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Modelling Dependencies in Credit Risk Management

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Abstract

We commence with an overview of the three most widely used credit risk models developed by KMV, J.P. Morgan (CreditMetrics) and Credit Suisse First Boston (CreditRisk$^+$). The mathematical essentials of each model lie in the way the joint distribution of the so-called 'default indicators' is modeled, a vector of Bernoulli random variables. With the focus on these vectors we will investigate two general frameworks for modelling such binary random events. We will also show how the KMV and CreditMetrics methodology can be translated into the framework of CreditRisk$^+$.

The credit risk models are then compared for 'homogeneous' portfolios using Monte Carlo simulation. As two of the three models use the multivariate normal distribution for their 'latent variables', we investigate the impact when proceeding to the broader class of elliptical distributions. A so-called t-model, incorporating a t-copula for the latent vector, shall be used to show the consequences of a possible generalisation. In this context we introduce the notion of tail dependence. Comparison of the extended t-model with the 'normal' two credit risk models will again be performed for the same types of portfolios used for the previous comparison.

Lastly, we will study the portfolio loss distributions for the various models due to increased portfolio size.

Contents

Introduction

Consider a portfolio of N loans or bonds subject to default. Managers of such a portfolio are typically interested in the portfolio value at a certain future time T, say one year. There are essentially two possible states a firm can be in after this time period, default or non-default. Thus we can model the state of each company i at our time horizon T as a Bernoulli random variable X_i, a so-called 'default indicator', defined by

$$X_i = \begin{cases} 1 & \text{if firm } i \text{ is in default at time } T \\ 0 & \text{else} \end{cases}$$

and p_i shall denote the corresponding probability of default, i.e. $p_i := P[X_i = 1]$ for $i = 1, \dots, N$.

In the event of default, the lender receives only a percentage of the total debt. This percentage, called the recovery rate, is non-deterministic but depends on the seniority of the loan or bond. By r_i we denote the recovery rate and by L_i the loan size of company i. Then the portfolio loss L at time T is given by

$$L = \sum_{i=1}^{N} X_i(1 - r_i)L_i.$$

Hence, as soon as we have chosen a multivariate distribution for the random vector $(X_1, \dots, X_N, r_1, \dots, r_N)$, the overall portfolio loss distribution at T is fully specified. To date, the dependence among the recovery rates and the default indicators is not understood well enough to structure in a model. Due to this fact, all three benchmark credit risk models (KMV, CreditMetrics, CreditRisk$^+$) assume the recovery rates to be independent of each other and independent of the default indicators $(X_i)_i$ as well. This leaves us with the multivariate distribution of $(X_1, \dots, X_N)$. All three models attempt to determine this distribution.

From a statistical point of view we are faced with the following three problems, when modelling losses on credit portfolios:

- dependence among default events
- dimension of the portfolio
- lack of historical data

History has revealed that the assumption of independence among the $(X_i)_i$ is far too strong and would yield very poor and inaccurate results. The number of entities in a typical credit portfolio varies from a few thousand to a few million. So even for a very 'small' portfolio standard statistical techniques to fit multivariate distributions are totally inappropriate. The problem that very little historical data is available lies within the nature of credit events and must be accepted. The task of developing a model which can handle all three problems is indeed a challenge.

In the early 90's interest in credit risk management rose substantially due to the temporarily high number of defaults (above historical average) which occurred after the global economic downturn in the late 80's. Banks and investment companies took tremendous losses on their credit portfolios. This empirical knowledge produces a feature which every credit risk model should be able to incorporate: time dependent default probabilities.

So far none of the three benchmark models has fulfilled this task. For obvious reasons it is extremely difficult to develop and especially to calibrate a model, where the key drivers of default are viewed as stochastic processes. In fact, even the identification of observable key drivers of default is a difficult task. But it is crucial to recognize that such a model would automatically give reasonably 'high' probabilities to extreme events (such as many simultaneous defaults) in times of a (global) economic recession.

Although current static credit risk models could incorporate a feature called 'extremal dependence', which captures at least some of the desired properties, this is unfortunately not the case for two of the three benchmark models. We will develop a possible extension of these models and show the massive impact of this 'additional dependence' on the loss distribution of the portfolio.

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Chapter 1

The KMV model

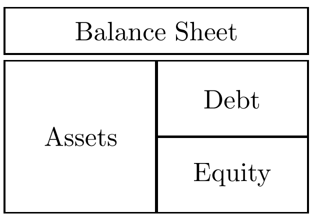
1.1 Overview

KMV uses the well known and understood framework of Merton to model the asset process of each firm. Viewing equity as a call option on the firms assets, a relationship can be established between observable market data of the firms equity, the unobservable asset value and its volatility using the option pricing formula by Black & Scholes. A so-called 'distance-to-default' for every firm is calculated and using historical data a default probability is assigned. Dependence among default events is induced through dependent Brownian Motions, the drivers of the asset process, which yields in multivariate normally distributed asset log-returns. To estimate the corresponding covariance matrix a factor model is calibrated.

1.2 Setup

For our purposes it is sufficient to consider only companies who's structures are fairly easy to deal with. Under the following assumption the essential mathematical parts of the model will still remain the same.

It is assumed that each firms balance sheet looks as follows:



As well it is assumed that all debt D_i of company i needs to be serviced at our fixed time horizon T and remains constant up till T. Additionally we assume the firms equity to be traded at an exchange.

Seeing two of the three entries in the balance sheet are time dependent, we define the following processes:

- A_t^i, the total value of assets of firm i at time t
- E_t^i, the total market value of equity of firm i at time t

KMV declares a firm to be in default at time T when the asset value is insufficient to cover the firms liabilities. This definition yields to the following relationship:

$$X_i = 1 \iff A_T^i < D_i$$

where the Bernoulli random variable X_i stands for the default indicator of firm i (as defined in the introduction).

Note that this definition of default allows for the asset value A_t^i to drop below D_i, as long as $t < T$. Thus X_i is not dependent on the path of A_t^i in the time interval $[0, T[$.

1.3 Option nature of equity

If a firm goes default the limited liability feature of equity means that the equity holders have the right but not the obligation, to pay off the debt holders and take over the remaining assets of the firm. That is, the debt holders essentially own the firm until their liabilities are paid off in full by the equity holders. Taking again our fixed time horizon T into account, equity can be viewed as a call option on the firms assets with strike price equal to the book value of the firms debts (payable at time T).

Defining $Call_t^i$ as the value of this particular call option for firm i at time t with strike D_i and maturity T, $i = 1, \ldots, N$, the following relationship holds:

$$\begin{aligned} Call_T^i &= \max(A_T^i - D_i, 0) \\ &= E_T^i. \end{aligned}$$

To get analytic results for the pricing formula of $Call_t^i$, $t \in [0, T[$, KMV uses the framework of Merton. This approach in conjunction with the option pricing formula established by Black & Scholes will allow to solve the unobservable current $t = 0$ asset value of each firm i, A_0^i, using the observable market value of the firms equity E_0^i.

1.4 Default probability of a single obligor

1.4.1 The Merton model

Merton tackled the problem of pricing and hedging a European Call option on a non-dividend paying stock if the stock value followed a geometric Brownian Motion. If the reader requires more details as a reminder of this section, see [15] and [17]. Under the 'no-arbitrage' assumption there exists a unique self-financing trading strategy which replicates the value of the Call at maturity, assuming trading is possible in a continuous manner and is only allowed in the stock (the risky asset) and in a risk-free asset, the bank account. The approach taken by KMV is to apply the Merton model to the asset value of the firm (consider the asset value of the firm as the risky asset) and to think of equity as a call option on the assets, as mentioned in the previous section. The value of the riskless asset at time t we denote by B_t.

Seeing trading is not possible in the firm's assets, KMV actually violates the setup proposed by Merton. This fact causes our 'market' to be incomplete and hence there exist infinitely many risk neutral probability measures under which the discounted asset process is a martingale (see [12]). Each measure is a candidate for pricing the option and reflects different views and investment opinions of the investors. Using the same measure as in the proper Merton setup does not yield in the 'correct' price of the option!

KMV is very well aware of this fact. As we will see later, this approach is only taken to calculate a default-index called 'distance-to-default', which (according to KMV) has predictive power on upcoming default events and allows to calculate very accurate marginal default probabilities p_i for each firm i.

Henceforth, we assume that continuous trading in the asset value of the firm is possible. Then the following two differential equations (ordinary and stochastic) describe the dynamics of the Merton model:

$$\begin{aligned} dB_t &= rB_t dt & (1.1) \\ dA_t^i &= A_t^i(\mu^{A,i}dt + \sigma^{A,i}dW_t^i) & (1.2) \end{aligned}$$

with

$$\begin{aligned} (W_t^i)_{t\in[0,T]} &= \text{a standard Brownian Motion (BM)} \\ \mu^{A,i} &= \text{const.} \\ &\cong \text{drift of the asset value of firm } i \\ \sigma^{A,i} &= \text{const.} \\ &\cong \text{volatility of the asset value of firm } i \\ r &= \text{const.} \\ &\cong \text{continuously comp. risk free interest rate} \\ B_0 &:= 1 \\ &\cong \text{initial value of bank account} \end{aligned}$$

The differential equations (1.1), (1.2) are valid for $t \in [0,T]$, i.e. from this point on until our fixed time horizon T. (1.1) is solved by integration and (1.2) by using Itô Calculus. The solutions to the equations are then given by

$$\begin{aligned} B_t &= \exp(rt) \\ A_t^i &= A_0^i \exp\left(\mu^{A,i}t - \frac{(\sigma^{A,i})^2}{2}t + \sigma^{A,i}W_t^i\right). & (1.3) \end{aligned}$$

Under the 'no-arbitrage' assumption if follows that the price of the call option $Call_t^i$ at any time $t \in [0,T[$ is given by the well known Black-Scholes formula

$$Call_t^i = A_t^i \Phi(d_1^i) - D_i \exp\big(-r(T-t)\big)\Phi(d_2^i) \qquad (1.4)$$

with

$$\begin{aligned} d_1^i &:= \frac{\log\left(\frac{A_t^i}{D_i}\right) + \left(r + \frac{(\sigma^{A,i})^2}{2}\right)(T-t)}{\sigma^{A,i}\sqrt{T-t}} \\ d_2^i &:= d_1^i - \sigma^{A,i}\sqrt{T-t} \\ \Phi(d) &:= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} \exp\left(\frac{-x^2}{2}\right)dx. \end{aligned}$$

1.4.2 Relationship between assets and equity

By joining the results of the last two sections we find that

$$E_t^i = A_t^i \Phi(d_1^i) - D_i \exp\big(-r(T-t)\big)\Phi(d_2^i) \tag{1.5}$$

and in particular the current asset value of firm i, A_0^i, is determined by

$$E_0^i = A_0^i \Phi(d_1^i) - D_i \exp(-rT)\Phi(d_2^i). \tag{1.6}$$

Applying Itô's Lemma to equation (1.5) we retrieve the following relationship between the volatilities at time $t = 0$:

$$\begin{aligned} \sigma^{E,i}(E_0^i) &= \frac{A_0^i}{E_0^i}\sigma^{A,i}\Phi(d_1^i) \\ &=: \sigma^{E,i}, \qquad i = 1, \ldots, N \end{aligned} \tag{1.7}$$

Given an estimate of $\sigma^{E,i}$ using historical data of the firms equity prices we can now solve (1.6) and (1.7) simultaneously for the unknown current asset value A_0^i of firm i and it's volatility $\sigma^{A,i}$. Note that the drift term $\mu^{A,i}$ does not appear anymore in those two equations. The reason lies within the risk neutral valuation principle used by Black & Scholes to price the Call option (see [12]).

