions enables explicit computations of interesting quantities such as the coefficients of tail dependence and spectral measures associated with regularly varying random vectors (see Definition 2.6). From our explicit formula for the coefficient of tail dependence we conclude that it is fully determined by the corresponding linear correlation coefficient (as defined in Definition 3.2) and the tail index of the radial random variable in the general representation (see Theorem 3.1) of elliptically distributed random vectors. For this class of multivariate distributions, regular variation and tail dependence are closely related. Existence of tail dependence of the bivariate marginals and of regular variation is equated to regular variation of the radial random variable in the general representation.

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Standard estimators of the linear correlation coefficient for elliptical distributions are based on the assumption of finite second moments. Kendall's tau and Spearman's rho (and their sample versions) do not rely on the existence of certain moments. It has been proved in Lindskog, McNeil and Schmock [8] that Kendall's tau is invariant in the class of elliptical distributions with continuous univariate marginals and a fixed dispersion matrix (up to a positive constant factor). This implies that the robust estimator of Kendall's tau can be used to estimate linear correlation coefficients without any other assumption on the underlying distribution than that of continuity of the univariate margins and joint ellipticality. One might also expect Spearman's rho to be invariant in the class of elliptical distributions with continuous marginals and a fixed dispersion matrix. We give a counterexample showing that this is not true.

It is known that sums of independent elliptically distributed random vectors with the same dispersion matrix are elliptical. In Theorem 4.1 we prove that sums of elliptically distributed random vectors with the same dispersion matrix are elliptical if they are dependent only through their radial parts. This result has applications to multivariate time series. It should be noted that the dispersion matrices are allowed to differ by a positive constant factor, see Remark 3.1(b) for details.

In this paper we use the spectral measure to answer questions about dependence of extremes for regularly varying elliptically distributed random vectors. In doing so it is crucial to consider a spectral measure with respect to a norm which corresponds to the question one is trying to answer. We discuss and exemplify this in Section 5. In particular, for a bivariate elliptically distributed random vector, we compute the spectral measure with respect to the Euclidean 2-norm and with respect to the max-norm.

The paper is organised as follows. In Section 2 we recall the definitions of various dependence concepts. Section 3 introduces elliptical distributions, in particular we give the general stochastic representation of elliptically distributed random vectors. This representation is fundamental for the subsequent analysis. Section 4 contains the main results and in Section 5 we discuss the interpretation of the spectral measure with respect to different norms. All proofs are given in Section 6.

2. Preliminaries

To begin with we recall the definitions of the concordance measures Kendall's tau and Spearman's rho.

Definition 2.1. *Kendall's tau* for the random vector $(X_1, X_2)^{\mathrm{T}}$ is defined as

$$\tau(X_1, X_2) \triangleq \mathbb{P}\{(X_1 - X_1')(X_2 - X_2') > 0\} - \mathbb{P}\{(X_1 - X_1')(X_2 - X_2') < 0\},$$

where $(X_1', X_2')^{\mathrm{T}}$ is an independent copy of $(X_1, X_2)^{\mathrm{T}}$.

Definition 2.2. *Spearman's rho* for the random vector $(X_1, X_2)^{\mathrm{T}}$ is defined as

$$\varrho_S(X_1, X_2) \triangleq 3\left(\mathbb{P}\{(X_1 - X_1')(X_2 - X_2'') > 0\} - \mathbb{P}\{(X_1 - X_1')(X_2 - X_2'') < 0\}\right),$$

where $(X_1', X_2')^{\mathrm{T}}$ and $(X_1'', X_2'')^{\mathrm{T}}$ are independent copies of $(X_1, X_2)^{\mathrm{T}}$.

An important property of Kendall's tau and Spearman's rho is that they are invariant under strictly increasing transformations of the underlying random variables. If $(X_1, X_2)^{\mathrm{T}}$ is a random vector with continuous univariate marginal distributions and T_1 and T_2 are strictly increasing transformations on the range of

X_1 and X_2 respectively, then $\tau(T_1(X_1), T_2(X_2)) = \tau(X_1, X_2)$. The same property holds for Spearman's rho. Note that this implies that Kendall's tau and Spearman's rho do not depend on the (marginal) distributions of X_1 and X_2.

Next we introduce two measures of dependence of multivariate extremes. Perhaps the most commonly encountered measure of dependence of bivariate extremes is the coefficient of upper (lower) tail dependence.

Let F be a univariate distribution function. We define the generalised inverse of F as $F^{-1}(u) \triangleq \inf\{x \in \mathbb{R} \mid F(x) \geq u\}$ for all u in $(0, 1)$.

Definition 2.3. Let $(X_1, X_2)^{\mathrm{T}}$ be a random vector with marginal distribution functions F_1 and F_2. The *coefficient of upper tail dependence* of $(X_1, X_2)^{\mathrm{T}}$ is defined as

$$\lambda_U(X_1, X_2) \triangleq \lim_{u \nearrow 1} \mathbb{P}\{X_2 > F_2^{-1}(u) \mid X_1 > F_1^{-1}(u)\},$$

provided that the limit $\lambda_U \in [0, 1]$ exists. The *coefficient of lower tail dependence* is defined as

$$\lambda_L(X_1, X_2) \triangleq \lim_{u \searrow 0} \mathbb{P}\{X_2 \leq F_2^{-1}(u) \mid X_1 \leq F_1^{-1}(u)\},$$

provided that the limit $\lambda_L \in [0, 1]$ exists. If $\lambda_U > 0$ ($\lambda_L > 0$), then we say that $(X_1, X_2)^{\mathrm{T}}$ have upper (lower) tail dependence.

For a pair of random variables, upper (lower) tail dependence is a measure of *joint* extremes. That is, it measures the probability that one component is extremely large (small) given that the other one is extremely large (small), relative to the marginal distributions.

The second measure of dependence in multivariate extremes that we discuss in this paper is the spectral measure associated with a regularly varying random vector (see Definition 2.6 below). Let us first recall the definition of regular variation for a (univariate) random variable.

Definition 2.4. The random variable R is said to be *regularly varying at* ∞ *with index* $\alpha > 0$ if for all $x > 0$,

$$\lim_{t \to \infty} \frac{\mathbb{P}\{R > tx\}}{\mathbb{P}\{R > t\}} = x^{-\alpha}.$$

Throughout the paper we use the shorter "regularly varying with index $\alpha > 0$" for "regularly varying at ∞ with index $\alpha > 0$".

To prepare for the definition of regular variation for random vectors, we recall the concept of vague convergence. Let $\mathcal{X}$ be a separable metric space. A set $B \subset \mathcal{X}$ is said to be relatively compact if its closure $\overline{B}$ is compact. Let $\mathcal{B}(\mathcal{X})$ be the Borel σ-algebra on $\mathcal{X}$. A measure μ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is called a Radon measure if $\mu(B) < \infty$ for all relatively compact sets $B \in \mathcal{B}(\mathcal{X})$.

Definition 2.5. Let $\mu, \mu_1, \mu_2, \ldots$ be Radon measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. We say that $\{\mu_n\}_{n \in \mathbb{N}}$ *converges to* μ *vaguely*, written $\mu_n \xrightarrow{v} \mu$, if

$$\lim_{n \to \infty} \int_{\mathcal{X}} f(s)\mu_n(\mathrm{d}s) = \int_{\mathcal{X}} f(s)\mu(\mathrm{d}s)$$

for all continuous functions $f : \mathcal{X} \to \mathbb{R}_+$ with compact support.

A useful equivalent formulation of vague convergence is given in the following theorem.

Theorem 2.1. *Let $\mu, \mu_1, \mu_2, \dots$ be Radon measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Then the following statements are equivalent.*

(1) $\mu_n \xrightarrow{v} \mu$ *as* $n \to \infty$.
(2) $\lim_{n\to\infty} \mu_n(B) = \mu(B)$ *for all relatively compact* $B \in \mathcal{B}(\mathcal{X})$ *with* $\mu(\partial B) = 0$.

For a proof, see Kallenberg [7] p. 169. For further details about vague convergence we refer to [7].

We denote by $\mathbb{S}^{d-1}$ the unit hypersphere in $\mathbb{R}^d$ with respect to a norm $|\cdot|$, and by $\mathcal{B}(\mathbb{S}^{d-1})$ the Borel σ-algebra on $\mathbb{S}^{d-1}$. We write $\mathbb{S}_2^{d-1} = \{z \in \mathbb{R}^d \,|\, z^{\mathrm{T}} z = 1\}$ for the unit hypersphere in $\mathbb{R}^d$ with respect to the Euclidean 2-norm $|\cdot|_2$.

Definition 2.6. The d-dimensional random vector X is said to be *regularly varying with index* $\alpha > 0$ if there exists a random vector Θ with values in $\mathbb{S}^{d-1}$ a.s. such that for all $x > 0$, as $t \to \infty$,

$$\frac{\mathbb{P}\{|X| > tx, X/|X| \in \cdot\}}{\mathbb{P}\{|X| > t\}} \xrightarrow{v} x^{-\alpha}\, \mathbb{P}\{\Theta \in \cdot\}.$$

The distribution of Θ is referred to as the *spectral measure* of X and α is referred to as the *tail index* of X.

In the definition we did not specify the choice of norm $|\cdot|$. The reason for this is that whether a random vector is regularly varying or not does not depend on the choice of norm in Definition 2.6. This is stated in the following lemma which proof is given in Section 6.

Lemma 2.1. *Let $|\cdot|_A$ and $|\cdot|_B$ be two norms on $\mathbb{R}^d$ and let X be a d-dimensional random vector. Then X is regularly varying with index $\alpha > 0$ with respect to the norm $|\cdot|_A$ if and only if X is regularly varying with index $\alpha > 0$ with respect to the norm $|\cdot|_B$.*

It is clear that the corresponding spectral measures do not coincide for different norms, see Section 5 for explicit examples. When we want to emphasise the choice of norm, we say that the distribution of Θ is the spectral measure of X with respect to the norm $|\cdot|$.

The following result on the effect of adding a constant vector to a regularly varying random vector turns out to be useful in the study of regular variation properties of elliptical distributions. The proof is given in Section 6.

Lemma 2.2. *Let X be a d-dimensional regularly varying random vector with tail index $\alpha > 0$ and spectral measure $\mathbb{P}\{\Theta \in \cdot\}$ with respect to the norm $|\cdot|$, and let $b \in \mathbb{R}^d$ be a constant vector. Then $X + b$ is regularly varying with the same tail index and the same spectral measure with respect to the norm $|\cdot|$.*

3. Elliptical distributions

The main topic of this paper is to understand various measures of dependence through elliptical distributions. In this section we introduce the class of elliptically distributed random vectors and give some of their properties. For further details about elliptical distributions we refer to Fang, Kotz and Ng [6] and Cambanis, Huang and Simons [1].

Definition 3.1. If X is a d-dimensional random vector and, for some vector $\mu \in \mathbb{R}^d$, some $d\times d$ non-negative definite symmetric matrix Σ and some function $\phi : [0,\infty) \to \mathbb{R}$, the characteristic function $\varphi_{X-\mu}$ of $X-\mu$ is of the form $\varphi_{X-\mu}(t) = \phi(t^{\mathrm{T}}\Sigma t)$, we say that X has an *elliptical distribution* with parameters μ, Σ and ϕ, and we write $X \sim E_d(\mu,\Sigma,\phi)$.

The function ϕ is referred to as the *characteristic generator* of X. When $d=1$, the class of elliptical distributions coincides with the class of one-dimensional symmetric distributions.

For elliptically distributed random vectors, we have the following general representation theorem.

Theorem 3.1. *$X \sim E_d(\mu,\Sigma,\phi)$ with $\mathrm{rank}(\Sigma) = k$ if and only if there exist a non-negative random variable R independent of U, a k-dimensional random vector uniformly distributed on the unit hypersphere $\mathbb{S}_2^{k-1} = \{z \in \mathbb{R}^k \,|\, z^{\mathrm{T}}z = 1\}$, and a $d\times k$ matrix A with $AA^{\mathrm{T}} = \Sigma$, such that*

$$X \stackrel{\mathrm{d}}{=} \mu + RAU. \tag{3.1}$$

For the proof of Theorem 3.1 and details about the relation between R and ϕ, see Fang, Kotz and Ng [6] or Cambanis, Huang and Simons [1].

Remark 3.1. (a) Note that the representation (3.1) is not unique: if $\mathcal{O}$ is an orthogonal $k\times k$ matrix, then (3.1) also holds with $A' \triangleq A\mathcal{O}$ and $U' \triangleq \mathcal{O}^{\mathrm{T}}U$.
(b) Note that elliptical distributions with different parameters can be equal: if $X \sim E_d(\mu,\Sigma,\phi)$, then $X \sim E_d(\mu, c\Sigma, \phi_c)$ for every $c > 0$, where $\phi_c(s) \triangleq \phi(s/c)$ for all $s \geq 0$.

Example 3.1. Classical examples of elliptical distributions are the multivariate normal and the multivariate t-distributions. Let $X \stackrel{\mathrm{d}}{=} \mu + RAU \sim E_d(\mu,\Sigma,\phi)$, where $\mathrm{rank}(\Sigma) = d$. Then X is normally distributed if and only if $R^2 \sim \chi^2_d$, and X is t-distributed with ν degrees of freedom if and only if $R^2/d \sim F(d,\nu)$, where $F(d,\nu)$ denotes an F-distribution with d and ν degrees of freedom.

If the elliptically distributed random vector X has finite second moments, then we can always find a representation such that $\mathrm{Cov}(X) = \Sigma$. To see this we use Theorem 3.1 to obtain

$$\mathrm{Cov}(X) = \mathrm{Cov}(\mu + RAU) = A\,\mathbb{E}(R^2)\,\mathrm{Cov}(U)\,A^{\mathrm{T}},$$

i.e. $\mathrm{Cov}(X)$ exists if and only if $\mathbb{E}(R^2) < \infty$. To compute $\mathrm{Cov}(U)$ let $Y \sim \mathcal{N}_d(0, I_d)$. Then $Y \stackrel{\mathrm{d}}{=} |Y|_2 U$, where $|Y|_2$ and U are independent. Furthermore $|Y|_2^2 \sim \chi^2_d$, so $\mathbb{E}(|Y|_2^2) = d$. Since $\mathrm{Cov}(Y) = I_d$ we see that if U is uniformly distributed on the unit hypersphere in $\mathbb{R}^d$, then $\mathrm{Cov}(U) = I_d/d$. Thus $\mathrm{Cov}(X) = AA^{\mathrm{T}}\mathbb{E}(R^2)/d$. By choosing the characteristic generator $\phi^*(s) = \phi(s/c)$, where $c = \mathbb{E}(R^2)/d$, we get $\mathrm{Cov}(X) = \Sigma$.

The following result provides the basis of most applications of elliptical distributions.

Lemma 3.1. *Let $X \sim E_d(\mu,\Sigma,\phi)$, let B be a $q\times d$ matrix and let $b \in \mathbb{R}^q$. Then*

$$b + BX \sim E_q(b + B\mu, B\Sigma B^{\mathrm{T}}, \phi).$$

Proof. By Theorem 3.1, $b + BX$ has a stochastic representation

$$b + BX \stackrel{\mathrm{d}}{=} b + B\mu + RBAU$$

and the conclusion follows from Definition 3.1. □

If we partition X, μ and Σ into

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where X_1 and μ_1 are $r \times 1$ vectors and Σ_{11} is a $r \times r$ matrix, then we have the following consequence of Lemma 3.1.

Corollary 3.1. *Let $X \sim E_d(\mu, \Sigma, \phi)$. Then*

$$X_1 \sim E_r(\mu_1, \Sigma_{11}, \phi), \quad X_2 \sim E_{d-r}(\mu_2, \Sigma_{22}, \phi).$$

Hence, marginal distributions of elliptical distributions are elliptical and of the same type (with the same characteristic generator).

Next we introduce the linear correlation coefficient for a pair of random variables with a joint elliptical distribution.

Definition 3.2. Let $X \sim E_d(\mu, \Sigma, \phi)$. For $i, j \in \{1, \dots, d\}$, if $\Sigma_{ii} > 0$ and $\Sigma_{jj} > 0$, then we call

$$\varrho_{ij} \triangleq \Sigma_{ij} / \sqrt{\Sigma_{ii}\Sigma_{jj}} \tag{3.2}$$

the *linear correlation coefficient* for X_i and X_j.

If $\operatorname{Var}(X_i), \operatorname{Var}(X_j) in (0, \infty)$, then $\varrho_{ij} = \operatorname{Cov}(X_i, X_j)/\sqrt{\operatorname{Var}(X_i)\operatorname{Var}(X_j)}$. We want to emphasise that the linear correlation coefficient as defined by (3.2) is an extension of the usual definition in terms of variances and covariances. We want to interpret the linear correlation coefficient as a scalar measure of dependence and, as such, it should not rely on finiteness of certain moments. Clearly (3.2) only makes sense for elliptical distributions. On the other hand, linear correlation is not always a meaningful measure of dependence for non-elliptical distributions, whereas Kendall's tau and Spearman's rho remain meaningful; see for example Embrechts, McNeil and Straumann [5] p. 25.

In this paper we primarily consider elliptically distributed random vectors having components with continuous distributions. It is therefore relevant to present necessary and sufficient conditions for the components of an elliptically distributed random vector to be continuous random variables. Throughout the paper we say that a random variable is continuous whenever it has a continuous distribution function. The proof of Lemma 3.2 is given in Section 6.

Lemma 3.2. *Let $X \sim E_d(\mu, \Sigma, \phi)$, with $\mathbb{P}\{X_i = \mu_i\} < 1$ for all $i \in \{1, \dots, d\}$ and with representation $X \stackrel{d}{=} \mu + RAU$ according to Theorem 3.1. If $\operatorname{rank}(\Sigma) = 1$, then $X_1, \dots, X_d$ are continuous random variables if and only if R is continuous. If $\operatorname{rank}(\Sigma) \geq 2$, then $X_1, \dots, X_d$ are continuous random variables if and only if $\mathbb{P}\{X_i = \mu_i\} = 0$ for all i, or equivalently, if and only if $\mathbb{P}\{R = 0\} = 0$.*

4. Main Results

The sum of two independent elliptical random vectors with the same dispersion matrix is elliptical. The next theorem shows that the sum of two dependent elliptical random vectors with the same dispersion matrix, which are dependent only through their radial parts, is also elliptical.

Theorem 4.1. *Let R and $\widetilde{R}$ be non-negative random variables and let $X \triangleq \mu + RZ \sim E_d(\mu, \Sigma, \phi)$ and $\widetilde{X} \triangleq \widetilde{\mu} + \widetilde{R}\widetilde{Z} \sim E_d(\widetilde{\mu}, \Sigma, \widetilde{\phi})$, where $(R, \widetilde{R}), Z, \widetilde{Z}$ are independent. Then $X + \widetilde{X} \sim E_d(\mu + \widetilde{\mu}, \Sigma, \phi^*)$. Moreover, if R and $\widetilde{R}$ are independent, then $\phi^*(u) = \phi(u)\widetilde{\phi}(u)$.*

For the expression of the characteristic generator, ϕ^*, we refer to the proof in Section 6.

A natural application of Theorem 4.1 is in the context of a multivariate time series.

Example 4.1. Let $X_t = \sigma_t Z_t$, $t \in \mathbb{Z}$, where the random vectors $Z_t \sim E_d(0, \Sigma, \phi_t)$ are mutually independent and independent of the non-negative (univariate) random variables σ_t for all t. The σ_t's are allowed to be dependent. Then for every $t \in \mathbb{Z}$, X_t is elliptically distributed with dispersion matrix Σ, and so are all partial sums $S_n = \sum_{t=1}^{n} X_t$.

The relations (given below) between Kendall's tau, Spearman's rho and the linear correlation coefficient are well known for bivariate normally distributed random vectors. As stated in the next theorem the relation between Kendall's tau and the linear correlation coefficient holds more generally for all elliptically distributed random vectors with continuous univariate marginals.

Theorem 4.2. *Let $X \sim E_d(\mu, \Sigma, \phi)$, where for $i, j \in \{1, \dots, d\}$, X_i and X_j are continuous. Then,*

$$\tau(X_i, X_j) = \frac{2}{\pi} \arcsin \varrho_{ij}. \tag{4.1}$$

For a proof of an extended version, see Lindskog, McNeil and Schmock [8]. As a consequence we have the following well-known result for Spearman's rho, for which we give an easy proof in Section 6.

Corollary 4.1. *Let $X \sim \mathcal{N}_d(\mu, \Sigma)$, where for $i, j \in \{1, \dots, d\}$, $\Sigma_{ii}, \Sigma_{jj} > 0$. Then*

$$\varrho_S(X_i, X_j) = \frac{6}{\pi} \arcsin(\varrho_{ij}/2). \tag{4.2}$$

In the light of Theorem 4.2 one might expect Spearman's rho to be invariant in the class of elliptical distributions with continuous univariate marginals and a fixed dispersion matrix. However, the counterexample below shows this to be *not* true.