From a probabilistic point of view the key driver of default, the asset value process A_t^i is almost fully described. The only unknown parameter in (1.3) is the drift term $\mu^{A,i}$. If there is historical asset value data available, then $\mu^{A,i}$ can be estimated empirically. Otherwise it is not clear what kind of value should be considered. Now we assume the parameter to be set. Then the default probability p_i can be calculated as follows:

$$\begin{aligned} p_i &= P[A_T^i < D_i] \\ &= P\left[A_0^i \exp\left(\mu^{A,i}T - \frac{(\sigma^{A,i})^2}{2}T + \sigma^{A,i}W_T^i\right) < D_i\right] \\ &= P\left[W_T^i < \frac{\log\left(\frac{D_i}{A_0^i}\right) + \left(\frac{(\sigma^{A,i})^2}{2} - \mu^{A,i}\right)T}{\sigma^{A,i}}\right] \\ &= P\Bigg[Z_i < \underbrace{\frac{\log\left(\frac{D_i}{A_0^i}\right) + \left(\frac{(\sigma^{A,i})^2}{2} - \mu^{A,i}\right)T}{\sigma^{A,i}\sqrt{T}}}_{=:-DD_i}\Bigg], \qquad Z_i \sim \mathcal{N}(0,1) \end{aligned} \tag{1.8}$$

We call the quantile $-DD_i$ the default point of firm i and KMV defines the 'distance-to-default' of company i as DD_i.

1.4.3 Actual default probabilities

As already pointed out before, KMV uses the Black-Scholes formula for option pricing only to calculate the 'default' index DD_i for each firm i. Seeing the Merton setup is violated (assets of companies are not tradeable), the corresponding default probabilities p_i given by (1.8) are anyhow wrong. KMV uses the $(DD_i)_i$ to estimate 'actual' default probabilities. For each company i historical data is used to search for all companies which at one stage in their history had (approximately) the same distance-to-default as firm i. Then the observed default frequency is converted into an actual probability $\hat{p}_i$. KMV names the $(\hat{p}_i)_i$ Expected Default Frequencies or EDF's. This estimation procedure is valid since all $(-DD_i)_i$ are quantiles of the same distribution (by (1.8)). To incorporate this correction of the theoretical marginal default probability of each entity in our portfolio, we simply adjust the distances-to-default such that they meet the actual EDF's, i.e. for $i = 1, \dots, N$ find $(\hat{DD}_i)$ which yields

$$\begin{aligned} P[Z_i < -\hat{DD}_i] &= \hat{p}_i \\ &= \Phi^{-1}(-\hat{DD}_i). \end{aligned}$$

1.5 The joint default distribution

Dependence between the default indicators $(X_i)_i$ is induced by making the asset value processes $\left((A_t^i)_{t\in[0,T]}\right)_i$ dependent. To do this the following fact is used: $\forall M \in \mathbb{N}$, $\left((W_t^j)_{t\in[0,T]}\right)_{j=1,\dots,M}$ standard and independent BM's and for constant weights $(c_j)_j$ the sum

$$\sum_{j=1}^{M} c_j W_t^j$$

is still a BM. So lets fix the integer M and consider the following stochastic differential equations for the asset processes:

$$dA_t^i = A_t^i \left(\mu^{A,i} dt + \sum_{j=1}^{M} \sigma_j^{A,i} dW_t^j \right), \qquad i = 1, \dots, N \tag{1.9}$$

or equivalently in vector notation ($\underline{x} := (x_1, \dots, x_N)^T$)

$$d\underline{A}_t = \underline{A}_t(\underline{\mu}^A dt + \boldsymbol{\sigma}^A d\underline{W}_t). \tag{1.10}$$

By comparing (1.10) with (1.2) we can almost guess what the solution of (1.9) must look like:

$$A_t^i = A_0^i \exp\left(\mu^{A,i} t - \frac{(\sigma^{A,i})^2}{2} t + \sum_{j=1}^{M} \sigma_j^{A,i} W_t^j \right), \qquad i = 1, \dots, N, \tag{1.11}$$

where $\sigma^{A,i}$ is defined as

$$(\sigma^{A,i})^2 := \sum_{j=1}^{M} (\sigma_j^{A,i})^2.$$

Using equation (1.11) the event of default is now described by

$$
\begin{aligned}
A_T^i < D_i \quad &\Longleftrightarrow \quad \sum_{j=1}^{M} \sigma_j^{A,i} W_T^j < \log\left(\frac{D_i}{A_0^i}\right) + \left(\frac{(\sigma^{A,i})^2}{2} - \mu^{A,i}\right) T \\
&\Longleftrightarrow \quad \sum_{j=1}^{M} \sigma_j^{A,i} \mathcal{E}_j < \frac{\log\left(\frac{D_i}{A_0^i}\right) + \left(\frac{(\sigma^{A,i})^2}{2} - \mu^{A,i}\right) T}{\sqrt{T}}, \qquad (\mathcal{E}_j)_j \stackrel{iid}{\sim} \mathcal{N}(0,1) \\
&\Longleftrightarrow \quad \frac{\sum_{j=1}^{M} \sigma_j^{A,i} \mathcal{E}_j}{\sigma^{A,i}} < \frac{\log\left(\frac{D_i}{A_0^i}\right) + \left(\frac{(\sigma^{A,i})^2}{2} - \mu^{A,i}\right) T}{\sigma^{A,i}\sqrt{T}} \\
&\Longleftrightarrow \quad \frac{\sum_{j=1}^{M} \sigma_j^{A,i} \mathcal{E}_j}{\sigma^{A,i}} < -DD_i.
\end{aligned}
$$

Defining $Z_i := \frac{\sum_{j=1}^{M} \sigma_j^{A,i} \mathcal{E}_j}{\sigma^{A,i}}$ we can conclude that $\underline{Z} \sim \mathcal{N}(\underline{0}, \Sigma)$, $Var(Z_i) = 1 \; \forall i$, and the correlation matrix Σ is given by

$$
[\Sigma]_{ij} = \frac{\sum_{s=1}^{M} \sigma_s^{A,i} \sigma_s^{A,j}}{\sigma^{A,i} \sigma^{A,j}}. \tag{1.12}
$$

The remaining task is the estimation of M and Σ. It is straightforward to calculate that

$$
\operatorname{Corr}\left(\log\left(\frac{A_t^i}{A_s^i}\right), \log\left(\frac{A_t^j}{A_s^j}\right)\right) = \frac{\sum_{k=1}^{M} \sigma_k^{A,i} \sigma_k^{A,j}}{\sigma^{A,i} \sigma^{A,j}} \stackrel{(1.12)}{=} [\Sigma]_{ij}.
$$

Hence the $(Z_i)_i$ represent standardized asset log-returns.

In practice we face the following problems when estimating the pairwise asset log-return correlations:

- lack of historical data on asset values
- computational burden: for N loans $\frac{N}{2}(N-1)$ correlations must be estimated. Impossible task if N is as large as one thousand or more
- the estimated correlation matrix will almost surely not be positive definite, but positive definiteness is needed to simulate multivariate normally distributed random variables and hence to simulate the portfolio loss distribution.

A way around these problems is to impose a certain structure on the correlation matrix Σ. The idea is to view K of the M BM's ($K \ll M$) as so-called key drivers of default (interpretable as non-observable macro-economic variables) common to all companies. Plus we assume that every company has its own idiosyncratic (company-specific) risk driver which is independent of all other risks. Translated to model of KMV this yields $M = K + N$, where $\sigma_j^{A,i} = 0 \quad \forall j > K, j \neq i$. Hence the correlation matrix is given by

$$
[\Sigma]_{ij} = \begin{cases} \dfrac{\sum_{s=1}^{K} \sigma_s^{A,i} \sigma_s^{A,j}}{\sigma^{A,i} \sigma^{A,j}} & i \neq j \\[2ex] \dfrac{\sum_{s=1}^{K} (\sigma_s^{A,i})^2 + (\sigma_{K+i}^{A,i})^2}{(\sigma^{A,i})^2} & i = j. \end{cases}
$$

An easy calculation shows that Σ is of the form $\mathbf{A}\mathbf{A}^T + \mathbf{D}$ with $[\mathbf{A}]_{ij} = a_{ij}$ a $(N \times K)$-matrix and $[\mathbf{D}]_{ij} = d_{ij}$ a diagonal $(N \times N)$-matrix given by

$$
\begin{aligned}
a_{ij} &= \frac{\sigma_j^{A,i}}{\sigma^{A,i}} \\
d_{ij} &= \begin{cases} \left(\frac{\sigma_{K+i}^{A,i}}{\sigma^{A,i}} \right)^2 & i = j \\ 0 & i \neq j. \end{cases}
\end{aligned}
$$

Using the nice properties of the Gaussian distribution we immediately see that $\underline{Z}$ can be represented as

$$
Z_i \stackrel{d}{=} \sum_{j=1}^{K} a_{ij} R_j + \mathcal{E}_i \qquad i = 1, \ldots, N \tag{1.13}
$$

where

$$
\begin{aligned}
\{(R_j)_j, (\mathcal{E}_i)_i\} &\text{ independent} \\
R_j \sim \mathcal{N}(0,1), &\qquad j = 1, \ldots, K \\
\mathcal{E}_i \sim \mathcal{N}(0, d_{ii}), &\qquad i = 1, \ldots, N.
\end{aligned} \tag{1.14}
$$

(1.13) is (for obvious reasons) called a factor model. The $(R_j)_j$ stand for the risk factors common to all firms and $\mathcal{E}_i$ is the idiosyncratic (firm-specific) risk factor which only influences company i. (1.14) can even be relaxed (without loss of generality) to multivariate normally distributed risk factors $(R_j)_j$ with standard normal marginals (key: Cholesky decomposition).

To calculate Σ we need to specify $\mathbf{A}$ and $\mathbf{D}$. This is done by identifying the common factors $(R_j)_j$. KMV proposes to take observable country, industry and global economic factors. To fit those into our factor model we assume that their log-returns are multivariate normally distributed.

1.6 The loan-loss distribution

By assuming some simplifying assumptions to the loan portfolio a closed form asymptotic portfolio loss distribution can be derived, asymptotic in the sense of letting the number of loans N tending to infinity. The core assumptions are the following:

- all loans mature at time T
- all loans have equal dollar amount
- all companies have the same marginal default probability, i.e. $p_i = p_j =: p \quad \forall i, j = 1, \ldots, N$
- the asset log-returns are equicorrelated, i.e. $[\Sigma]_{ij} = \rho \quad \forall i \neq j$

When interpreting the default indicator X_i as the gross loss before recovery on the i-th loan KMV derives analytically the asymptotic distribution of the portfolio percentage gross loss L which is defined as

$$L = \frac{1}{N} \sum_{i=1}^{N} X_i.$$

Now we focus on calculating the distribution of L, i.e. the probabilities

$$P\left[L = \frac{k}{N}\right] \quad k = 0, 1, \ldots, N.$$

Leaving the details aside KMV shows that

$$L \xrightarrow{d} \mathcal{L} \quad \text{for } N \to \infty, \quad \mathcal{L} \sim \mathcal{NI}(\rho, p)$$

where $\mathcal{NI}(a, b)$ denotes the normal-inverse distribution with parameters a, b. The normal-inverse cumulative distribution function NI is defined as

$$NI(x; a, b) = \Phi\left(\frac{1}{\sqrt{b}}\left(\sqrt{1-b}\,\Phi^{-1}(x) - \Phi^{-1}(a)\right)\right).$$

KMV points out that this limiting distribution also describes the loss distribution of a large, diversified and heterogeneous portfolio very well. At this point it is not quite clear how KMV derives the portfolio loss distribution. By the author's understanding the normal inverse distribution is directly calibrated to the heterogeneous portfolio using a special technique to estimate the parameters a, b. The correlation matrix Σ is only needed for portfolio management purposes, i.e. analysis of the portfolio structure and risk concentrations.

Chapter 2

CreditMetrics (CM)

2.1 Overview

The framework established by J.P. Morgan to evaluate bond portfolios is based on a rating system model. Changes in portfolio value are only related to the eventual migration in credit quality of each obligor, both up and downgrades, as well as default. Assuming interest rates to behave deterministically, each bond at our time horizon is re-valued, using the zero-curve corresponding to the bond's rating category. Transition probabilities are estimated using historical data and under the assumption of stationarity. Dependence between credit migration of different companies arises as with the KMV model when the firms asset values are represented by correlated geometric Brownian Motions. To estimate asset log-return correlations again a factor model is applied.

2.2 Rating systems and transition probabilities

The core ingredient to the model is the mapping of firms to a rating system. This categorisation consists of D classes $A_1, \dots, A_D$. The first $D-1$ groups describe the possible non-default states of the firms. When declared default a company falls into class A_D. Out of the $D-1$ rating categories, we take A_1 as the one characterising the highest credit quality and A_{D-1} as the lowest.