Counterexample. Let $X \sim \mathcal{N}_2(\mu, \Sigma)$, where $\Sigma_{11}, \Sigma_{22} > 0$. According to Theorem 3.1 X has a stochastic representation $X \stackrel{\mathrm{d}}{=} \mu + RAU$, where $R \sim \chi_2^2$. We construct a counterexample by deriving a relation between Spearman's rho and the linear correlation coefficient for the bivariate elliptically distributed random vector $W \triangleq AU$. The relation is given by

$$\varrho_S(W_1, W_2) = 3\left(\frac{\arcsin \varrho}{\pi}\right) - 4\left(\frac{\arcsin \varrho}{\pi}\right)^3,$$

where $\varrho = \Sigma_{12}/\sqrt{\Sigma_{11}\Sigma_{22}}$. This relation differs from the relation (4.2) between Spearman's rho and the linear correlation coefficient for a bivariate normal distribution. The difference $\varrho_S(X_1, X_2) - \varrho_S(W_1, W_2)$ as a function of the linear correlation coefficient ϱ is plotted in Figure 2. We see that the difference is small but clearly not equal to zero. For more details on this counterexample we refer to Section 6. It should be noted that there are other choices of R (other than $R^2 \sim \chi_2^2$) for which the difference $\varrho_S(X_1, X_2) - \varrho_S(W_1, W_2)$ becomes much bigger.

In Section 2 we introduced two concepts for measuring dependence of multivariate extremes of random vectors, the coefficient of tail dependence and the spectral measure associated with a regularly varying random vector. In the next theorem we clarify the connection between these two concepts. We also derive an explicit expression for the coefficient of tail dependence for two random variables with a joint elliptical distribution.

Theorem 4.3. *Let* $X \stackrel{\mathrm{d}}{=} \mu + RAU \sim E_d(\mu, \Sigma, \phi)$, *with* $\Sigma_{ii} > 0$ *for* $i = 1, \ldots, d$, $|\varrho_{ij}| < 1$ *for all* $i \neq j$, *and where* μ, R, A *and* U *are as in Theorem 3.1. Then the following statements are equivalent.*

(1) *R is regularly varying with index $\alpha > 0$.*
(2) *X is regularly varying with index $\alpha > 0$.*
(3) *For all $i \neq j$, $(X_i, X_j)^{\mathrm{T}}$ has tail dependence.*

Moreover, if R is regularly varying with index $\alpha > 0$, then for all $i \neq j$,

$$\lambda_U(X_i, X_j) = \lambda_L(X_i, X_j) = \frac{\int_{(\pi/2 - \arcsin \varrho_{ij})/2}^{\pi/2} \cos^{\alpha} t dt}{\int_0^{\pi/2} \cos^{\alpha} t dt}.$$

Remark 4.1. Note that (1) and (2) are equivalent even if the condition $|\varrho_{ij}| < 1$ for all $i \neq j$ is not satisfied.

From the theorem above we can conclude that whether the bivariate marginals of an elliptically distributed vector X have tail dependence or not depends only on whether the radial random variable R in the representation $X \stackrel{\mathrm{d}}{=} \mu + RAU$ is regularly varying or not. The linear correlation coefficient ϱ_{ij} only effects the magnitude of the coefficient of tail dependence. An interesting consequence is that if $X \sim E_d(\mu, \Sigma, \phi)$, then $(X_i, X_j)^{\mathrm{T}}$ can have a coefficient of tail dependence significantly larger than zero even if the linear correlation coefficient for $(X_i, X_j)^{\mathrm{T}}$ is zero or negative. In Figure 1 we have plotted the coefficient of tail dependence for an elliptically distributed bivariate random vector with uncorrelated components as a function of the tail index α.

5. Multivariate extremes for elliptical distributions

In this section we discuss how to interpret the spectral measure with respect to different norms. The discussion is general but in the case of elliptical distributions we can explicitly compute the spectral measure with respect to different norms and compare different choices. Real data, e.g. financial asset price log returns, often indicate that the underlying distribution is elliptical or at least close to elliptical, and many statistical models are based on the assumption of ellipticality. Hence the following discussion should be relevant for many applications, especially in risk management.

By Lemma 2.1 we know that if a random vector X is regularly varying with respect to some norm on $\mathbb{R}^d$, then it is regularly varying with respect to every norm on $\mathbb{R}^d$. For every choice of the norm the spectral measure is a measure of dependence between extreme values. However, the choice of norm becomes essential when interpreting the spectral measure. The choice of norm must be related to the question we are trying to answer. A natural question would be: *What is the dependence between the components of a random vector given that at least one of its components is extreme?* In the literature (see e.g. Stărică [9]) most authors consider the Euclidean 2-norm, $|\cdot|_2$. However, if we want a measure of dependence between the components - given that at least one of the components is extreme -

then we should use the max-norm $|X|_\infty \triangleq \max\{|X_1|,\dots,|X_d|\}$. Clearly, if we take $x=1$ in Definition 2.6, we have that

$$\begin{aligned}\mathbb{P}\{\Theta_\infty \in \cdot\} &= \lim_{t\to\infty} \mathbb{P}\{X/|X|_\infty \in \cdot \mid |X|_\infty > t\}\\ &= \lim_{t\to\infty} \mathbb{P}\{X/|X|_\infty \in \cdot \mid |X_1| > t \cup \cdots \cup |X_d| > t\},\end{aligned}$$

from which it is seen that the max-norm corresponds to the question posed. However, if the components are not identically distributed, then it might be more natural to condition on the events $\{|X_1| > G_1^{-1}(u) \cup \cdots \cup |X_d| > G_d^{-1}(u)\}$, where G_i is the distribution function of $|X_i|$ and $u \nearrow 1$. For $X \sim E_d(0,\Sigma,\phi)$ this is achieved by considering the weighted max-norm $|X|_{\infty,\Sigma} \triangleq \max\{|X_1|/\sqrt{\Sigma_{11}},\dots,|X_d|/\sqrt{\Sigma_{dd}}\}$, since in this case,

$$\begin{aligned}\mathbb{P}\{\Theta_{\infty,\Sigma} \in \cdot\} &= \lim_{t\to\infty} \mathbb{P}\{X/|X|_{\infty,\Sigma} \in \cdot \mid |X|_{\infty,\Sigma} > t\}\\ &= \lim_{u\nearrow 1} \mathbb{P}\{X/|X|_{\infty,\Sigma} \in \cdot \mid |X_1| > G_1^{-1}(u) \cup \cdots \cup |X_d| > G_d^{-1}(u)\},\end{aligned}$$

is the spectral measure of X with respect to the norm $|\cdot|_{\infty,\Sigma}$. The corresponding question in this case would be: *What is the dependence between the components of a random vector given that at least one of its components is extreme relative to its marginal distribution?*

In the following two examples we compute the spectral measure with respect to the Euclidean 2-norm and the max-norm for bivariate regularly varying elliptical distributions. This can also be done for elliptical distributions of higher dimension, but the corresponding computations in spherical coordinates become quite tedious.

Example 5.1. Let $X \sim E_2(0,\Sigma,\phi)$, with $\Sigma_{11},\Sigma_{22} > 0$, be regularly varying with index $\alpha > 0$, and let $X \stackrel{d}{=} RAU$ be a stochastic representation according to Theorem 3.1. Without loss of generality we can choose A and U such that

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \stackrel{d}{=} R \begin{pmatrix} \sqrt{\Sigma_{11}} & 0 \\ \sqrt{\Sigma_{22}}\varrho_{12} & \sqrt{\Sigma_{22}}\sqrt{1-\varrho_{12}^2} \end{pmatrix} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix},$$

where $\varphi \sim U(-\pi/2, 3\pi/2)$, i.e.

$$\begin{aligned}(X_1,X_2)^{\mathrm{T}} &\stackrel{d}{=} (\sqrt{\Sigma_{11}}R\cos\varphi, \sqrt{\Sigma_{22}}\varrho_{12}R\cos\varphi + \sqrt{\Sigma_{22}}\sqrt{1-\varrho_{12}^2}R\sin\varphi)^{\mathrm{T}}\\ &= (\sqrt{\Sigma_{11}}R\cos\varphi, \sqrt{\Sigma_{22}}R\sin(\arcsin\varrho_{12}+\varphi))^{\mathrm{T}},\end{aligned}$$

where $\varphi \sim U(-\pi/2, 3\pi/2)$. Let

$$\begin{aligned} f(t) &\triangleq \left(\Sigma_{11}\cos^2 t + \Sigma_{22}\sin^2(\arcsin\varrho_{ij}+t)\right)^{1/2},\\ g(t) &\triangleq \begin{cases} -\pi/2 & \text{for } t = -\pi/2,\\ \arctan\left(\sqrt{\frac{\Sigma_{22}}{\Sigma_{11}}}(\varrho_{12} + \sqrt{1-\varrho_{12}^2}\tan t)\right) & \text{for } t \in (-\pi/2,\pi/2),\\ g(t-\pi)+\pi & \text{for } t \in [\pi/2, 3\pi/2).\end{cases}\end{aligned}$$

Then,

$$RAU = R|AU|_2\frac{AU}{|AU|_2} \stackrel{d}{=} Rf(\varphi)\begin{pmatrix}\cos g(\varphi)\\ \sin g(\varphi)\end{pmatrix}.$$

Since X is regularly varying and $X/|X|_2$ has continuous distribution on $\mathbb{S}_2^1$, there exists a random vector Θ such that for every $x>0$ and every $S \in \mathcal{B}(\mathbb{S}_2^1)$,

$$\lim_{z\to\infty} \frac{\mathbb{P}\{R|AU|_2 > zx, AU/|AU|_2 \in S\}}{\mathbb{P}\{R|AU|_2 > z\}} = x^{-\alpha}\,\mathbb{P}\{\Theta \in S\}.$$

Moreover, by Theorem 4.2, R is regularly varying which implies that there exists a slowly varying function L, i.e., a positive, Lebesgue measurable function on $(0,\infty)$ satisfying $\lim_{t\to\infty} L(tx)/L(t) = 1$, for $x > 0$, such that $\mathbb{P}\{R > x\} = x^{-\alpha}L(x)$. Let $S_{\theta_1,\theta_2} = \{(\cos t, \sin t)^{\mathrm{T}} \mid \theta_1 < t < \theta_2\}$, where by symmetry we can assume that $-\pi/2 < \theta_1 < \theta_2 < \pi/2$. The case $|\varrho_{12}| = 1$ is trivial, so we consider only the case $|\varrho_{12}| < 1$. Then,

$$g^{-1}(t) = \arctan\left(\frac{1}{\sqrt{1-\varrho_{12}^2}}\left(\frac{\sqrt{\Sigma_{11}}}{\sqrt{\Sigma_{22}}}\tan t - \varrho_{12}\right)\right) \quad \text{for } -\pi/2 < t < \pi/2,$$

and

$$\begin{aligned}
\lim_{z\to\infty}\frac{\mathbb{P}\{R|AU|_2 > zx, AU/|AU|_2 \in S\}}{\mathbb{P}\{R|AU|_2 > z\}} &= \lim_{z\to\infty}\frac{\int_{g^{-1}(\theta_1)}^{g^{-1}(\theta_2)} z^{-\alpha}x^{-\alpha}f(t)^{\alpha}L(zx/f(t))\mathrm{d}t}{\int_0^{2\pi} z^{-\alpha}f(t)^{\alpha}L(z/f(t))\mathrm{d}t} \\
&= x^{-\alpha}\lim_{z\to\infty}\frac{\int_{g^{-1}(\theta_1)}^{g^{-1}(\theta_2)} f(t)^{\alpha}L(zx/f(t))/L(z)\mathrm{d}t}{\int_0^{2\pi} f(t)^{\alpha}L(z/f(t))/L(z)\mathrm{d}t} = x^{-\alpha}\frac{\int_{g^{-1}(\theta_1)}^{g^{-1}(\theta_2)} f(t)^{\alpha}\mathrm{d}t}{\int_0^{2\pi} f(t)^{\alpha}\mathrm{d}t} \\
&= x^{-\alpha}\frac{\int_{g^{-1}(\theta_1)}^{g^{-1}(\theta_2)}\big(\Sigma_{11}\cos^2 t + \Sigma_{22}\sin^2(\arcsin\varrho_{12}+t)\big)^{\alpha/2}\mathrm{d}t}{\int_0^{2\pi}\big(\Sigma_{11}\cos^2 t + \Sigma_{22}\sin^2(\arcsin\varrho_{12}+t)\big)^{\alpha/2}\mathrm{d}t}.
\end{aligned}$$

The third equality follows from the fact that $L(tx)/L(t) \to 1$ uniformly on intervals $[a,b]$, $0 < a \le b < \infty$ (see Theorem A3.2 in Embrechts, Klüppelberg and Mikosch [4] p. 566) and from the fact that there are constants $0 < c_1 < c_2 < \infty$ such that $c_1 < 1/f(t) < c_2$ for all $t \in [-\pi/2, \pi/2]$. Now we can identify the spectral measure as

$$\mathbb{P}\{\Theta \in S_{\theta_1,\theta_2}\} = \frac{\int_{g^{-1}(\theta_1)}^{g^{-1}(\theta_2)}\big(\Sigma_{11}\cos^2 t + \Sigma_{22}\sin^2(\arcsin\varrho_{12}+t)\big)^{\alpha/2}\mathrm{d}t}{\int_0^{2\pi}\big(\Sigma_{11}\cos^2 t + \Sigma_{22}\sin^2(\arcsin\varrho_{12}+t)\big)^{\alpha/2}\mathrm{d}t}.$$

Note that the spectral measure depends on the tail index α. Furthermore, note that $\lim_{\alpha\to 0}\mathbb{P}\{\Theta \in S\} = \mathbb{P}\{AU/|AU|_2 \in S\}$ for all $S \in \mathcal{B}(\mathbb{S}_2^{d-1})$. We see that the spectral measure is absolutely continuous and hence it has a density. The density is plotted in Figure 3 for bivariate regularly varying elliptical distributions with $(\Sigma_{11}, \Sigma_{22}, \varrho_{12}) = (1, 1, 0.5)$ and with tail indices $\alpha = 0, 2, 4, 8, 16$ (we write $\alpha = 0$ for the limit measure $\lim_{\alpha\to 0}\mathbb{P}\{\Theta \in \cdot\}$). From this figure it can be seen that as α increases - that is as the tails become lighter - the probability mass becomes *more concentrated* in the main directions of the ellipse (in this case $\pi/4$ and $5\pi/4$).

Example 5.2. Let us now compute the spectral measure with respect to the norm $|\cdot|_\infty$. Proceeding analogously to the previous example but replacing the function f by

$$\widetilde{f}(t) \triangleq \max\{\sqrt{\Sigma_{11}}\,|\cos t|, \sqrt{\Sigma_{22}}\,|\sin(\arcsin\varrho_{12}+t)|\,\},$$

we find that

$$RAU = R|AU|_\infty\frac{AU}{|AU|_\infty}$$

where $AU/|AU|_\infty \stackrel{\mathrm{d}}{=} \widetilde{f}(\varphi)$ with $\varphi \sim U(-\pi/2, 3\pi/2)$. Following the computations in the previous example we obtain the spectral measure with respect to the max-norm as

$$\mathbb{P}\{\Theta \in \widetilde{S}_{\theta_1,\theta_2}\} = \frac{\int_{g^{-1}(\theta_1)}^{g^{-1}(\theta_2)} \left(\max\{\sqrt{\Sigma_{11}}\,|\cos t|, \sqrt{\Sigma_{22}}\,|\sin(\arcsin \varrho_{12} + t)|\,\}\right)^{\alpha} \mathrm{d}t}{\int_0^{2\pi} \left(\max\{\sqrt{\Sigma_{11}}\,|\cos t|, \sqrt{\Sigma_{22}}\,|\sin(\arcsin \varrho_{12} + t)|\,\}\right)^{\alpha} \mathrm{d}t},$$

where $\widetilde{S}_{\theta_1,\theta_2}$ is the radial projection of S_{θ_1,θ_2} (see Example 5.1) on $\mathbb{S}_{\infty}^{d-1}$, the the unit circle with respect to the max-norm. The density of the spectral measure is plotted in Figure 4 for bivariate regularly varying elliptical distributions with $(\Sigma_{11}, \Sigma_{22}, \varrho_{12}) = (1, 1, 0.5)$ and with tail indices $\alpha = 0, 2, 4, 8, 16$. From this figure it can be seen that as α increases - that is, as the tails become lighter - the probability mass becomes *less concentrated* in the main directions of the ellipse (in this case $\pi/4$ and $5\pi/4$). This is quite intuitive, for (bivariate) regularly varying elliptical distributions with lighter tails, the probability of joint extremes (that both components are extreme) becomes very small compared to the probability that one component is extreme. This can be seen from the fact that the coefficient of tail dependence tends to zero as the tail index increases (see Remark 4.1 and Figure 1).

Note the striking difference between the spectral measure with respect to the Euclidean norm and the spectral measure with respect to the max-norm. By choosing a norm which does not correspond to the question one is trying to answer, one might draw completely wrong conclusions about dependences between extremes. The best illustration of this is the comparison of Figure 3 with Figure 4.

6. Proofs

There exist several definitions of (multivariate) regular variation equivalent to Definition 2.6 (see Davis, Mikosch and Basrak [2]), one of which will be useful in the proof of Lemma 2.1 and Theorem 4.3. A statement similar to the following has been proved in [2], but we include a proof for completeness.

Let $\overline{\mathbb{R}^d} \triangleq \{(r, \theta) \mid r \in [0, \infty], \theta \in \mathbb{S}^{d-1}\}$ be a compactification of the Euclidean space $\mathbb{R}^d$ in polar coordinates. For $(r, \theta) \in \overline{\mathbb{R}^d}$ and $c \in [0, \infty]$, let $c(r, \theta) \triangleq (cr, \theta)$, where we use the convention $\infty \cdot 0 = 0$ and $\infty \cdot c = \infty$ for $c > 0$. Furthermore, for $u > 0$ and $S \in \mathcal{B}(\mathbb{S}^{d-1})$, let $V_{u,S} \triangleq \{(r, \theta) \mid r \in (u, \infty], \theta \in S\}$.

Lemma 6.1. *The following statements are equivalent.*
(1) *X is regularly varying with index $\alpha > 0$ in the sense of Definition 2.6.*
(2) *There exists a non-zero Radon measure μ on $\overline{\mathbb{R}^d}\backslash\{0\}$ and a relatively compact set $E \in \mathcal{B}(\overline{\mathbb{R}^d}\backslash\{0\})$ with $\mathbb{P}\{X \in tE\} > 0$ for all $t \in (0, \infty)$ such that*

(a) $\mu_t(\cdot) \triangleq \mathbb{P}\{X \in t\,\cdot\}/\,\mathbb{P}\{X \in tE\} \xrightarrow{v} \mu(\cdot)$ *as* $t \to \infty$,

(b) *there exists an $\alpha > 0$ such that for all $x > 0$ and $D \in \mathcal{B}(\overline{\mathbb{R}^d}\backslash\{0\})$* $\mu(xD) = x^{-\alpha}\mu(D)$.

Remark 6.1. If $(2a)$ holds, then

$$\widetilde{\mu}_t(\cdot) \triangleq \mathbb{P}\{X \in t\,\cdot\}/\,\mathbb{P}\{X \in t\widetilde{E}\} \xrightarrow{v} \widetilde{\mu}(\cdot) \text{ as } t \to \infty$$

holds with $\widetilde{\mu}(\cdot) = \mu(\cdot)/\mu(\widetilde{E})$ for any relatively compact set $\widetilde{E} \in \mathcal{B}(\overline{\mathbb{R}^d}\backslash\{0\})$ such that $\mu(\widetilde{E}) > 0$ and $\mu(\partial\widetilde{E}) = 0$. This follows directly from the fact that

$$\frac{\mathbb{P}\{X \in t\cdot\}}{\mathbb{P}\{X \in tE\}} \frac{\mathbb{P}\{X \in tE\}}{\mathbb{P}\{X \in t\widetilde{E}\}} \xrightarrow{v} \widetilde{\mu}(\cdot) \text{ as } t \to \infty$$

if the set $\widetilde{E}$ satisfies the above conditions.