We will see that the CM framework allows without any additional difficulties for a multi-period model. We are interested in the portfolio value at the time horizons $T = \Delta t, 2\Delta t, \dots$. Credit migration, i.e. a company moving from one credit quality A_j to another A_k will only be allowed at the time horizons $(n\Delta t)_{n\geq 1}$. By X_i^n we denote the rating (state) variable of company i at time $n\Delta t$, $n \in \mathbb{N}$, taking values in $\{1, \dots, D\}$.

Example 2.1. *A rating table proposed by Standard & Poors is*

$$\begin{array}{lll} A_1 = AAA & A_4 = BBB & A_7 = CCC \\ A_2 = AA & A_5 = BB & A_8 = Default \\ A_1 = A & A_6 = B & \end{array}$$

The companies are re-rated typically every year, i.e. $T = 1$.

The following assumptions stated by J.P. Morgan are the core assumptions of the CM model:

- all bond issuers are credit-homogeneous within the same rating class, i.e. they share the same transition probabilities at all times $n\Delta t$, $n \geq 1$.

- the transition probability of every firm depends only on the rating category the company is in now.
- the transition probabilities are stationary, i.e. not time dependent.

Mathematically speaking these three assumptions yield

1. $X_i^n = X_j^n \implies P[X_i^{n+1} = A_k] = P[X_j^{n+1} = A_k] \quad \forall i, j, k$
2. $P[X_i^{n+1} = A_{k_{n+1}} | X_i^n = A_{k_n}] = P[X_i^{n+1} = A_{k_{n+1}} | X_i^n = A_{k_n}, \dots, X_i^0 = A_{k_0}]$
3. $P[X_i^{n+1} = A_{k_{n+1}} | X_i^n = A_{k_n}] = P[X_i^n = A_{k_{n+1}} | X_i^{n-1} = A_{k_n}]$.

If we define $p_{jk}^n = P[X_i^{n+1} = A_k | X_i^n = A_j]$, the above mentioned 3rd assumption gives $p_{jk}^n = p_{jk}^{n+1} =: p_{jk} \ \forall j, k = 1, \dots, D$.

We conclude that if the processes $(X_i^n)_{n \geq 0}$ are viewed independently of each other, then the 'state'-process of each firm i describes a Markov-process with state space $\{1, \dots, D\}$ and transition matrix $[p_{jk}]_{jk}$. Thus by the Lemma of Chapman-Kolmogorov (see [20], pages 72–74) the transition matrix from now to $t = m\Delta t$ equals $[p_{jk}]^m$, i.e., the m-th power of the one-decade matrix $[p_{jk}]$.

2.3 Valuation of a single bond

Since the CM framework models the portfolio value and not portfolio losses only it allows for future increases in portfolio value as well. If an obligor's credit quality improves during $[0, T]$, i.e. moves at time T up into a better rating class, then his bond yield spread over Treasury will tighten and hence his bond value will increase. The same argument holds for credit quality deterioration, but values will move in the opposite direction.

We will take the following three steps to derive a valuation technique of a single bond subject to default: specification of the credit risk horizon T, definition of the forward pricing model and derivation of the forward distribution of changes in bond value.

2.3.1 The credit risk horizon

The risk horizon can be chosen arbitrarily (typically 1 year), but must be consistent with the specified transition matrix. That is, if our transition matrix gives the rating transition distribution from today till Δt, then the risk horizon T must be an integer multiple of Δt, i.e. $\exists m \in \mathbb{N} : T = m\Delta t$.

2.3.2 The forward pricing model

Seeing interest rates are assumed constant, the only uncertainty in the future bond value comes from possible credit-migration. Thus the valuation of a bond is derived from the zero-curve corresponding to the rating of the issuer. Since there are $D - 1$ possible credit qualities apart from 'default', $D - 1$ spread curves are required to price the bond in all possible states, all obligors within the same rating class being marked-to-market with the same curve (according to assumption (1)). If our time horizon T is measured in years, then we will need the forward zero-curve T years ahead, which is then applied to the residual cash flows of the bond from T years to the maturity of the bond.

In case of default, the value of the instrument is set at a percentage, the recovery rate, of the face value of the bond. The recovery rate is assumed to follow a Beta distribution. The parameters are estimated using historical data of defaulted bonds for all seniority classes.

2.3.3 Forward distribution of changes in bond value

Given the possible future time T values of the bond for all states and under the assumption that our bond is at the moment in rating category A_j, the forward distribution of changes in bond value is given by

$$\Delta V_{jk}^T := V_k^T - V_j^T \qquad \text{with probability } p_{jk},$$

where V_k^T stands for the value of our bond at time T in rating category A_k.

2.4 Valuation of a bond portfolio

To generate dependence among credit events, J.P. Morgan takes the same approach for modelling the asset processes as KMV. Calculations in the previous chapter have suggested, that the asset log-returns of the firms follow a multivariate normal distribution. Again without loss of generality we can assume the margins to be standard normal. At this point we encounter two problems:

1. how to merge the information of the transition probabilities with the multivariate asset log-return distribution?
2. how to estimate the asset log-return correlation matrix Σ?

Question number one is easily answered. We only need to slice the x-axis under the marginal standard normal distribution for every firm i into bands, each band standing for a rating category after a possible rating migration, such that when we draw a possible asset log-return of firm i randomly, the probability of the variate to lie within a band is equal to the companies migration probability to the corresponding rating grade. That is, if the rating of company i today is A_j, i.e. $X_i^0 = j$ and we know that the probability of moving to rating grade A_k at time T is given by p_{jk}, we then choose $D-1$ thresholds $Z_l^j \in \mathbb{R}$, $Z_l^j < Z_{l+1}^j$, $l = 1, \ldots, (D-1)$ so that

$$P[X_i^T = k] = p_{jk} = \Phi(Z_{k+1}^j) - \Phi(Z_k^j).$$

Because the obligors within one rating class are taken to be ex-changeable (by assumption (1)), we don't need to choose $D-1$ thresholds for every obligor i but only for every non-default rating class A_l, $l = 1, \ldots, (D-1)$. We assume that if a company once defaulted in the past, it remains default, i.e. stays in the rating class A_D (i.e. $p_{DD} = 1$).

To estimate the asset log-return correlation matrix J.P. Morgan also uses a factor model. Calibrating is done by taking country and industry equity indices as common factors. As before it is assumed that the log-returns of these indices follow a multivariate normal distribution.

Chapter 3

CreditRisk$^+$ (CR$^+$)

3.1 Overview

In statistics the framework of CR$^+$, developed by Credit Suisse First Boston (CSFB), is known as a mixture type model. The default indicator X_i of each firm i is taken conditionally independent on its Bernoulli parameter p_i, where p_i itself is random and described by a factor model. Correlation among default events is induced by dependence of the $(p_i)_i$ on a set of common factors. Instead of simulating the portfolio loss distribution an analytic approach is taken. By discretization of the losses given default, which are assumed to be independent of the default events, the probability generating function of the portfolio losses can be approximated.

3.2 Setup

As in the previous models we describe the state of company i at our time horizon T by the Bernoulli random variable X_i, where

$$X_i = \begin{cases} 1 & \text{if firm } i \text{ is in default at time } T \\ 0 & \text{else.} \end{cases}$$

and $P[X_i = 1] = p_i$. The Bernoulli parameter p_i is taken stochastic as well and the $(X_i)_i$ conditionally independent on these parameters. That is

$$(X_i \mid p_1 \dots p_n)_i \quad \text{independent } \sim \text{Ber}(p_i).$$

It is assumed that there exist K risk factors $R_1, \dots, R_K$ which describe the variability of the default probabilities p_i. These factors are taken to be independent Gamma distributed. The link between the $(p_i)_i$ and the $(R_j)_j$ is given by the following factor model:

$$p_i = \bar{p}_i \sum_{j=1}^{K} a_{ij} R_j, \qquad i = 1, \dots, N \tag{3.1}$$

and

$$\begin{aligned} &(R_j)_j \quad \text{independent } \sim \text{Gam}(1, \sigma_j^2) \\ &\sum_{j=1}^{K} a_{ij} = 1 \qquad \forall i. \end{aligned}$$

It is clear that the factor loadings a_{ij} measure the sensitivity of obligor i to the risk R_j. Seeing $E(p_i) = \bar{p}_i$, $\bar{p}_i$ stands for an average default probability over any time period $[0, T]$ of obligor i and thus could be estimated easily if obligor i was mapped to any credit rating system. Of course this last statement is only valid under the assumption of stationary default probabilities with respect to time (within each rating class). But this is implicitly assumed in the way the factor model is set up.

In CR$^+$ the loss given defaults $(LGD_i)_i$ are modeled as a constant fraction of loan size and the loss exposure amounts are expressed as an integer multiple of a fixed base unit of loss (e.g. one million dollars). We define $\forall i = 1, \ldots, N$

$$\begin{aligned} L_i &:= \text{loan size of obligor } i \\ \lambda_i &:= \text{expected percentage loss given default, } (\in]0,1[) \\ &= 1 - \text{expected recovery rate} \\ v_0 &:= \text{base unit of loss.} \end{aligned}$$

Then we have that

$$\begin{aligned} LGD_i &= \lambda_i L_i \\ &\approx v_i v_0 \qquad v_i \in \mathbb{N}, \end{aligned}$$

if v_i is given by

$$v_i := \text{round}\left(\frac{\lambda_i L_i}{v_0}\right).$$

v_i stands for the nearest integer value to $\frac{\lambda_i L_i}{v_0}$. Now every LGD can be expressed (approximately) as a fixed multiple of a predefined base unit of loss.

Because of the fact that for any discrete random variable it's distribution function and its probability generating function (pgf) contain the same amount of information, it makes sense to calculate whichever is easier to handle. In this case CSFB chose the second possibility which will allow for an approximate analytical result for the portfolio loss distribution.

In the next section a brief introduction to pgf's is given with their relevant properties, which shall be used when deriving the portfolio loss pgf.

3.3 Introduction to probability generating functions

For our further investigation on CR$^+$ we will only need to deal with non-negative integer valued random variables, for which we now will give some properties of their pgf's.

In this section the random variables X, Y are assumed to take non-negative, integer values only. On their distribution itself we do not impose any conditions.

Definition 3.1. *The pgf of X is defined as $G_X(s) = \mathbb{E}(s^X) = \sum_i s^i P[X = i]$.*

From the definition it immediately follows that

$$P[X = k] = \frac{1}{k!} G_X^{(k)}(0), \qquad k \in \mathbb{N}_0. \tag{3.2}$$

We see that given the pgf of a random variable allows easy derivation of it's distribution. Here are two examples of pgf's, which we will meet again later on in this chapter:

Example 3.1.

1. *Bernoulli random variable. If* $P[X = 1] = p$ *and* $P[X = 0] = 1 - p$ *then*
$$G_X(s) = 1 + p(s-1). \tag{3.3}$$

2. *Poisson random variable. If* X *has a Poisson distribution with parameter* λ *then*
$$G_X(s) = \sum_{k=0}^{\infty} s^k \frac{\lambda^k}{k!} \exp(-\lambda) = \exp\big(\lambda(s-1)\big). \tag{3.4}$$

The following two properties will appear to be very useful:

Proposition 3.1.

1. X, Y *two independent random variables. Then*
$$G_{X+Y}(s) = G_X(s)G_Y(s)$$

2. *Let* $G_{X|Y}(s)$ *be the pgf of* $X|Y$ *and* $Y \sim F$*. Then*
$$G_X(s) = \int G_{X|Y=y}(s)F(dy).$$

Proof.