Proof of Lemma 6.1. (2) $\Rightarrow$ (1) We first show that the set $V_{1,\mathbb{S}^{d-1}}$ satisfies the conditions in Remark 6.1. Then applying (2) to sets $V_{x,S}$ for $x>0$ and $S \in \mathcal{B}(\mathbb{S}^{d-1})$ will prove the claim. Let INF $\triangleq \{(\infty,\theta) \mid \theta \in \mathbb{S}^{d-1}\}$. Since INF $\in \mathcal{B}(\overline{\mathbb{R}^d}\backslash\{0\})$ and INF is relatively compact $\mu(\mathrm{INF}) < \infty$. Since $x\mathrm{INF} = \mathrm{INF}$ for all $x>0$ and $\mu(x\mathrm{INF}) = x^{-\alpha}\mu(\mathrm{INF})$ we conclude that $\mu(\mathrm{INF}) = 0$. Hence $\mu(\partial V_{u,\mathbb{S}^{d-1}}) = \mu(\{u\theta \mid \theta \in \mathbb{S}^{d-1}\} \cup \mathrm{INF}) = \mu(\{u\theta \mid \theta \in \mathbb{S}^{d-1}\})$. Suppose that $\mu(\partial V_{1,\mathbb{S}^{d-1}}) = c > 0$. Then, if $\mathbb{Q}$ denotes the rational numbers,

$$\begin{aligned}\mu(V_{1,\mathbb{S}^{d-1}}\backslash V_{2,\mathbb{S}^{d-1}}) &\geq \mu(\cup_{q\in\mathbb{Q}\cap(1,2]}\partial V_{q,\mathbb{S}^{d-1}}) = \sum_{q\in\mathbb{Q}\cap(1,2]}\mu(\partial V_{q,\mathbb{S}^{d-1}})\\ &\geq c2^{-\alpha}\sum_{q\in\mathbb{Q}\cap(1,2]} 1 = \infty.\end{aligned}$$

Since $V_{1,\mathbb{S}^{d-1}}\backslash V_{2,\mathbb{S}^{d-1}}$ is relatively compact this is a contradiction and it follows that $\mu(\partial V_{1,\mathbb{S}^{d-1}}) = 0$. Since E is relatively compact, E is bounded away from $\{0\}$ so there exists a $u>0$ such that $E \subset V_{u,\mathbb{S}^{d-1}}$ and $\mu(V_{1,\mathbb{S}^{d-1}}) = u^{\alpha}\mu(V_{u,\mathbb{S}^{d-1}}) > 0$ since $\mu(E) > 0$. Furthermore, $V_{1,\mathbb{S}^{d-1}}$ is relatively compact. Hence, by Remark 6.1, we may put $E = V_{1,\mathbb{S}^{d-1}}$. For any $x>0$ and $S \in \mathcal{B}(\mathbb{S}^{d-1})$ with $\mu(\partial V_{x,S}) = 0$,

$$\lim_{t\to\infty} \mathbb{P}\{|X| > tx, X/|X| \in S\}/\,\mathbb{P}\{|X| > t\} = \lim_{t\to\infty}\mu_t(V_{x,S}) = \mu(V_{x,S}) = x^{-\alpha}\mu(V_{1,S}).$$

Since $\mu(V_{1,\cdot})$ is a probability measure on $\mathbb{S}^{d-1}$ there exists a random vector Θ with this distribution. Hence X is regularly varying with index $\alpha > 0$.

(1) $\Rightarrow$ (2) Let $\mathcal{P} \triangleq \{V_{u,S} \mid u > 0, S \in \mathcal{B}(\mathbb{S}^{d-1})\}$. Then $\mathcal{P}$ is a π-system, i.e. $A, B \in \mathcal{P} \Rightarrow A \cap B \in \mathcal{P}$, and $\sigma(\mathcal{P}) = \mathcal{B}(\overline{\mathbb{R}^d}\backslash\{0\})$. For $x>0$ define

$$\mu_t(V_{x,\cdot}) \triangleq \mathbb{P}\{|X| > tx, X/|X| \in \cdot\}/\,\mathbb{P}\{|X| > t\} \quad \text{and} \quad \mu(V_{x,\cdot}) \triangleq x^{-\alpha}\,\mathbb{P}\{\Theta \in \cdot\}.$$

Since $\sigma(\mathcal{P}) = \mathcal{B}(\overline{\mathbb{R}^d}\backslash\{0\})$ it follows that μ_t and μ are well defined as Radon measures on $(\overline{\mathbb{R}^d}\backslash\{0\}, \mathcal{B}(\overline{\mathbb{R}^d}\backslash\{0\}))$. By definition of μ, $\mu(V_{x,\cdot}) = x^{-\alpha}\mu(V_{1,\cdot})$ for all $x>0$. Hence, on $\mathcal{P}$ μ agrees with the product measure ν, given by $\nu(\mathrm{d}r\times\mathrm{d}\theta) = \alpha r^{-\alpha-1}\mathrm{d}r\times \mathrm{d}F_\Theta(\theta)$, where F_Θ is the distribution function of Θ. Since $V_{x,\mathbb{S}^{d-1}} \nearrow \overline{\mathbb{R}^d}\backslash\{0\}$ as $x \searrow 0$ and $\mu(V_{x,\mathbb{S}^{d-1}}), \nu(V_{x,\mathbb{S}^{d-1}}) < \infty$ for all $x>0$, it follows (see e.g. Durrett [3] p. 448) that μ and ν agree on $\mathcal{B}(\overline{\mathbb{R}^d}\backslash\{0\})$. Hence $\mu(xD) = x^{-\alpha}\mu(D)$ for all $x>0$ and $D \in \mathcal{B}(\overline{\mathbb{R}^d}\backslash\{0\})$. By definition μ is a non-zero Radon measure. Since X is regularly varying $\mu_t(V_{x,\cdot}) \xrightarrow{v} \mu(V_{x,\cdot})$ as $t \to \infty$ on $(\mathbb{S}^{d-1}, \mathcal{B}(\mathbb{S}^{d-1}))$ for all $x>0$. Let $\mathcal{A}$ be the sets of the form $V_{u,S}\backslash V_{v,S}$ where $u \leq v$ and $S \in \mathcal{B}(\mathbb{S}^{d-1})$, i.e. $\mathcal{A}$ contains the rectangles, in polar coordinates, of $\overline{\mathbb{R}^d}\backslash\{0\}$. Since for all $x>0$ and $D \in \mathcal{B}(\overline{\mathbb{R}^d}\backslash\{0\})$ $\mu(xD) = x^{-\alpha}\mu(D)$, we have, by arguing as in the (2) $\Rightarrow$ (1) part, that $\mu(\partial V_{u,S}) = 0$ if and only if $\mu(V_{u,\partial S}) = 0$. Since $\mu_t(V_{x,\cdot}) \xrightarrow{v} \mu(V_{x,\cdot})$ as $t\to\infty$ on $(\mathbb{S}^{d-1}, \mathcal{B}(\mathbb{S}^{d-1}))$ it follows that for $u \leq v$ and $S \in \mathcal{B}(\mathbb{S}^{d-1})$

$$\mu_t(V_{u,S}\backslash V_{v,S}) = \mu_t(V_{u,S}) - \mu_t(V_{v,S}) \to \mu(V_{u,S}) - \mu(V_{v,S}) = \mu(V_{u,S}\backslash V_{v,S})$$

as $t\to\infty$ if $\mu(V_{u,\partial S}) = 0$. Since

$$\partial(V_{u,S}\backslash V_{v,S}) = \{(u,\theta) \mid \theta \in S\} \cup \{(v,\theta) \mid \theta \in S\} \cup V_{u,\partial S}\backslash V_{v,\partial S}$$

and since $\mu(xD) = x^{-\alpha}\mu(D)$ for all $x>0$ and $D \in \mathcal{B}(\overline{\mathbb{R}^d}\backslash\{0\})$ it follows that $\mu(\partial(V_{u,S}\backslash V_{v,S})) = \mu(V_{u,\partial S}\backslash V_{v,\partial S})$. Furthermore,

$$\mu(V_{u,\partial S}\backslash V_{v,\partial S}) = \mu(V_{u,\partial S}) - \mu(V_{v,\partial S}) = (1 - (v/u)^{-\alpha})\mu(V_{u,\partial S})$$

and hence $\mu(\partial(V_{u,S}\backslash V_{v,S})) = 0$ if and only if $\mu(V_{u,\partial S}) = 0$. That is, $\mu_t(A) \to \mu(A)$ as $t \to \infty$ for all $A \in \mathcal{A}$ such that $\mu(\partial A) = 0$. Since $\mathcal{A}$ contains all rectangles of $\overline{\mathbb{R}^d}\backslash\{0\}$, $\mu_t \xrightarrow{v} \mu$ as $t \to \infty$ on $(\overline{\mathbb{R}^d}\backslash\{0\}, \mathcal{B}(\overline{\mathbb{R}^d}\backslash\{0\}))$.

□

Proof of Lemma 2.1. Assume that X is regularly varying with index $\alpha > 0$ with respect to the norm $|\cdot|_A$. Then statement (2) of Lemma 6.1 holds for some μ. Proceeding as in the first part of the proof of Lemma 6.1 with the norm $|\cdot|_B$ proves the claim. □

Proof of Lemma 2.2. Let $x > 0$ be arbitrary but fixed and let $S \in \mathcal{B}(\mathbb{S}^{d-1})$ be arbitrary but fixed with $\mathbb{P}\{\Theta \in \partial S\} = 0$. For $t > 0$ let

$$\begin{aligned} A_t &\triangleq \frac{\mathbb{P}\{|X+b| > tx, (X+b)/|X+b| \in S\}}{\mathbb{P}\{|X+b| > t\}}, \\ L_t &\triangleq \frac{\mathbb{P}\{|X| > tx + |b|, (X+b)/|X+b| \in S\}}{\mathbb{P}\{|X| > t - |b|\}}, \\ U_t &\triangleq \frac{\mathbb{P}\{|X| > tx - |b|, (X+b)/|X+b| \in S\}}{\mathbb{P}\{|X| > t + |b|\}}. \end{aligned}$$

Then

$$\begin{aligned} L_t &= \frac{\mathbb{P}\{|X| > tx + |b|, (X+b)/|X+b| \in S\}}{\mathbb{P}\{|X| > t + |b|/x\}} \frac{\mathbb{P}\{|X| > t + |b|/x\}}{\mathbb{P}\{|X| > t - |b|\}} \triangleq L_t^{(1)} L_t^{(2)}, \\ U_t &= \frac{\mathbb{P}\{|X| > tx - |b|, (X+b)/|X+b| \in S\}}{\mathbb{P}\{|X| > t - |b|/x\}} \frac{\mathbb{P}\{|X| > t - |b|/x\}}{\mathbb{P}\{|X| > t + |b|\}} \triangleq U_t^{(1)} U_t^{(2)}. \end{aligned}$$

Since $L_t \le A_t \le U_t$ for all $t > 0$, $L_t^{(2)}, U_t^{(2)} \to 1$ as $t \to \infty$ and $L_t^{(1)} = U_s^{(1)}$ with $s = t + 2|b|/x$,

$$\begin{aligned} \lim_{t\to\infty} A_t &= \lim_{t\to\infty} L_t = \lim_{t\to\infty} U_t = \lim_{t\to\infty} \frac{\mathbb{P}\{|X| > tx, (X+b)/|X+b| \in S\}}{\mathbb{P}\{|X| > t\}} \\ &= \lim_{t\to\infty} \frac{\mathbb{P}\{|X| > tx, X/|X| \in S\}}{\mathbb{P}\{|X| > t\}} \\ &\quad + \lim_{t\to\infty} \frac{\mathbb{P}\{|X| > tx, (X+b)/|X+b| \in S, X/|X| \notin S\}}{\mathbb{P}\{|X| > t\}} \\ &\quad - \lim_{t\to\infty} \frac{\mathbb{P}\{|X| > tx, (X+b)/|X+b| \notin S, X/|X| \in S\}}{\mathbb{P}\{|X| > t\}}. \end{aligned}$$