1. X, Y independent $\Longrightarrow s^X, s^Y$ independent $\Longrightarrow \mathbb{E}\left[s^{X+Y}\right] = \mathbb{E}\left[s^X\right]\mathbb{E}\left[s^Y\right]$
2. $G_X(s) = \mathbb{E}[s^X] = \int \mathbb{E}[s^X|Y = y]F(dy) = \int G_{X|Y=y}(s)F(dy)$

□

3.4 Calculation of the portfolio loss pgf

First we derive the conditional pgf of $Z := X_1 + \ldots + X_N$ given $\underline{R} = (R_1, \ldots, R_K)$, where Z stands for the total number of defaulted portfolio entities at time T and the $(X_i)_i$ for the default indicators. We know that given $\underline{R}$, the $(X_i)_i$ are independent $\sim$ Ber(p_i). Hence by (3.3) their pgf is

$$G_{X_i|\underline{R}}(s) = 1 + p_i(s-1), \qquad i = 1, \ldots, N.$$

At this point CSFB approximates each pgf $G_{X_i|\underline{R}}(s)$ using a Taylor series expansion of the function $log(1 + x) = x + \mathrm{O}(x^2)$ at $x_0 = 0$. Thus

$$\begin{aligned} G_{X_i|\underline{R}}(s) &= 1 + p_i(s-1) \\ &= \exp\Big(\log\big(1 + p_i(s-1)\big)\Big) \\ &\approx \exp\big(p_i(s-1)\big) \qquad \text{for } p_i \approx 0, s \approx 1. \end{aligned} \tag{3.5}$$

Now if we compare (3.5) with (3.4) we conclude that this approximation is equal to saying the $(X_i)_i$ were $Pois(p_i)$ distributed. That's why this approximation is called the Poisson approximation. For further calculations the CR$^+$ model regards the $(X_i)_i$ as Poisson distributed random variables. The idea is that as long as p_i is small, we can ignore the constraint that a single obligor i can default only once (seeing the probability of defaulting 'twice' or even more within the time interval $[0, T]$ is very small). Note that the exponential form in (3.4) of the Poisson pgf will be essential to the computational facility of the model!

Conditional on $\underline{R}$, the $(X_i)_i$ are independent. In conjunction with proposition 3.1 it follows that

$$\begin{aligned} G_{Z|\underline{R}}(s) &= G_{X_1|\underline{R}} \cdots G_{X_N|\underline{R}} \\ &= \prod_{i=1}^{N} \exp\bigl(p_i(s-1)\bigr) \\ &= \exp\bigl(\mu(s-1)\bigr), \qquad \mu := \sum_{i=1}^{N} p_i. \end{aligned}$$

Using (3.1) and the fact that the $(R_j)_j$ are independent Gamma distributed we calculate the unconditional pgf of Z. $f_j(x)$ will stand for the probability density function of the Gamma random variable R_j.

$$\begin{aligned} G_Z(s) &= \int_{\mathbb{R}^+} \cdots \int_{\mathbb{R}^+} G_{Z|\underline{R}=(x_1,\dots,x_K)}(s) f_1(x_1) \cdots f_K(x_K) dx_1 \cdots dx_K \\ &= \int_{\mathbb{R}^+} \cdots \int_{\mathbb{R}^+} \exp\left(\sum_{i=1}^{N} \left(\bar{p}_i \sum_{j=1}^{K} a_{ij} x_j \right) (s-1) \right) f_1(x_1) \cdots f_K(x_K) dx_1 \cdots dx_K \\ &= \int_{\mathbb{R}^+} \cdots \int_{\mathbb{R}^+} \exp\left((s-1) \sum_{j=1}^{K} \left(\sum_{i=1}^{N} \bar{p}_i a_{ij} \right) x_j \right) f_1(x_1) \cdots f_K(x_K) dx_1 \cdots dx_K \\ &= \int_{\mathbb{R}^+} \cdots \int_{\mathbb{R}^+} \exp\bigl((s-1)\mu_1 x_1\bigr) f_1(x_1) dx_1 \cdots \exp\bigl((s-1)\mu_K x_K\bigr) f_K(x_K) dx_K \\ &= \prod_{j=1}^{K} \left(\frac{1-\delta_j}{1-\delta_j s} \right)^{\frac{1}{\sigma_j^2}}, \qquad \delta_j := \frac{\sigma_j^2 \mu_j}{1+\sigma_j^2 \mu_j} \quad \text{and} \quad \mu_j := \sum_{i=1}^{N} \bar{p}_i a_{ij} \end{aligned}$$

Now let's take a look at the loss contribution of obligor i, $Loss_i$, to the overall portfolio loss measured in base units v_0. Given $\underline{R}$ he adds

$$Loss_i|\underline{R} = v_i X_i, \qquad v_i := round\left(\frac{\lambda_i L_i}{v_0}\right).$$

Because the $(X_i|\underline{R})_i$ are independent it follows immediately that the $(Loss_i|\underline{R})_i$ are independent as well. So given the pgf of $Loss_i|\underline{R}$, which is

$$G_{Loss_i|\underline{R}}(s) = G_{X_i|\underline{R}}(s^{v_i}), \tag{3.6}$$

we get, conditional on $\underline{R}$, for the overall portfolio loss pgf $G_{Loss|\underline{R}}(s)$ (again using proposition 3.1)

$$\begin{aligned} G_{Loss|\underline{R}}(s) &= \prod_{i=1}^{N} G_{Loss_i|\underline{R}}(s) \\ &\overset{(3.6)}{=} \prod_{i=1}^{N} G_{X_i|\underline{R}}(s^{v_i}) \\ &= \exp\left(\sum_{j=1}^{K} R_j \left(\sum_{i=1}^{N} \bar{p}_i a_{ij} (s^{v_i} - 1)\right)\right). \end{aligned}$$

As before, the unconditional total portfolio loss pgf is attained by integrating out the risk factors R_j. This yields

$$G_{Loss}(s) = \prod_{j=1}^{K} \left(\frac{1-\delta_j}{1-\delta_j P_j(s)}\right)^{\frac{1}{\sigma_j^2}}, \qquad P_j(s) := \frac{1}{\mu_j} \sum_{i=1}^{N} \bar{p}_i a_{ij} s^{v_i}.$$

To calculate the probability of incurring a portfolio loss of l standard units v_0 we simply use (3.2). The CR$^+$ manual [4] provides an easy to calculate recurrence relationship for $l = 0, 1, \ldots$.

3.5 Mixed Poisson distributions

At a first glance at the CR$^+$ model one might ask why the default volatility drivers $(R_j)_j$ are taken to be Gamma distributed? This assumption was made because the model was developed using a Poisson approximation technique common in insurance mathematics.

Once again we take a closer look at the overall number of defaults described by the random variable $Z = \sum_{i=1}^{N} X_i$. To evaluate $P[Z = m]$ easily, we apply again the previously mentioned Poisson approximation, i.e. $(X_i|\underline{R})_i$ independent $\sim Pois(p_i(\underline{R}))$. Thus

$$P[Z = m|\underline{R}] = \frac{\lambda^m(\underline{R})}{m!} \exp\big(\lambda(\underline{R})\big)$$

with

$$\lambda(\underline{R}) := \sum_{i=1}^{N} p_i(\underline{R}) = \sum_{i=1}^{N} \bar{p}_i \sum_{j=1}^{K} a_{ij} R_j. \tag{3.7}$$

By rearranging the order of summation in (3.7) we can also write

$$\lambda(\underline{R}) = \sum_{j=1}^{K} \mu_j R_j, \qquad \mu_j := \sum_{i=1}^{N} \bar{p}_i a_{ij}.$$

Now consider K random variables Z_j

$$(Z_j|\underline{R})_j \text{ independent}, \qquad Z_j|\underline{R} \sim Pois(\mu_j R_j).$$

Obviously, Z and $\sum_{j=1}^{K} Z_j$ have the same conditional and hence the same unconditional distribution, i.e.

$$Z \stackrel{d}{=} \sum_{j=1}^{K} Z_j.$$

We notice that the $(Z_j)_j$ are even unconditionally independent, as each Z_j depends only on R_j and these are independent. Moreover, each Z_j, $j = 1, \ldots, K$ has by construction a mixed Poisson distribution with parameter $\mu_j R_j$ such that the total number of defaults in the portfolio is equal in distribution to a sum of K independent mixed Poisson distributed random variables.

At this stage CSFB models the $(R_j)_j$ as Gamma random variables, to ensure that an analytic derivation of the distribution of Z_j is possible. Then Z_j has a Negative Binomial distribution (for all $j = 1, \ldots, K$) when integrating out the conditional Gamma distribution. This result is well known in the field of actuarial mathematics. Thus Z can be represented as the sum of K independent Negative Binomial distributed random variables and hence its pgf can be calculated easily, as already seen.

3.6 Sector analysis & factor models

If the portfolio is broken down into different groups of obligors, say by industry sectors and by rating categories, then the factors $R_j, j = 1, \ldots, K$, can be 'made' portfolio specific, i.e. one can estimate the variances σ_j^2.

Assuming homogeneity in default behaviour for all firms in the same rating class, we can estimate the average long term default frequency and also the average default volatilities $(\sigma_j)_j$ using historical data. We also need to fix the weightings w_{ij}, which represent our judgement of the extent to which the state of sector j (country or industry specific) influences the fortunes of obligor i. Then the factor model (3.1) imposes the following relationship between the variances ($\sigma^2(X)$ stands for the variance of X):

$$\sigma^2(p_i) = \bar{p}_i^2 \sum_{j=1}^{K} a_{ij}^2 \sigma^2(R_j) \qquad i = 1, \ldots, N. \tag{3.8}$$

If we define $U_i := \frac{\sigma^2(p_i)}{\bar{p}_i^2}$, $V_j := \sigma^2(R_j)$ the equivalent vector equation to (3.8) is $\underline{U} = \mathbf{A}\underline{V}$, $[\mathbf{A}]_{ij} = a_{ij}^2$, and hence an ordinary least square solution is given by

$$(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\underline{U} = \underline{V} \qquad \text{if } Rank(\mathbf{A}) = K,\ K < N.$$

Chapter 4

Summary of the models

From a mathematical point of view, the KMV and the CM model are the same if we restrict CM to model only the portfolio losses and if the one-period version is considered only. Both approaches model the asset log-returns using a multivariate normal distribution, declaring a firm to be in default at time T if the asset value (at time T) has fallen beneath a certain threshold or equivalently when the standardised asset log-returns have taken values below the default point $-DD$. The only difference between the two models is how the distances-to-default $(DD_i)_i$ are calculated. While KMV establishes a relationship to option pricing J.P. Morgan uses their rating system. Even estimation of the asset log-return correlation matrix is performed in the same manner, namely by using a factor model.

Because of this mathematical equivalence of the two frameworks we will not distinguish between the two models anymore for the rest of this study. From now on we will focus on the comparison of the KMV/CM and the CR^+ model.

Throughout the paper, whenever we investigate one of the 'two' methodologies, the following notation will be used:

- KMV/CM

$$\begin{aligned}
\underline{Z} &= \text{standardised asset log-returns} \\
&\sim \mathcal{N}(\underline{0}, \Sigma), \qquad \Sigma = \text{correlation matrix} \\
Z_i &= \sum_{j=1}^{K} a_{ij} R_j + \mathcal{E}_i \qquad \text{(factor model)} \\
X_i &= \text{default indicator of firm } i \\
&= I_{\{Z_i < DD_i\}} = I_{\{Z_i < \Phi^{-1}(p_i)\}} \\
p_i &= \text{default probability of firm } i \text{ over time period } [0, T]
\end{aligned}$$

- CR^+

$$\begin{aligned}
X_i &= \text{default indicator of firm } i, \qquad (X_i|p_i)_i \text{ indep. } \sim Ber(p_i) \\
p_i &= \bar{p}_i \sum_{j=1}^{K} a_{ij} R_j \qquad \text{(factor model)}, \sum_{j=1}^{K} a_{ij} = 1 \quad \forall i \\
\bar{p}_i &= \text{default probability of firm } i \text{ over time period } [0, T]
\end{aligned}$$

Both, the KMV/CM and the CR^+ methodologies have one thing more in common. They all model the loss given default's $(LGD_i)_i$ independently of the default events. So for any further investigations on the structure of the models it actually suffices to analyze the joint distribution of the default indicators $(X_i)_i$.

We have chosen the same notation for some of the random variables and parameters for both models (such as the default indicators $(X_i)_i$) to emphasize the structures of the models. Note, that from a mathematical point of view they do not have anything in common. Whenever it is not quite clear which random variables or parameters of what model we are talking about, we write for example X_i^{KMV} or X_i^{CR} instead of X_i. Some of the letters used for these random variables shall be used later on in other contexts and should not be interpreted as in the summary of the models, unless explicitly mentioned.

Chapter 5

Modelling dependencies

In this chapter we will discuss all relevant topics on dependence among random variables needed for further investigation on the models. Seeing factor models have come up in all three credit risk models, we will first take a short look at those. In the following section copulas will be introduced, which allow breaking down a multivariate distribution into its margins and dependence structure. Then we shall discuss two very general ways of modeling multivariate binary (Bernoulli) random outcomes, seeing KMV,CM and CR$^+$ only focus on modelling the multivariate Bernoulli default events $(X_i)_i$ and the loss given defaults are independent of the default events.

5.1 Factor models

Factor analysis is a mathematical model which attempts to explain the correlation between a large set of variables in terms of a small number of underlying factors. A major assumption of factor analysis is that it is not possible to observe these factors directly; the variables depend upon the factors but are also subject to random errors.

5.1.1 Setup

Let $\underline{Z} = (Z_1, \ldots, Z_n)^T$ be a random vector with mean $\underline{\mu}$ and covariance matrix Σ. Then we say that the k-factor model holds for $\underline{Z}$ $(k \ll n)$ if $\underline{Z}$ can be written in the form

$$\underline{Z} = \mathbf{A}\underline{R} + \underline{\mathcal{E}} + \underline{\mu}, \tag{5.1}$$

where $\mathbf{A}$ is a non-random $(n \times k)$-matrix, $\underline{R} = (R_1, \ldots, R_k)^T$ and $\underline{\mathcal{E}} = (\mathcal{E}_1, \ldots, \mathcal{E}_n)^T$ are random vectors. The $(R_j)_j$ are called common factors and the $(\mathcal{E}_i)_i$ specific or unique factors. It is additionally assumed that

$$\begin{aligned} &\mathbb{E}[R_j] = 0 \quad \forall j, && \mathrm{Cov}[R_i, R_j] = \delta_{ij} \\ &\mathbb{E}[\mathcal{E}_i] = 0 \quad \forall i, && \mathrm{Cov}[\mathcal{E}_i, \mathcal{E}_j] = \psi_i^2 \delta_{ij} \\ &\mathrm{Cov}[R_j, \mathcal{E}_i] = 0 \quad \forall i, j \end{aligned}$$

and denote the covariance matrix of $\underline{\mathcal{E}}$ by $\Psi = \mathrm{diag}(\psi_1^2, \ldots, \psi_n^2)$. δ_{ij} stands for the Kronecker function. By definition all of the factors are uncorrelated and the common factors are each standardised to have variance 1.

The validity of the k-factor model can be expressed in terms of a simple condition on Σ without imposing any restricions on the distribution of $\underline{Z}$.

Theorem 5.1. *A k-factor model holds for $\underline{Z} = (Z_1, \dots, Z_n)$, $k < n$ if and only if*

$$\exists \mathbf{A}, \Psi : \quad \Sigma = \mathbf{A}\mathbf{A}^T + \Psi \tag{5.2}$$

where $\mathbf{A} = (n \times k) - matrix$, $Rank(\mathbf{A}) = k$ and Ψ is of the form $diag(\psi_1^2, \dots, \psi_n^2)$.

Proof. see [19] □

5.1.2 Factor models applied to credit risk

The first step when deriving a factor model is to estimate the covariance matrix Σ out of a set of m independent observations $\hat{Z}_1, \dots \hat{Z}_m$ of the vector $\underline{Z}$. Then we would try to estimate $\mathbf{A}, \Psi$ and the distributions of the factors (mostly chosen to be multivariate normal). The models proposed by KMV and J.P. Morgan follow a different approach: the factor models are used to produce the covariance (in our case correlation) matrix. The matrix $\mathbf{A}$ is statistically estimated by identifying observable risk factors $(R_j)_j$ and the unexplainable random noise is incorporated in Ψ. Even the major assumption in classical factor analysis of unobservable common factors is violated.

As we see, the estimation procedure is not a standard technique in classical factor analyis.

As in most other cases the multivariate normal distribution plays again an exceptional role also with factor models. It is straight forward to check that if $\underline{Z} \sim \mathcal{N}(\underline{\mu}, \Sigma)$ and Σ is of the form $\mathbf{A}\mathbf{A}^T + \Psi$, then the easiest possible factor model is retrieved by setting the factors $R_1, \dots R_k$ iid $\sim \mathcal{N}(0,1)$, $(\mathcal{E}_i)_i$ independent $\sim \mathcal{N}(0, \psi_i^2)$, $\{R_j, \mathcal{E}_i\}_{i,j}$ independent.

So all factors can be chosen to be normally distributed and most of all independent among each other. This is due to the fact that multivariate normally distributed random variables with zero correlation are independent. We will use this property later on explicitly.

5.2 Copulas

Consider n continuous real-valued random variables $Z_1, \dots, Z_n$ with marginal distribution functions $F_1, \dots, F_n$. Their dependence is completely described by their joint distribution function

$$F(z_1, \dots, z_n) = P[Z_1 \leq z_1, \dots, Z_n \leq z_n].$$

The idea of separating F into a part which describes the dependence structure and parts which contain all information on the marginal behaviour, has led to the concept of a copula.

Definition 5.1. *An n-copula is the distribution function of a random vector in $\mathbb{R}^n$ with uniform$-(0,1)$ marginals or equivalently an n-copula is any function $C : [0,1]^n \to [0,1]$ which has the following three properties:*

1. *$C(x_1, \dots, x_n)$ is increasing in each component x_i*
2. *$C(1, \dots, 1, x_i, 1, \dots, 1) = x_i, \quad \forall i \in [1, \dots n]$, $x_i \in [0,1]$*
3. *$\forall$ $(a_1, \dots, a_n)$, $(b_1, \dots, b_n) \in [0,1]^n$ with $a_i \leq b_i$ we have*

$$\sum_{i_1=1}^{2} \dots \sum_{i_n=1}^{2} (-1)^{i_1+\dots+i_n} C(x_{1i_1}, \dots, x_{ni_n}) \geq 0 \tag{5.3}$$

where $x_{j1} = a_j$, $x_{j2} = b_j$ $\forall j \in \{1, \dots n\}$

The sum (5.3) can be interpreted as $P[a_1 \leq X_1 \leq b_1, \ldots, a_n \leq X_1 \leq b_n]$. The following proposition will give the link between copulas and joint distribution functions:

Proposition 5.1. *Let X be a random variable with distribution function G. Let G^{-1} be the quantile transform of G, i.e. $G^{-1}(\alpha) = \inf\{x|G(x) \geq \alpha\}$, $\alpha \in]0,1[$. Then*

1. *If $U \sim Unif(0,1)$ we have $G^{-1}(U) \sim G$*

2. *If G is continuous, then $G(X) \sim$ Unif$(0,1)$*

Proof. See [21], page 59. □

We conclude that $F_i(Z_i) \sim Unif(0,1)$. Seeing the $(F_i)_i$ are strictly monotone increasing on $[0,1]$ we can rewrite (5.2) as

$$\begin{aligned} F(z_1, \ldots, z_n) &= P[F_1(Z_1) \leq F_1(z_1), \ldots, F_n(Z_n) \leq F_n(z_n)] \\ &=: C(F_1(z_1), \ldots, F_n(z_n)). \end{aligned} \tag{5.4}$$

The following Theorem by Sklar states that C is indeed a distribution function on $[0,1]^n$ and even states a uniqueness property.

Theorem 5.2. *Let H be an n-dimensional distribution function with marginals $H_1, \ldots H_n$. Then there exists an n-copula C such that $\forall \underline{x} \in \mathbb{R}^n$*

$$H(x_1, \ldots, x_n) = C(H_1(x_1), \ldots, H_n(x_n)).$$

If all $(H_i)_i$ are continuous, then C is unique.

Conversely, if C is an n-copula and $H_1, \ldots H_n$ are distribution functions, then the function H defined above is an n-dimensional distribution function with marginals $H_1, \ldots H_n$.

Proof. see [22] □

Thus the representation (5.4) defines a unique copula C. This theorem shows clearly, why the copula associated with $\underline{Z}$ is interpreted as the dependence structure among the $(Z_i)_i$.
Below are examples of three different copulas:

- copula of n independent random variables

$$C(u_1, \ldots, u_n) = u_1 \cdot \ldots \cdot u_n$$

- standard bivariate Gaussian or normal copula

$$C_\rho^{Ga}(u_1, u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp\left(-\frac{(s^2 - 2\rho st + t^2)}{2(1-\rho^2)}\right) dsdt$$

where

$$\begin{aligned} \rho &= \text{linear correlation parameter}, \quad -1 < \rho < 1 \\ \Phi &= \text{univariate standard normal distribution function} \end{aligned}$$

- standard bivariate t-copula with ν degrees of freedom

$$C^t_{\nu,\rho}(u_1,u_2) \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{1}{2\pi\sqrt{(1-\rho^2)}} \left(1+\frac{s^2-2\rho st+t^2}{\nu(1-\rho^2)}\right)^{-\frac{\nu+2}{2}} dsdt$$

where

$$\begin{aligned} \rho &= \text{linear correlation parameter}, \quad -1<\rho<1 \\ \Phi &= \text{univariate standard t-distribution function} \end{aligned}$$

We illustrate these examples with the following three pictures.

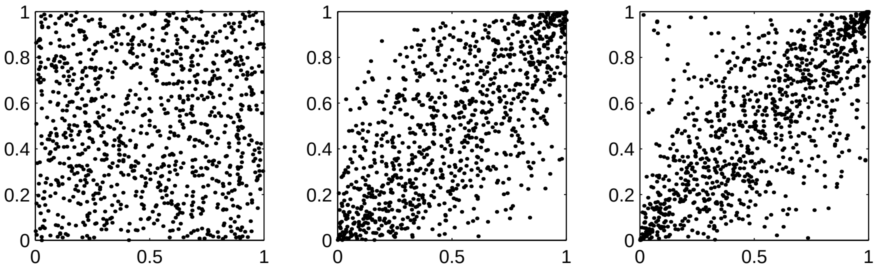


Figure 5.1: *Generation of 1000 random variates from the following bivariate copulas: 1) copula of independent random variables, 2) normal copula with correlation $\rho = 0.7$, 3) t-copula with correlation $\rho = 0.7$ and $\nu = 3$.*

An attractive feature of the copula representation of dependence is that the copula itself is invariant under increasing and continuous transformations of the marginals.

Proposition 5.2. *If $(Z_1,\dots,Z_n)^T$ has copula C and $T_1,\dots T_n$ are increasing and continuous functions, then $(T_1(Z_1),\dots,T_n(Z_n))^T$ also has copula C.*

Proof. see [7], page 6 □

Note, that if H is an n-dimensional distribution function with marginals $H_1,\dots H_n$ and copula C we will also write equivalently $C(H_1,\dots,H_n)$ for H.

5.3 Multivariate binary distributions

In this section we will focus on two ways of how to generate multivariate binary distributions: a latent variable model and a mixture type model. We will show that the two previously discussed KMV/CM and CR$^+$ models each fit into one of the two frameworks.

5.3.1 Latent variable models

A general approach to induce dependence among binary random variables is to discretise a continuous n-variate random vector using a set of 'cutoff'-points. The so-called latent variable model is given by

$$\begin{aligned} \underline{Z} &= (Z_1,\dots,Z_n)^T, \qquad \underline{Z} \sim C(F,\dots,F) \\ X_i &= I_{\{Z_i<F^{-1}(p_i)\}}, \qquad i=1,\dots n \end{aligned}$$

with

$$\begin{aligned} F &= \text{a continuous distribution function} \\ F^{-1}(p_i) &= \text{cutoff-points.} \end{aligned}$$

The discrete distribution of $\underline{X}$ can be calculated as follows. Let $x_i \in \{0,1\}\ \forall i$, then

$$\begin{aligned} P[X_1 = x_1, \ldots, X_n = x_n] &= P[a_1 \le Z_1 \le b_1, \ldots, a_n \le Z_n \le b_n] \\ &\stackrel{(5.3)}{=} \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\ldots+i_n} C(y_{1i_1}, \ldots, y_{ni_n}) \end{aligned}$$

with

$$\begin{aligned} a_i &= \begin{cases} -\infty & \text{if } x_i = 1 \\ F^{-1}(p_i) & \text{else} \end{cases} \\ b_i &= \begin{cases} F^{-1}(p_i) & \text{if } x_i = 1 \\ \infty & \text{else} \end{cases} \end{aligned}$$

and $y_{i1} = a_i$, $y_{i2} = b_i\ \forall i \in \{1, \ldots n\}$. For the marginal probabilities we get trivially that $P[X_i = 1] = C(1, \ldots, 1, p_i, 1, \ldots, 1) = p_i$.