The second to last term can be written as

$$\begin{aligned} &\lim_{t\to\infty} \frac{\mathbb{P}\{|X| > tx, (X+b)/|X+b| \in S, X/|X| \notin S\}}{\mathbb{P}\{|X| > t\}} \\ &= \lim_{t\to\infty} \frac{\mathbb{P}\{|X| > tx\}}{\mathbb{P}\{|X| > t\}} \mathbb{P}\{(X+b)/|X+b| \in S, X/|X| \notin S \mid |X| > tx\} \\ &\le \lim_{t\to\infty} \frac{\mathbb{P}\{|X| > tx\}}{\mathbb{P}\{|X| > t\}} \mathbb{P}\{X/|X| \in \partial S \mid |X| > tx\} = x^{-\alpha}\, \mathbb{P}\{\Theta \in \partial S\} = 0, \end{aligned}$$

and similarly for the last term, from which the conclusion follows. □

Proof of Lemma 3.2. Let $X \stackrel{\mathrm{d}}{=} \mu + RAU$ be a stochastic representation according to Theorem 3.1. Suppose $\operatorname{rank}(\Sigma) = 1$, then A is a $d \times 1$ matrix and U is symmetric $\{1, -1\}$-valued. Furthermore, $\mathbb{P}\{X_i = \mu_i\} < 1$ implies $A_{i1} \neq 0$. Hence, if $\operatorname{rank}(\Sigma) =$

1, then $X_1,\dots,X_d$ are continuous random variables if and only if R is continuous. Now suppose $\mathrm{rank}(\Sigma)=k\geq 2$. Define $A_i \triangleq (A_{i1},\dots,A_{ik})$ and $a\triangleq A_iA_i^{\mathrm{T}}$. Since $\mathbb{P}\{X_i=\mu_i\}<1$, the case $a=0$ is excluded. By choosing an orthogonal $k\times k$ matrix $\mathcal{O}$ whose first column is A_i^{T}/a and using Remark 3.1(a) if necessary, we may assume that $A_i=(a,0,\dots,0)$, hence $X_i\overset{\mathrm{d}}{=}\mu_i+aRU_1$. Note that U_1 is a continuous random variable because $k\geq 2$. Hence $\mathbb{P}\{aRU_1=x\}=0$ for all $x\in\mathbb{R}\backslash\{0\}$. Hence, if $\mathrm{rank}(\Sigma)\geq 2$, then $X_1,\dots,X_d$ are continuous random variables if and only if $\mathbb{P}\{X_i=\mu_i\}=0$ for $i=1,\dots,d$, or equivalently, if and only if $\mathbb{P}\{R=0\}=0$. □

Proof of Theorem 4.1. Let $\widetilde{\phi}^{(r)}$ be the characteristic generator of $(\widetilde{R}\mid R=r)\widetilde{Z}$, let ϕ' be the characteristic generator of Z, and let F_R be the distribution function of R. Then for all $t\in\mathbb{R}^d$,

$$\begin{aligned}\varphi_{RZ+\widetilde{R}\widetilde{Z}}(t) &= \int_0^\infty \varphi_{rZ}(t)\varphi_{(\widetilde{R}|R=r)\widetilde{Z}}(t)\mathrm{d}F_R(r)\\ &= \int_0^\infty \phi'(r^2t^{\mathrm{T}}\Sigma t)\widetilde{\phi}^{(r)}(t^{\mathrm{T}}\Sigma t)\mathrm{d}F_R(r),\end{aligned}$$

from which it follows that $X+\widetilde{X}\sim E_d(\mu+\widetilde{\mu},\Sigma,\phi^*)$, with

$$\phi^*(u)=\int_0^\infty \phi'(r^2u)\widetilde{\phi}^{(r)}(u)\mathrm{d}F_R(r).$$

Moreover, if R and $\widetilde{R}$ are independent, then $\widetilde{\phi}^{(r)}(u)=\widetilde{\phi}(u)$ and

$$\phi^*(u)=\int_0^\infty \phi'(r^2u)\widetilde{\phi}^{(r)}(u)\mathrm{d}F_R(r)=\widetilde{\phi}(u)\int_0^\infty \phi'(r^2u)\mathrm{d}F_R(r)=\phi(u)\widetilde{\phi}(u).$$

□

Proof of Corollary 4.1. Recall that $\varrho_{ij}\triangleq\Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}}$. Let $\widetilde{X}_i\overset{\mathrm{d}}{=}X_i$ for $i=1,\dots,d$ be mutually independent, and independent of X. Then $\widetilde{X}\sim\mathcal{N}_d(\mu,\widetilde{\Sigma})$, where $\widetilde{\Sigma}=\mathrm{diag}(\Sigma_{11},\dots,\Sigma_{dd})$. Hence, $X^*\triangleq X-\widetilde{X}\sim\mathcal{N}_d(0,\Sigma^*)$, where $\Sigma^*=\Sigma+\widetilde{\Sigma}$. Let $\varrho^*_{ij}\triangleq\Sigma^*_{ij}/\sqrt{\Sigma^*_{ii}\Sigma^*_{jj}}$. Then,

$$\varrho_S(X_i,X_j)=3\tau(X_i^*,X_j^*)=3\left(\frac{2}{\pi}\arcsin\varrho^*_{ij}\right)=\frac{6}{\pi}\arcsin(\varrho_{ij}/2),$$

where the second equality follows from Theorem 4.2 and the fact that the dispersion matrix of a sum of two independent identically distributed elliptical random vectors differs from those of the terms by at most a positive constant factor. □

Proof of Theorem 4.3. By Lemma 2.2 we can without loss of generality assume that $\mu=0$. (1) $\iff$ (2) If $\mathrm{rank}(\Sigma)=k<d$, denote by $\Sigma^{(-1)}\triangleq(A^{(-1)})^{\mathrm{T}}A^{(-1)}$ the generalised inverse of Σ, where $A^{(-1)}\triangleq(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$, i.e. $A^{(-1)}$ solves $A^{(-1)}A=I_k$, where I_k denotes the $k\times k$ identity matrix. Note that $\Sigma^{(-1)}=\Sigma^{-1}$ if $\mathrm{rank}(\Sigma)=d$. By choosing the norm $|x|_\Sigma\triangleq(x^{\mathrm{T}}\Sigma^{(-1)}x)^{1/2}$ in the definition of regular variation (by Lemma 2.1 we are allowed to choose any norm), we obtain

$$\frac{\mathbb{P}\{|X|_\Sigma>tx, X/|X|_\Sigma\in\cdot\}}{\mathbb{P}\{|X|_\Sigma>t\}}=\frac{\mathbb{P}\{R>tx, AU\in\cdot\}}{\mathbb{P}\{R>t\}}=\frac{\mathbb{P}\{R>tx\}\,\mathbb{P}\{AU\in\cdot\}}{\mathbb{P}\{R>t\}}.$$

If R is regularly varying, then $\lim_{t\to\infty}\mathbb{P}\{R>tx\}/\mathbb{P}\{R>t\}=x^{-\alpha}$, and hence $\mathbb{P}\{|X|_\Sigma>tx, X/|X|_\Sigma\in\cdot\}/\mathbb{P}\{|X|_\Sigma>t\}\xrightarrow{v}x^{-\alpha}\,\mathbb{P}\{AU\in\cdot\}$ as $t\to\infty$. Conversely, if $\mathbb{P}\{|X|_\Sigma > tx, X/|X|_\Sigma \in \cdot\}/\mathbb{P}\{|X|_\Sigma > t\} \xrightarrow{v} x^{-\alpha}\,\mathbb{P}\{\Theta \in \cdot\}$ as $t \to \infty$, then we must have $\Theta \stackrel{\mathrm{d}}{=} AU$ and $\lim_{t\to\infty} \mathbb{P}\{R > tx\}/\mathbb{P}\{R > t\} = x^{-\alpha}$.

(1) $\iff$ (3) First note that if $X \sim E_d(0, \Sigma, \phi)$, with $X_i \sim F_i$ and $X_j \sim F_j$, then $F_i^{-1}(u) = \sqrt{\Sigma_{ii}/\Sigma_{jj}}F_j^{-1}(u)$ for $u \in (0,1)$. Secondly, if $\lim_{u\nearrow 1} F_i^{-1}(u) < \infty$, i.e. if X_i is a bounded random variable, then there exists a $u_0 \in (0,1)$ such that the events $\{X_i > F_i^{-1}(u)\}$ and $\{X_j > F_j^{-1}(u)\}$ are disjoint for $u > u_0$, and hence

$$\lim_{u\nearrow 1} \frac{\mathbb{P}\{X_i > F_i^{-1}(u), X_j > F_j^{-1}(u)\}}{\mathbb{P}\{X_i > F_i^{-1}(u)\}} = 0.$$

Therefore, without loss of generality, we only consider random vectors whose marginal distributions are such that $\lim_{u\nearrow 1} F_i^{-1}(u) = \infty$, i.e. random variables with unbounded support. Then,

$$\lambda_U(X_i, X_j) = \lim_{z\to\infty} \frac{\mathbb{P}\{X_i > \sqrt{\Sigma_{ii}}z, X_j > \sqrt{\Sigma_{jj}}z\}}{\mathbb{P}\{X_i > \sqrt{\Sigma_{ii}}z\}}.$$

Since $X \stackrel{\mathrm{d}}{=} RAU$,

$$\begin{pmatrix} X_i \\ X_j \end{pmatrix} \stackrel{\mathrm{d}}{=} R \begin{pmatrix} \sqrt{\Sigma_{ii}} & 0 \\ \sqrt{\Sigma_{jj}}\varrho_{ij} & \sqrt{\Sigma_{jj}}\sqrt{1-\varrho_{ij}^2} \end{pmatrix} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix},$$

where $\varphi \sim U(-\pi, \pi)$, i.e.

$$\begin{aligned} (X_i, X_j)^{\mathrm{T}} &\stackrel{\mathrm{d}}{=} (\sqrt{\Sigma_{ii}}R\cos\varphi, \sqrt{\Sigma_{jj}}\varrho_{ij}R\cos\varphi + \sqrt{\Sigma_{jj}}\sqrt{1-\varrho_{ij}^2}R\sin\varphi)^{\mathrm{T}} \\ &= (\sqrt{\Sigma_{ii}}R\cos\varphi, \sqrt{\Sigma_{jj}}R\sin(\arcsin\varrho_{ij} + \varphi))^{\mathrm{T}}, \end{aligned}$$

with $\varphi \sim U(-\pi, \pi)$. Hence

$$\lambda_U(X_i, X_j) = \lim_{z\to\infty} \frac{\mathbb{P}\{R\cos\varphi > z, R\sin(\arcsin\varrho_{ij} + \varphi) > z\}}{\mathbb{P}\{R\cos\varphi > z\}}.$$