We see that the latent variable approach allows for a very broad dependence structure among the binary random variables $(X_i)_i$. An example of such a model is KMV/CM. In both models the latent variable vector $\underline{Z}$ stands for the asset log-returns (refer to page 19), i.e.

$$\begin{aligned} F &= \text{cumulative standard normal distribution function} \\ C &= \text{normal copula with correlation matrix } \Sigma \\ p_i &= \text{marginal default probability of firm } i. \end{aligned}$$

5.3.2 Mixture type models

In the following mixture type model dependence between a set of Bernoulli random variables $X_i \sim Ber(p_i)$ comes from a conditional independence in which the Bernoulli parameters $(p_i)_i$ are random and dependent on each other. That is

$$\begin{aligned} &(X_i|p_i)_i \text{ independent } \sim Ber(p_i), \qquad i = 1, \ldots n \\ &\underline{p} \sim G, \qquad G \text{ a distribution function on } [0,1]^n. \end{aligned}$$

We then get for the conditional probability distribution of $\underline{X}$ that

$$P[X_1 = x_1, \ldots, X_n = x_n|\underline{p}] = \prod_{i=1}^{n} p_i^{x_i}(1-p_i)^{1-x_i}$$

and hence the unconditional

$$P[X_1 = x_1, \ldots, X_n = x_n] = \int_{[0,1]^n} \prod_{j=1}^{n} p_i^{x_i}(1-p_i)^{1-x_i} G(d\underline{p}).$$

We see that CR$^+$ fits almost perfectly into the framework of mixture type models. The only difference to the setup is that the support of $(p_i)_i$ is defined on $[0,\infty[$ instead on $[0,1]$. Due to this fact we will encounter some difficulties in our further investigations on the model.

5.3.3 Mapping KMV/CM to the framework of CR$^+$

Apart from the fact that the latent variable and the mixture type models are two very general methodologies to impose dependence among binary random variables, the two setups do not seem to have anything in common. In general, an analytic relationship between the two models can not be established. But because KMV/CM generate the latent variables (i.e. the asset log-returns) by a normal factor model, it is possible in this special case to map the latent variable type KMV/CM model to the mixture model of CR$^+$ (as already shown in [10], page 125-126). To do this we start again with the factor model for the asset log-returns (refer to page 19):

$$\begin{aligned} Z_i &= \sum_{j=1}^{K} a_{ij} R_j + \mathcal{E}_i, \qquad i = 1, \dots N \\ X_i &= I_{\{Z_i < \Phi^{-1}(p_i)\}}. \end{aligned}$$

By $\mathcal{F}$ we define the sigma-field generated by $R_1, \dots R_K$, i.e. $\mathcal{F} = \sigma(R_1, \dots R_K)$, and by σ_i^2 the variance of the idiosyncratic risk $\mathcal{E}_i$, $i = 1, \dots N$. Then we calculate the following conditional probability:

$$\begin{aligned} P[X_i = 1|\mathcal{F}] &= P\left[\sum_{j=1}^{K} a_{ij} R_j + \mathcal{E}_i < \Phi^{-1}(p_i)|\mathcal{F}\right] \\ &= P\left[\mathcal{E}_i < \Phi^{-1}(p_i) - \sum_{j=1}^{K} a_{ij} R_j|\mathcal{F}\right] \\ &= P\left[\tilde{\mathcal{E}}_i < \frac{\Phi^{-1}(p_i) - \sum_{j=1}^{K} a_{ij} R_j}{\sigma_i}|\mathcal{F}\right], \qquad (\tilde{\mathcal{E}}_i)_i \overset{iid}{\sim} \mathcal{N}(0,1) \\ &= \Phi\left(\frac{\Phi^{-1}(p_i) - \sum_{j=1}^{K} a_{ij} R_j}{\sigma_i}\right) \\ &=: \tilde{p}_i. \end{aligned}$$

Noting that $(X_i|\tilde{p}_i) = (X_i|\mathcal{F})$ we retrieve the following mixture model:

$$\begin{aligned} &(X_i|\tilde{p}_i)_i \quad \text{independent} \quad \sim Ber(\tilde{p}_i) \\ &\Phi^{-1}(\tilde{p}_i) = \frac{1}{\sigma_i}\left(\Phi^{-1}(p_i) - \sum_{j=1}^{K} a_{ij} R_j\right). \end{aligned}$$

This is the translated version of the KMV/CM model to the CR$^+$ framework. We still even have a factor model, not for the Bernoulli parameter $\tilde{p}_i$ itself, but for $\Phi^{-1}(\tilde{p}_i)$.

It is crucial to see that the only reason why we could establish this mapping is because of the normality of the factor model. If the $(\mathcal{E}_i)_i$ were uncorrelated only, as in general factor model theory, the $(X_i|\tilde{p}_i)_i$ would not be independent.

Chapter 6

Comparison of the models

Even though we can now write the KMV/CM and the CR$^+$ model both in in terms of a mixture type model, there is no obvious answer to the question on how to compare the two models, when applied to a heterogeneous portfolio. Because either models strongly depend on the weighting's $(a_{ij}^{KMV}), (a_{ij}^{CR})$ (refer to page 19), which determine the dependence structure, and no reasonable link in between can be established, simplifying assumptions to the portfolio are needed.

We will consider so-called 'homogeneous' portfolios of loans and bonds. Portfolio homogeneity we define to be the following: $\forall k \in \{1, \dots n\}$ and for all permutations s of $\{1, \dots k\}$

$$(X_1, \dots, X_k) \stackrel{d}{=} (X_{s(1)}, \dots, X_{s(k)}). \tag{6.1}$$

When considering homogeneous portfolios we will use from now on the following notation:

$$\begin{aligned} \pi &:= P[X_i = 1] = p_i \qquad \forall i = 1, \dots n \\ \rho^X &:= \operatorname{Corr}[X_i, X_j] \quad \in]0,1[\ \ \forall i \neq j \end{aligned}$$

Our aim is to compare KMV/CM with CR$^+$ by evaluating the models for various homogeneous portfolios. Each portfolio shall be characterised by a pair (π, ρ_X).

We start off by giving some properties of the latent variable and the mixture models in the homogeneous case and show that the homogeneity structure on the portfolio is not too strong to give a reasonable comparison of the models.

6.1 Homogeneous latent variable models

Using the same notation as in the introduction to general latent variable models (refer to page 24) we get for the homogeneous case

$$\begin{aligned} \underline{Z} &\sim C(F, \dots, F), && C \text{ an ex-changeable copula} \\ X_i &= I_{\{Z_i < F^{-1}(\pi)\}}, && \pi \in]0,1[, \ i = 1, \dots n. \end{aligned}$$

By an ex-changeable copula we mean the property $C(x_1, \dots, x_n) = C(x_{s(1)}, \dots, x_{s(n)})$, where s stands for any permutation of $\{1, \dots n\}$, $x_i \in [0,1]$. This additional condition on the copula is needed to guarantee (6.1).

The higher order joint default probabilities and the default indicator correlations are then given by

$$\begin{aligned}
\pi_k &:= P[X_1 = 1, \dots, X_k = 1] \\
&= P[X_{s(1)} = 1, \dots, X_{s(k)} = 1], \quad s \text{ any permutation of } \{1, \dots \text{ n}\} \\
&= C(\pi, \dots, \pi, 1, \dots, 1) \quad (k\text{-times } \pi,\ (n-k)\text{-times } 1),\ k \in \{1, \dots n\} \qquad (6.2) \\
\rho^X &= \frac{\pi_2 - \pi^2}{\pi - \pi^2} \qquad (6.3)
\end{aligned}$$

6.2 Homogeneous mixture type models

The structure of the homogeneous mixture type model is

$$\begin{aligned}
&(X_i|p)_i \text{ independent } \sim Ber(p) \\
&p \sim G, \quad G \text{ a distribution function on } [0,1].
\end{aligned}$$

Immediately we have $\pi = \mathbb{E}[p]$, $\pi_k = \mathbb{E}[p^k]$, $\rho^X = \frac{\pi_2 - \pi^2}{\pi - \pi^2}$ and that

$$\begin{aligned}
P\left[\sum_{i=1}^n X_i = k\right] &= \binom{n}{k} P[X_1 = 1, \dots, X_k = 1, X_{k+1} = 0, \dots, X_n = 0] \\
&= \binom{n}{k} \int_0^1 p^k (1-p)^{n-k} G(dp) \\
&= \binom{n}{k} \int_0^1 p^k \sum_{i=0}^{n-k} \binom{n-k}{i} (-1)^i p^i G(dp) \\
&= \binom{n}{k} \sum_{i=0}^{n-k} \binom{n-k}{i} (-1)^i \pi_{k+i}, \qquad k \in \{0, 1, \dots n\} \qquad (6.4)
\end{aligned}$$

6.3 Validity of homogeneous portfolios

At a first glance the homogeneity condition (6.1) on our portfolio seems very strong. But fortunately it is not too strong in the sense that when fixing the parameters π, ρ^X the higher order joint default probabilities $(\pi_3, \pi_4, \dots)$ still depend on the model and are not automatically determined by the two parameters.

To make this statement clear we give an example of a homogeneous portfolio for a mixture type model (like CR^+). The idea is to define a model with 3 (or more) parameters such that π and ρ^X are determined as soon as 2 of the 3 parameters are set. The 3rd parameter then allows for the variability in the higher order joint default probabilities. So let's consider the following model:

$$\begin{aligned}
&(X_i|p)_i \text{ independent } \sim Ber(p) \\
&p = \pi(wR + 1 - 0.5w) \\
&R \sim G, \qquad G \text{ a distribution function on } [0,1] \\
&\mathbb{E}[R] = 0.5, \qquad \mathrm{Var}[R] = \sigma^2
\end{aligned}$$

The parameters of this model are w, π, σ and it is straightforward to calculate the following relationships:

$$
\begin{aligned}
\mathbb{E}[p] &= \pi \\
\mathbb{E}[p^2] &= \pi^2(1+w^2\sigma^2) \\
&= \pi_2
\end{aligned}
\tag{6.5}
$$

In the last section we have seen that fixing (π, ρ^X) is equivalent to fixing (π, π_2). We immediately see that for any given pair (π, π_2) there exist infinitely many pairs (w, σ) which solve equation (6.5) and hence variation in the higher order joint default probabilities π_3, π_4, etc. is still possible.

6.4 Parameter estimation

Now we need to check for both frameworks KMV/CM and CR$^+$, that any given $\pi \in]0,1[$ and $\rho^X \in]0,1[$ can be met by choosing the free parameters in the models appropriately.

6.4.1 KMV/CM

The free parameters in the KMV/CM model are the distances-to-default $(DD_i)_i$ and the latent variable (asset log-return) correlation matrix Σ. since

$$
\begin{aligned}
\pi &= P[Z_i < -DD_i] \\
&= \Phi(-DD_i)
\end{aligned}
\tag{6.6}
$$

We obviously have for the distances-to-default that $DD_i = DD_j =: DD$ and for all $\pi \in]0,1[$ $\exists! DD$ such that (6.6) is valid. The default indicator correlation ρ^X is given by (6.3) and hence we need to check that π_2 can possibly take every value in the range $]\pi^2, \pi[$. π_2 is given by

$$
\begin{aligned}
\pi_2 &= \int_{-\infty}^{DD}\int_{-\infty}^{DD} f_{\rho^Z_{ij}}(x,y)dxdy, \quad i \neq j \\
&=: \pi_2(\rho^Z_{ij})
\end{aligned}
\tag{6.7}
$$

where $f_\rho(x,y)$ is the standard bivariate normal density function with correlation coefficient ρ and ρ^Z_{ij} stands for $\mathrm{Corr}[Z_i, Z_j]$. Hence $\rho^Z_{ij} =: \rho^Z$ $\forall i \neq j$. Furthermore

$$
\begin{aligned}
\pi_2(0) &= \inf_{u \in [0,1[} \pi_2(u) \\
&= \pi^2 \\
\lim_{u \nearrow 1} \pi_2(u) &= \sup_{u \in [0,1[} \pi_2(u) \\
&= \pi,
\end{aligned}
$$

and by using continuity and strict monotony of the function $\pi_2(u)$ we conclude that for every pair (π, ρ^X) there exists in the KMV/CM model a unique pair (DD, ρ^Z) such that all the model parameters are specified.

6.4.2 CR$^+$

As for KMV/CM we check the relationships between the parameters in the CR$^+$ model. The free parameters are the default probabilities $(\bar{p}_i)_i$, the variances of the Gamma-factors $(\sigma_j^2)_j$ and their weighting's (a_{ij}) (refer to page 19). We have seen before that $\pi = \mathbb{E}(p_i) = \bar{p}_i \ \forall i$. Another conclusion out of the homogenity property of the portfolio is that the factor loadings need to be the same for every obligor i, i.e. $a_{ij} = a_{kj} =: a_j \ \forall i, k$. So our model reduces to

$$
\begin{aligned}
&(X_i|p)_i \text{ independent } \sim Ber(p)\\
&p = \pi \sum_{j=1}^{K} a_j R_j, \quad R_j \text{ independent } \sim Gam(1, \sigma_j^2).
\end{aligned}
$$

Unfortunately the distribution of $\sum_j a_j R_j$ is neither a Gamma nor any other common distribution with an analytic representation. We have found that for volatilities $(\sigma_j)_j$ in the interval $]0, 1]$, $\sum_j a_j R_j$ is very well approximated by a Gamma distribution again with mean 1 and variance $\sum_j a_j^2 \sigma_j^2$. To this chosen range of volatilities we will come back in the next section. Hence

$$
\pi \sum_{j=1}^{K} a_j R_j \overset{d}{\approx} \pi R, \qquad R \sim Gam(1, \sigma^2)
$$

for

$$
\sigma^2 := \mathrm{Var}\left[\sum_{j=1}^{K} a_j R_j\right] = \sum_{j=1}^{K} a_j^2 \sigma_j^2.
$$

We notice that this approximation yields the same result as when assuming only one common factor to all $(p_i)_i$. From now on we assume only one common factor R for the CR$^+$ factor model. Thus $a_1 = 1$ and for the probability π_2 we have

$$
\begin{aligned}
\pi_2 &= \mathbb{E}\left[(\pi R)^2\right]\\
&= \pi^2(\sigma^2 + 1)
\end{aligned}
$$

and hence for the default indicator correlation ρ^X we have

$$
\rho^X = \frac{\pi \sigma^2}{1 - \pi}.
$$

Again we conclude that for every pair (π, ρ^X) there exists exactly one pair (π, σ) such that the model is fully determined.

6.5 Setup

In the previous section we mentioned a range of values for the default rate volatility parameter σ. This range was selected because according to information provided by UBS, analysis of historical default rate volatilities for all different industrial and country specific sectors shows that σ never exceeded the value of 1. That's why we have chosen 1 as an upper boundary for σ.

The following table gives the values for (π, σ) which we shall consider for the comparison of the models:

π (in %)	0.01	0.06	0.15	0.5	2.5	7.5
σ	0.2	0.6	1			

This setup is valid because we proved that fixing (π, ρ^X) is equivalent to fixing (π, σ). Hence the choice of parameters leaves us with $6 \times 3 = 18$ homogeneous portfolios, characterized in the following table:

Nr	Group	π in %	σ	ρ^X in %	ρ^Z in %
1	A	0.01	0.2	0.000	0.288
2	A	0.01	0.6	0.004	2.040
3	A	0.01	1.0	0.010	4.660
4	B	0.06	0.2	0.002	0.319
5	B	0.06	0.6	0.022	2.575
6	B	0.06	1.0	0.060	5.951
7	C	0.15	0.2	0.006	0.375
8	C	0.15	0.6	0.054	2.987
9	C	0.15	1.0	0.150	6.957
10	D	0.50	0.2	0.020	0.472
11	D	0.50	0.6	0.181	3.798
12	D	0.50	1.0	0.503	8.922
13	E	2.50	0.2	0.103	0.723
14	E	2.50	0.6	0.923	5.893
15	E	2.50	1.0	2.564	14.10
16	F	7.50	0.2	0.324	1.111
17	F	7.50	0.6	2.919	9.212
18	F	7.50	1.0	8.108	22.55

Table 6.1: *The portfolios are numbered from 1 to 18 and categorized in groups A to F of equal marginal default probability π. σ is the standard deviation of the Gamma risk factor, ρ^X the corresponding default indicator correlations for the KMV/CM and for the CR^+ model respectively, ρ^Z the asset log-return correlations for the KMV/CM model.*

Moreover, we will assume that our portfolio consists of $N = 14$ loans or bonds. Unfortunately time constraints and limited computational power have not allowed for simulations of larger portfolios. The models will be compared by taking a look at the quantities P_k^X, which are defined as

$$P_k^X = P\left[\sum_{i=1}^{N} X_i = k\right], \qquad k = 0, 1, \ldots N.$$

Furthermore, we want to compare portfolio loss distributions as well. So we need to specify a distribution for the loss given defaults $(LGD_i)_i$. As CM we assume a Beta distribution for the recovery rate r with expectation 0.4 and variance $(0.25)^2$ for all portfolios. These parameters correspond approximately to maximum likelihood estimates on historical recovery rates for senior subordinated bonds (see [5], page 71). The loan sizes or face values of the bonds we set equal to \$100'000 and denote them by V. Hence the loss contribution L_i of obligor i is

$$L_i = \begin{cases} X_i LGD_i & \text{in the CM/KMV model} \\ Y_i LGD_i & \text{in the CR}^+ \text{ model} \end{cases}$$
$$LGD_i = (1 - r_i)V$$

with $(r_i)_i$ iid $\sim Beta(0.4, (0.25)^2)$. Remember, in all models the $(LGD_i)_i$ are independent of the default events $(X_i)_i$.

6.6 Simulation

For the KMV/CM model we solved (6.7) for the asset log-return correlations ρ^Z for each of the 18 portfolios and then used Monte Carlo Simulation with 0.5 million runs to count the joint defaults and hence to estimate the $(P_k^X)_k$.

The CR$^+$ model allows for an analytical calculation of the joint default probabilities P_k^X. Seeing πR is Gamma distributed we can solve for $\pi_k = \mathbb{E}[(\pi R)^k]$ using characteristic functions and then apply formula (6.4). Unfortunately, we encounter problems when applying (6.4) to portfolio No. 18. For this portfolio one of the $(P_k^X)_k$ turns out to be negative! This comes from the fact that the Gamma distribution of πR has its support on $\mathbb{R}^+$ and not only on $[0, 1]$, as with mixture type models in general. Hence for every parameter pair (π, σ) there exists a k_0 such that for all $k > k_0$ we have that

$$\mathbb{E}[(\pi R)^k] > \mathbb{E}[(\pi R)^{k_0}] \quad \Longleftrightarrow \quad \pi_k > \pi_{k_0}. \tag{6.8}$$

We immediately conclude that the second inequality in (6.8) does not make sense. Hence the portfolio where this problem occurred had to be taken out of consideration. Note that in the original CR$^+$ model this problem was eliminated by the Poisson approximation, one of the first steps when deriving the portfolio loss pgf.

The comparison of the two models will be produced in tables on the next few pages. First of all the default frequencies of the Monte Carlo simulation for KMV/CM will be shown. To make comparison to CR$^+$ easier, we transformed the analytically calculated joint default probabilities of CR$^+$ to default frequencies (by multiplying them with the factor 0.5 million and rounding to the nearest integer). The 3rd table will give the values of the so-called frequency multipliers, the ratio of KMV/CM and corresponding CR$^+$ frequencies. These were only calculated for the frequencies which were observed at least 100 times, since the variance of the Monte Carlo estimation otherwise grows too big.

After that, we compare the loss distributions arising from the two models. This will be performed by calculating the mean, variance, skewness, kurtosis, Value at Risk (VaR), at the levels 90%, 95%, 97%, 99%, 99.5% and Expected Shortfall (ES) at the VaR-levels.

Skewness (Skew) refers to whether the distribution is symmetrical with respect to its dispersion from the mean. Since loss distributions are by nature highly asymmetric, the measure of skewness will provide information on the lack of symmetry.

Kurtosis (Kurt) measures the weight of the tails of a distribution. Loss distributions have their support on $\mathbb{R}^+$ only and hence kurtosis will give an insight on the amount of mass in the upper right tail; the more mass in the tail, the higher the value of kurtosis. Kurtosis is (as well as skewness) a variance corrected measure. That is, if we denote by $\text{Kurt}(X)$ the kurtosis property of the distribution of the random variable X, then we have that

$$\text{Kurt}(cX) = \text{Kurt}(X) \qquad \forall c \in \mathbb{R}.$$

For further risk quantification in the tail we calculate VaR and ES for the homogeneous portfolios. By VaR at the level α, VaR_α, we simply mean the α-quantile of the corresponding loss distribution. So VaR_α will tell us how far out we must set the loss threshold such that with probability α we incur a loss on our portfolio of VaR_α or less. ES at the level VaR_α, ES_α, is defined as $\mathbb{E}[L|L > \text{VaR}_\alpha]$ and hence provides information on how large on average a potential loss will be, given the loss exceeds the level VaR_α.

Note, that ES is a coherent risk measure (according to [1]), where as VaR is not.

Seeing we can't calculate the loss distributions analytically, we will calculate the empirical estimators of the above mentioned properties from the simulated distributions. We will present the simulation results in groups of 3 (as in table 6.1), since always 3 out of the 18 portfolios share the same marginal default probability π.

6.7 Results and discussion

6.7.1 Analysis within groups

Tables 6.2 and 6.3 exhibit nicely that for constant marginal default probability π an increase in default correlation produces higher joint default probabilities. More precisely, the probability of having no defaults at all increases, but conditional on the event that at least one default occurred, the probability of k defaults ($k > 1$) increases substantially. The higher π, the bigger the impact.

Groups A, B, C, which reflect portfolios of firms with quite high credibility, show that increasing correlation within each group does not affect the loss distribution too much (refer to tables 6.4, 6.5). All properties of the distribution seem to remain stable. This changes with groups of deteriorating credit quality (D, E, F). Variance and most of all kurtosis starts to increase drastically; the higher π, the more. Check on table 6.4 that portfolio No. 18 has almost double the variance of portfolio No. 16 and even double the amount of kurtosis! This fact is also visible in all the values for VaR and ES.

6.7.2 Comparison of the groups

It follows by definition of the models that when credit quality decreases (i.e. π increases) the number of joint defaults increases. This fact is obviously confirmed by the tables 6.2, 6.3.

When focusing on the loss distributions (tables 6.4, 6.5) we see how the mean starts to move away from zero further out to the right and goes hand in hand with a drastic increase in variance.

For comparison of the tails of the distributions we take a look at the values of VaR and ES. Those figures show once again that the lower credit quality is, the more mass is shifted out in the right tail of the portfolio loss distributions. For example in the KMV/CM model, $\text{ES}_{99.5\%}$ of portfolio No. 18 is almost 10 times as large as the corresponding $\text{ES}_{99.5\%}$ of portfolio No. 3.

6.7.3 Multipliers

Table 6.6 gives the ratio of default frequencies for the two models: CR^+ default frequencies divided by corresponding KMV/CM default frequencies. The cells containing 'n/a', i.e. not available, show the big disadvantage of Monte Carlo simulation in general. Although 0.5 million is quite a large number of simulations, it is by far not enough for comparison of events which occure with extremely small probability (the event of 8 defaults for portfolio No. 6, for example). For a closer look at the mass contained in the very right tail of the KMV/CM loss distribution 'extreme value theory' should be applied, if possible (see [8]). Unfortunately this is beyond the scope of this paper.

Overall, the numbers available in table 6.6 seem to be very close 1, which suggests that the two models perform very similar for all considered portfolios. This result is further underlined by the multiplier table 6.7, which compares the loss distribution properties of the two models.

6.7.4 Conclusion

We see that there is no significant difference in performance in the default frequency behaviour or in the loss distribution of the 17 different homogeneous portfolios.

For comparison of larger portfolios we would need to use the Poisson approximation, as with the 'original' CR^+ model. Otherwise we would encounter for most larger portfolios the same problem as with portfolio No. 18.

For a comparison of the models for 4 heterogeneous portfolios of a larger size, see [10].