The numerator can be written as

$$\begin{aligned} &\mathbb{P}\{R\cos\varphi > z, R\sin(\arcsin\varrho_{ij} + \varphi) > z\} \\ &\quad = \mathbb{P}\{R > z\max(\frac{1}{\cos\varphi}, \frac{1}{\sin(\arcsin\varrho_{ij} + \varphi)}), \cos\varphi > 0, \sin(\arcsin\varrho_{ij} + \varphi) > 0\} \\ &\quad = \frac{1}{\pi}\int_{(\pi/2 - \arcsin\varrho_{ij})/2}^{\pi/2} \mathbb{P}\{R > z/\cos t\}\mathrm{d}t, \end{aligned}$$

and the denominator can be written as

$$\mathbb{P}\{R\cos\varphi > z\} = \mathbb{P}\{R > z/\cos\varphi, \cos\varphi > 0\} = \frac{1}{\pi}\int_0^{\pi/2} \mathbb{P}\{R > z/\cos t\}\mathrm{d}t.$$

Suppose R is regularly varying with index $\alpha > 0$. By Theorem A.3.2 in Embrechts, Klüppelberg and Mikosch [4] p. 566, for $a > 0$,

$$\mathbb{P}\{R > zx\}/\mathbb{P}\{R > z\} \to x^{-\alpha} \text{ as } z \to \infty,$$

uniformly in x on each $[a, \infty)$. In particular, with $x = 1/\cos t$,

$$\mathbb{P}\{R > z/\cos t\}/\mathbb{P}\{R > z\} \to \cos^\alpha t \text{ as } z \to \infty,$$

uniformly in t on $[0, \pi/2)$. Let

$$f_z(t) \triangleq \begin{cases} \mathbb{P}\{R > z/\cos t\}/\mathbb{P}\{R > z\} & \text{for } t \in [0, \pi/2), \\ 0 & \text{for } t = \pi/2. \end{cases}$$

Then $f_z(\cdot) \to \cos^\alpha(\cdot)$ uniformly on $[0, \pi/2]$, and hence

$$\lambda_U(X_i, X_j) = \lim_{z\to\infty} \frac{\int_{(\pi/2-\arcsin\varrho_{ij})/2}^{\pi/2} f_z(t)\mathrm{d}t}{\int_0^{\pi/2} f_z(t)\mathrm{d}t} = \frac{\int_{(\pi/2-\arcsin\varrho_{ij})/2}^{\pi/2} \cos^\alpha t\mathrm{d}t}{\int_0^{\pi/2} \cos^\alpha t\mathrm{d}t}.$$

Since

$$\lim_{\alpha\to\infty} \frac{\int_{(\pi/2-\arcsin\varrho_{ij})/2}^{\pi/2} \cos^\alpha t\mathrm{d}t}{\int_0^{\pi/2} \cos^\alpha t\mathrm{d}t} = 0,$$

we conclude that $\lambda_U(X_i, X_j) > 0$ if and only if R is regularly varying with index $\alpha > 0$ and in that case

$$\lambda_U(X_i, X_j) = \frac{\int_{(\pi/2-\arcsin\varrho_{ij})/2}^{\pi/2} \cos^\alpha t\mathrm{d}t}{\int_0^{\pi/2} \cos^\alpha t\mathrm{d}t}.$$

Moreover, because elliptically distributed random vectors are radially symmetric about μ, $\lambda_U(X_i, X_j) = \lambda_L(X_i, X_j)$. □

We close this section with a more detailed version of the counterexample already discussed in Section 4.

Counterexample. Let $X \overset{\mathrm{d}}{=} \mu + RAU \sim E_2(\mu, \Sigma, \phi)$, where $\Sigma_{11}, \Sigma_{22} > 0$ and μ, R, A and U are as in Theorem 3.1. To construct a counterexample we derive the relation between Spearman's rho and the linear correlation coefficient $\varrho = \Sigma_{12}/\sqrt{\Sigma_{11}\Sigma_{22}}$ for $W \triangleq AU$. We only consider the case with $\mathrm{rank}(\Sigma) = 2$, since the case with $\mathrm{rank}(\Sigma) = 1$ is trivial. From the invariance of Spearman's rho under componentwise strictly increasing transformations of the underlying random vector we can without loss of generality assume that $\Sigma_{11} = \Sigma_{22} = 1$ and $\Sigma_{12} = \Sigma_{21} = \varrho$. We show that the following relation holds,

$$\varrho_S(W_1, W_2) = 3\left(\frac{\arcsin\varrho}{\pi}\right) - 4\left(\frac{\arcsin\varrho}{\pi}\right)^3. \tag{6.1}$$

In the case of a bivariate normal distribution, i.e. $R \sim \chi_2^2$, we know from Corollary 4.1 that the relation between Spearman's rho and the linear correlation coefficient is

$$\varrho_S(X_1, X_2) = \frac{6}{\pi}\arcsin(\varrho/2). \tag{6.2}$$

Since these two relations differ (the difference is plotted in Figure 2) we conclude that, contrary to Kendall's tau, Spearman's rho is not invariant in the class of elliptical distributions with a fixed dispersion matrix. It remains to be shown that (6.1) holds. This can be done following the steps below.

Step 1. Let (W_1, W_2), (W_1', W_2') and (W_1'', W_2'') be independent copies. Then

$$\varrho_S(W_1, W_2) = 12\,\mathbb{P}\{W_1' \le W_1, W_2'' \le W_2\} - 3.$$

Step 2. For (W_1, W_2), W_1', W_2'' as above we have that,

$$\begin{aligned}\mathbb{P}\{W_1' \le W_1, W_2'' \le W_2\} = \frac{1}{2\pi}\int_0^{2\pi} \Bigg(&\frac{1}{2} + \frac{1}{\pi}\arcsin(\sin(\arcsin\varrho + t)) \\ &- \frac{1}{2\pi}\arccos(\cos t) \\ &- \frac{1}{\pi^2}\arccos(\cos t)\arcsin(\sin(\arcsin\varrho + t))\Bigg)\mathrm{d}t.\end{aligned}$$

Step 3. The following equalities hold:

(i) $\displaystyle\int_0^{2\pi} \arcsin(\sin(\arcsin\varrho + t))\mathrm{d}t = 0.$

(ii) $\displaystyle\int_0^{2\pi} \arccos(\cos t)\mathrm{d}t = \pi^2.$

(iii) $\displaystyle\int_0^{2\pi} \arccos(\cos t)\arcsin(\sin(\arcsin\varrho + t))\mathrm{d}t = \frac{2}{3}(\arcsin\varrho)^3 - \frac{\pi^2}{2}\arcsin\varrho.$

Combining Steps 1-3 yields (6.1),

$$\begin{aligned}\varrho_S(W_1, W_2) &= 12\,\mathbb{P}\{W_1' \le W_1, W_2'' \le W_2\} - 3\\ &= \frac{12}{2\pi}\left(\pi - \frac{\pi}{2} - \frac{2}{3\pi^2}(\arcsin\varrho)^3 + \frac{1}{2}\arcsin\varrho\right) - 3\\ &= 3\left(\frac{\arcsin\varrho}{\pi}\right) - 4\left(\frac{\arcsin\varrho}{\pi}\right)^3.\end{aligned}$$

Proof of Step 1. Straightforward computations of Spearman's rho for continuous random variables yields

$$\begin{aligned}\varrho_S(W_1, W_2) &= 3\,(2\,\mathbb{P}\{(W_1 - W_1')(W_2 - W_2'') > 0\} - 1)\\ &= 3\,(4\,\mathbb{P}\{W_1' \le W_1, W_2'' \le W_2\} - 1)\\ &= 12\,\mathbb{P}\{W_1' \le W_1, W_2'' \le W_2\} - 3.\end{aligned}$$

□

Proof of Step 2. Let $\varphi, \varphi', \varphi'' \sim U(0, 2\pi)$ be independent. Then

$$\begin{aligned}(W_1, W_2) &\stackrel{\mathrm{d}}{=} (\cos\varphi, \sin(\arcsin\varrho + \varphi)),\\ (W_1', W_2') &\stackrel{\mathrm{d}}{=} (\cos\varphi', \sin(\arcsin\varrho + \varphi')),\\ (W_1'', W_2'') &\stackrel{\mathrm{d}}{=} (\cos\varphi'', \sin(\arcsin\varrho + \varphi'')).\end{aligned}$$

Conditioning on φ yields,

$$\begin{aligned}&\mathbb{P}\{W_1' \le W_1, W_2'' \le W_2\} = \mathbb{P}\{\cos\varphi' \le \cos\varphi, \sin(\arcsin\varrho + \varphi'') \le \sin(\arcsin\varrho + \varphi)\}\\ &\quad = \frac{1}{2\pi}\int_0^{2\pi} \mathbb{P}\{\cos t - \cos\varphi' \ge 0\}\,\mathbb{P}\{\sin(\arcsin\varrho + t) - \sin(\arcsin\varrho + \varphi'') \ge 0\}\mathrm{d}t.\end{aligned}$$

The factors in the integrand can be written as

$$\begin{aligned}&\mathbb{P}\{\cos t - \cos\varphi' \ge 0\} = 1 - \frac{1}{2\pi}2\arccos(\cos t),\\ &\mathbb{P}\{\sin(\arcsin\varrho + t) - \sin(\arcsin\varrho + \varphi'') \ge 0\}\\ &\quad = 1 - \frac{1}{2\pi}\left(\pi - \arcsin(\sin(\arcsin\varrho + t)) - \arcsin(\sin(\arcsin\varrho + t))\right)\\ &\quad = \frac{1}{2} + \frac{1}{2\pi}2\arcsin(\sin(\arcsin\varrho + t)).\end{aligned}$$

Combining these expressions yields,

$$\begin{aligned}
&\mathbb{P}\{\cos t - \cos\varphi' \geq 0\}\,\mathbb{P}\{\sin(\arcsin\varrho + t) - \sin(\arcsin\varrho + \varphi'') \geq 0\}\\
&= \frac{1}{2} + \frac{1}{\pi}\arcsin(\sin(\arcsin\varrho + t)) - \frac{1}{2\pi}\arccos(\cos t)\\
&\quad - \frac{1}{\pi^2}\arccos(\cos t)\arcsin(\sin(\arcsin\varrho + t)).
\end{aligned}$$

□

Proof of Step 3. (i) and (ii) are elementary. To compute (iii) we first split the integral depending on arccos(cos t) and then use a variable transformation to obtain

$$\begin{aligned}
\mathcal{I} &\triangleq \frac{1}{\pi^2}\int_0^{2\pi}\arccos(\cos t)\arcsin(\sin(\arcsin\varrho + t))\mathrm{d}t\\
&= \frac{1}{\pi^2}\left(\int_0^{\pi} t\arcsin(\sin(\arcsin\varrho + t))\mathrm{d}t + \int_{\pi}^{2\pi}(2\pi - t)\arcsin(\sin(\arcsin\varrho + t))\mathrm{d}t\right)\\
&= \frac{1}{\pi^2}\left(\int_{\arcsin\varrho}^{\pi+\arcsin\varrho}(u - \arcsin\varrho)\arcsin(\sin u)\mathrm{d}u\right.\\
&\qquad\left. + \int_{\pi+\arcsin\varrho}^{2\pi+\arcsin\varrho}(2\pi - u + \arcsin\varrho)\arcsin(\sin u)\mathrm{d}u\right)\\
&= \frac{1}{\pi^2}\left(\underbrace{\int_0^{\pi}(u - \arcsin\varrho)\arcsin(\sin u)\mathrm{d}u}_{I} - \underbrace{\int_0^{\arcsin\varrho}(u - \arcsin\varrho)u\mathrm{d}u}_{II}\right.\\
&\qquad + \underbrace{\int_{\pi}^{\pi+\arcsin\varrho}(u - \arcsin\varrho)(\pi - u)\mathrm{d}u}_{III} + \underbrace{\int_{\pi}^{2\pi}(2\pi - u + \arcsin\varrho)\arcsin(sin u)\mathrm{d}u}_{IV}\\
&\qquad - \underbrace{\int_{\pi}^{\pi+\arcsin\varrho}(2\pi - u + \arcsin\varrho)(\pi - u)\mathrm{d}u}_{V}\\
&\qquad\left. + \underbrace{\int_{2\pi}^{2\pi+\arcsin\varrho}(2\pi - u + \arcsin\varrho)(u - 2\pi)\mathrm{d}u}_{VI}\right)
\end{aligned}$$

The different parts can now be computed separately.

$$\begin{aligned}
I &= \int_0^{\pi}(u - \arcsin\varrho)\arcsin(\sin u)\mathrm{d}u = \int_0^{\pi/2}(u - \arcsin\varrho)u\mathrm{d}u\\
&\quad + \int_{\pi/2}^{\pi}(u - \arcsin\varrho)(\pi - u)\mathrm{d}u = \pi^3/8 - \pi^2(\arcsin\varrho)/4\\
II &= \int_0^{\arcsin\varrho}(u - \arcsin\varrho)u\mathrm{d}u = -(\arcsin\varrho)^3/6\\
III &= \int_{\pi}^{\pi+\arcsin\varrho}(u - \arcsin\varrho)(\pi - u)\mathrm{d}u = -\pi(\arcsin\varrho)^2/2 + (\arcsin\varrho)^3/6
\end{aligned}$$

$$\begin{aligned}
IV &= \int_{\pi}^{2\pi} (2\pi - u + \arcsin\varrho)\arcsin(\sin u)\mathrm{d}u \\
&= \int_{\pi}^{3\pi/2} (2\pi - u + \arcsin\varrho)(\pi - u)\mathrm{d}u + \int_{3\pi/2}^{2\pi} (2\pi - u + \arcsin\varrho)(u - 2\pi)\mathrm{d}u \\
&= -\pi^3/8 - \pi^2(\arcsin\varrho)/4 \\
V &= \int_{\pi}^{\pi+\arcsin\varrho} (2\pi - u + \arcsin\varrho)(\pi - u)\mathrm{d}u = -\pi(\arcsin\varrho)^2/2 - (\arcsin\varrho)^3/6 \\
VI &= \int_{2\pi}^{2\pi+\arcsin\varrho} (2\pi - u + \arcsin\varrho)(u - 2\pi)\mathrm{d}u = (\arcsin\varrho)^3/6
\end{aligned}$$

Putting everything together yields

$$\begin{aligned}
\mathcal{I} &= \frac{1}{\pi^2}(I - II + III + IV - V + VI) = \frac{1}{\pi^2}\left(\frac{2}{3}(\arcsin\varrho)^3 - \frac{\pi^2}{2}\arcsin\varrho\right) \\
&= \frac{2}{3\pi^2}(\arcsin\varrho)^3 - \frac{1}{2}\arcsin\varrho.
\end{aligned}$$

□

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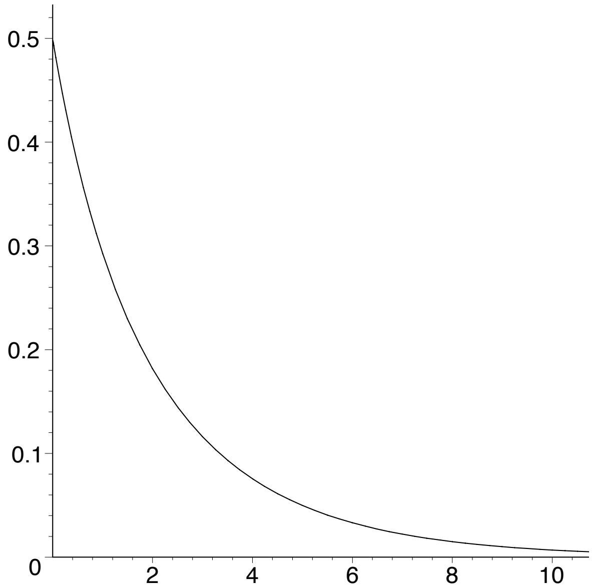

FIGURE 1. The coefficients of upper and lower tail dependence for regularly varying bivariate elliptical distributions with $(\Sigma_{11}, \Sigma_{22}, \varrho_{12}) = (1, 1, 0)$, as a function of the tail index α (see Remark 4.1).

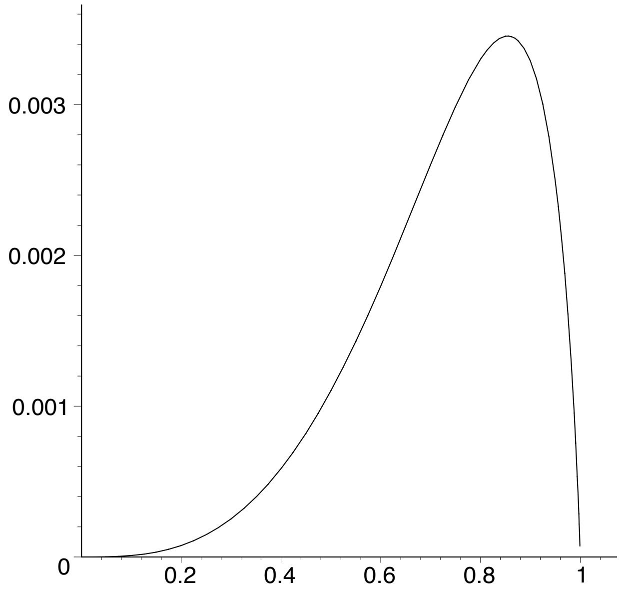

FIGURE 2. The difference between $(6/\pi)\arcsin(\varrho/2)$ and $(3/\pi)\arcsin\varrho - (4/\pi^3)(\arcsin\varrho)^3$ as a function of ϱ for $\varrho \in [0, 1]$ (see the counterexample in Section 4).

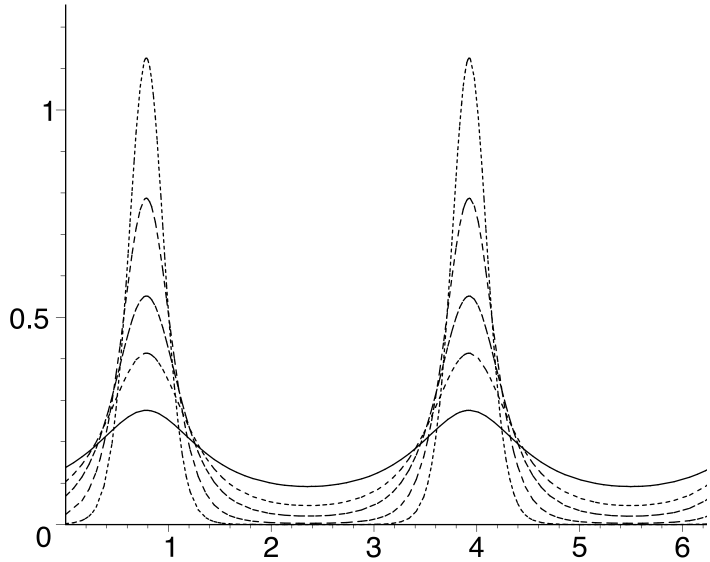

FIGURE 3. Densities of the spectral measure of $X \sim E_2(\mu, \Sigma, \phi)$ with respect to the Euclidean 2-norm, where $(\Sigma_{11}, \Sigma_{22}, \varrho_{12}) = (1, 1, 0.5)$, and tail index $\alpha = 0, 2, 4, 8, 16$. Larger tail indices correspond to higher peaks (see Example 5.1).

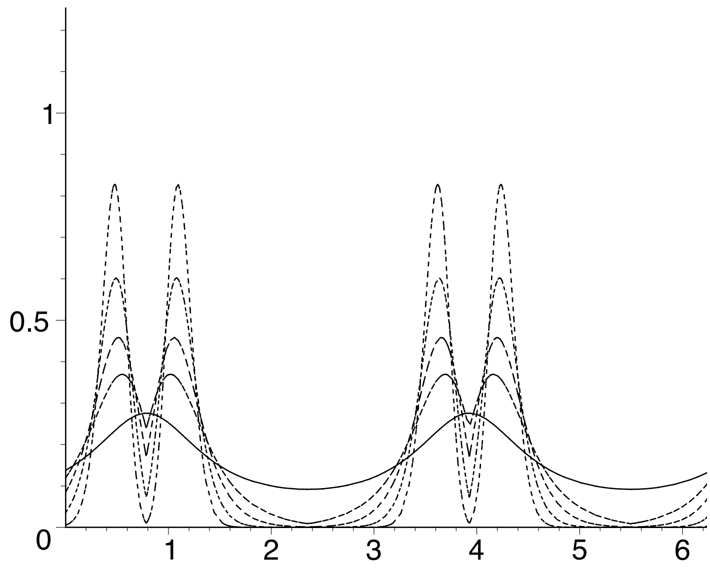

FIGURE 4. Densities of the spectral measure of $X \sim E_2(\mu, \Sigma, \phi)$ with respect to the max-norm, where $(\Sigma_{11}, \Sigma_{22}, \varrho_{12}) = (1, 1, 0.5)$, and tail index $\alpha = 0, 2, 4, 8, 16$. Larger tail indices correspond to higher peaks (see Example 5.2).