		KMV/CM Model: Number of Defaults														
Portfolios		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A	1	499267	733	0	0	0	0	0	0	0	0	0	0	0	0	0
	2	499304	696	0	0	0	0	0	0	0	0	0	0	0	0	0
	3	499316	682	2	0	0	0	0	0	0	0	0	0	0	0	0
B	4	495774	4204	22	0	0	0	0	0	0	0	0	0	0	0	0
	5	495871	4107	22	0	0	0	0	0	0	0	0	0	0	0	0
	6	495883	4082	35	0	0	0	0	0	0	0	0	0	0	0	0
C	7	489640	10250	109	1	0	0	0	0	0	0	0	0	0	0	0
	8	489769	10099	131	1	0	0	0	0	0	0	0	0	0	0	0
	9	489810	9980	204	6	0	0	0	0	0	0	0	0	0	0	0
D	10	466419	32492	1068	21	0	0	0	0	0	0	0	0	0	0	0
	11	466432	32176	1337	54	1	0	0	0	0	0	0	0	0	0	0
	12	467313	30676	1886	122	3	0	0	0	0	0	0	0	0	0	0
E	13	351323	124820	21191	2438	208	18	2	0	0	0	0	0	0	0	0
	14	357637	114798	23157	3842	501	55	8	2	0	0	0	0	0	0	0
	15	367730	99664	24586	6053	1501	345	89	23	9	0	0	0	0	0	0
F	16	171719	186205	98976	33258	8165	1426	220	27	4	0	0	0	0	0	0
	17	198713	161221	84535	35837	13357	4503	1294	421	87	27	4	1	0	0	0
	18	237758	129718	66144	33216	16797	8465	4236	2097	883	427	172	66	18	3	0

Table 6.2: *Number of defaults produced by Monte Carlo simulation from the KMV/CM model.*

Portfolios		CR$^+$ Model: Number of Defaults														
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A	1	499300	700	0	0	0	0	0	0	0	0	0	0	0	0	0
A	2	499300	699	1	0	0	0	0	0	0	0	0	0	0	0	0
A	3	499301	698	1	0	0	0	0	0	0	0	0	0	0	0	0
B	4	495817	4166	17	0	0	0	0	0	0	0	0	0	0	0	0
B	5	495822	4156	22	0	0	0	0	0	0	0	0	0	0	0	0
B	6	495833	4135	32	0	0	0	0	0	0	0	0	0	0	0	0
C	7	489606	10289	104	1	0	0	0	0	0	0	0	0	0	0	0
C	8	489638	10226	135	1	0	0	0	0	0	0	0	0	0	0	0
C	9	489701	10102	194	3	0	0	0	0	0	0	0	0	0	0	0
D	10	466158	32709	1109	24	0	0	0	0	0	0	0	0	0	0	0
D	11	466495	32060	1396	48	1	0	0	0	0	0	0	0	0	0	0
D	12	467146	30831	1907	110	6	0	0	0	0	0	0	0	0	0	0
E	13	351613	124516	21335	2341	184	10	1	0	0	0	0	0	0	0	0
E	14	357869	114315	23428	3798	521	62	6	1	0	0	0	0	0	0	0
E	15	368551	98257	25176	6174	1442	319	66	13	2	0	0	0	0	0	0
F	16	171812	186527	98401	33387	8129	1501	216	25	2	0	0	0	0	0	0
F	17	199259	160386	84622	36147	13384	4420	1317	355	86	19	4	1	0	0	0
F	18	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a

Table 6.3: *Number of expected defaults calculated for the CR$^+$ model. n/a stands for 'not available'.*

			KMV/CM Model: Loss Distribution Properties													
							Value at Risk (VaR)					Expected Shortfall (ES)				
			Mean	Var	Skew	Kurt	90%	95%	97%	99%	99.5%	90%	95%	97%	99%	99.5%
Portfolios	A	1	8.59E+01	5.99E+06	3.11E+01	1.03E+03	0	0	0	0	0	5.86E+04	5.86E+04	5.86E+04	5.86E+04	5.86E+04
		2	8.50E+01	6.04E+06	3.13E+01	1.03E+03	0	0	0	0	0	6.10E+04	6.10E+04	6.10E+04	6.10E+04	6.10E+04
		3	8.12E+01	5.69E+06	3.20E+01	1.08E+03	0	0	0	0	0	5.94E+04	5.94E+04	5.94E+04	5.94E+04	5.94E+04
	B	4	5.11E+02	3.60E+07	1.28E+01	1.75E+02	0	0	0	0	5.56E+04	6.04E+04	6.04E+04	6.04E+04	6.04E+04	7.81E+04
		5	4.96E+02	3.51E+07	1.30E+01	1.83E+02	0	0	0	0	5.38E+04	6.00E+04	6.00E+04	6.00E+04	6.00E+04	7.76E+04
		6	5.00E+02	3.56E+07	1.30E+01	1.82E+02	0	0	0	0	5.48E+04	6.07E+04	6.07E+04	6.07E+04	6.07E+04	7.81E+04
	C	7	1.26E+03	8.90E+07	8.12E+00	7.23E+01	0	0	0	6.45E+04	8.26E+04	6.09E+04	6.09E+04	6.09E+04	8.30E+04	9.23E+04
		8	1.25E+03	8.84E+07	8.19E+00	7.40E+01	0	0	0	6.47E+04	8.18E+04	6.11E+04	6.11E+04	6.11E+04	8.28E+04	9.22E+04
		9	1.25E+03	8.99E+07	8.38E+00	7.91E+01	0	0	0	6.41E+04	8.26E+04	6.13E+04	6.13E+04	6.13E+04	8.37E+04	9.40E+04
	D	10	4.15E+03	2.90E+08	4.44E+00	2.36E+01	0	4.22E+04	6.77E+04	8.97E+04	9.53E+04	6.18E+04	7.39E+04	8.62E+04	1.01E+05	1.10E+05
		11	4.21E+03	3.01E+08	4.52E+00	2.47E+01	0	4.24E+04	6.82E+04	9.06E+04	9.62E+04	6.27E+04	7.51E+04	8.77E+04	1.04E+05	1.15E+05
		12	4.18E+03	3.10E+08	4.73E+00	2.79E+01	0	4.10E+04	6.82E+04	9.12E+04	9.72E+04	6.40E+04	7.58E+04	8.95E+04	1.09E+05	1.24E+05
	E	13	2.10E+04	1.47E+09	1.98E+00	7.14E+00	8.22E+04	9.55E+04	1.12E+05	1.59E+05	1.77E+05	1.10E+05	1.31E+05	1.52E+05	1.86E+05	2.05E+05
		14	2.10E+04	1.59E+09	2.16E+00	8.28E+00	8.27E+04	9.71E+04	1.24E+05	1.68E+05	1.87E+05	1.15E+05	1.40E+05	1.63E+05	2.00E+05	2.23E+05
		15	2.10E+04	1.86E+09	2.61E+00	1.18E+01	8.35E+04	1.01E+05	1.38E+05	1.85E+05	2.23E+05	1.25E+05	1.58E+05	1.84E+05	2.35E+05	2.69E+05
	F	16	6.31E+04	4.30E+09	1.10E+00	4.18E+00	1.56E+05	1.87E+05	2.14E+05	2.60E+05	2.88E+05	2.01E+05	2.33E+05	2.56E+05	3.00E+05	3.27E+05
		17	6.31E+04	5.51E+09	1.45E+00	5.54E+00	1.66E+05	2.10E+05	2.42E+05	3.06E+05	3.44E+05	2.27E+05	2.69E+05	2.99E+05	3.60E+05	3.97E+05
		18	6.30E+04	7.85E+09	1.97E+00	8.01E+00	1.80E+05	2.44E+05	2.91E+05	3.86E+05	4.41E+05	2.70E+05	3.31E+05	3.74E+05	4.62E+05	5.14E+05

Table 6.4: *Properties of the loss distribution produced by Monte Carlo simulation from the KMV/CM model.*

			CR+ Model: Loss Distribution Properties													
							Value at Risk (VaR)					Expected Shortfall (ES)				
			Mean	Var	Skew	Kurt	90%	95%	97%	99%	99.5%	90%	95%	97%	99%	99.5%
Portfolios	A	1	8.29E+01	5.78E+06	3.17E+01	1.06E+03	0	0	0	0	0	5.93E+04	5.93E+04	5.93E+04	5.93E+04	5.93E+04
		2	8.35E+01	5.89E+06	3.17E+01	1.06E+03	0	0	0	0	0	5.97E+04	5.97E+04	5.97E+04	5.97E+04	5.97E+04
		3	8.48E+01	6.02E+06	3.14E+01	1.03E+03	0	0	0	0	0	6.06E+04	6.06E+04	6.06E+04	6.06E+04	6.06E+04
	B	4	5.00E+02	3.51E+07	1.29E+01	1.78E+02	0	0	0	0	5.43E+04	5.98E+04	5.98E+04	5.98E+04	5.98E+04	7.74E+04
		5	5.06E+02	3.61E+07	1.29E+01	1.80E+02	0	0	0	0	5.49E+04	6.06E+04	6.06E+04	6.06E+04	6.06E+04	7.86E+04
		6	5.00E+02	3.55E+07	1.31E+01	1.84E+02	0	0	0	0	5.36E+04	6.00E+04	6.00E+04	6.00E+04	6.00E+04	7.80E+04
	C	7	1.27E+03	8.96E+07	8.14E+00	7.32E+01	0	0	0	6.52E+04	8.25E+04	6.09E+04	6.09E+04	6.09E+04	8.32E+04	9.24E+04
		8	1.27E+03	8.99E+07	8.14E+00	7.28E+01	0	0	0	6.50E+04	8.29E+04	6.11E+04	6.11E+04	6.11E+04	8.34E+04	9.29E+04
		9	1.27E+03	9.19E+07	8.27E+00	7.65E+01	0	0	0	6.54E+04	8.30E+04	6.18E+04	6.18E+04	6.18E+04	8.43E+04	9.43E+04
	D	10	4.22E+03	2.97E+08	4.42E+00	2.34E+01	0	4.31E+04	6.84E+04	9.03E+04	9.57E+04	6.23E+04	7.48E+04	8.71E+04	1.02E+05	1.11E+05
		11	4.20E+03	3.01E+08	4.55E+00	2.52E+01	0	4.24E+04	6.83E+04	9.05E+04	9.61E+04	6.27E+04	7.50E+04	8.78E+04	1.05E+05	1.16E+05
		12	4.21E+03	3.12E+08	4.72E+00	2.78E+01	0	4.15E+04	6.85E+04	9.12E+04	9.72E+04	6.41E+04	7.60E+04	8.97E+04	1.09E+05	1.24E+05
	E	13	2.10E+04	1.46E+09	1.97E+00	7.04E+00	8.21E+04	9.55E+04	1.12E+05	1.58E+05	1.76E+05	1.10E+05	1.31E+05	1.51E+05	1.85E+05	2.04E+05
		14	2.10E+04	1.60E+09	2.18E+00	8.40E+00	8.28E+04	9.72E+04	1.24E+05	1.67E+05	1.88E+05	1.15E+05	1.41E+05	1.63E+05	2.01E+05	2.26E+05
		15	2.10E+04	1.86E+09	2.56E+00	1.11E+01	8.37E+04	1.03E+05	1.39E+05	1.85E+05	2.21E+05	1.25E+05	1.59E+05	1.84E+05	2.32E+05	2.64E+05
	F	16	6.30E+04	4.30E+09	1.10E+00	4.20E+00	1.56E+05	1.87E+05	2.14E+05	2.61E+05	2.89E+05	2.02E+05	2.33E+05	2.56E+05	3.00E+05	3.27E+05
		17	6.30E+04	5.48E+09	1.43E+00	5.43E+00	1.66E+05	2.10E+05	2.41E+05	3.04E+05	3.40E+05	2.26E+05	2.68E+05	2.97E+05	3.56E+05	3.92E+05
		18	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a

Table 6.5: *Properties of the loss distribution produced by Monte Carlo simulation from the CR$^+$ model. n/a stands for 'not available'.*

Portfolios		Default Frequency Multiplier: 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A	1	1.000	0.955	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	2	1.000	1.004	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	3	1.000	1.023	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
B	4	1.000	0.991	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	5	1.000	1.012	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	6	1.000	1.013	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
C	7	1.000	1.004	0.954	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	8	1.000	1.013	1.031	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	9	1.000	1.012	0.951	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
D	10	0.999	1.007	1.038	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	11	1.000	0.996	1.044	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	12	1.000	1.005	1.011	0.902	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
E	13	1.001	0.998	1.007	0.960	0.885	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	14	1.001	0.996	1.012	0.989	1.040	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	15	1.002	0.986	1.024	1.020	0.961	0.925	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
F	16	1.001	1.002	0.994	1.004	0.996	1.053	0.982	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	17	1.003	0.995	1.001	1.009	1.002	0.982	1.018	0.843	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	18	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a

Table 6.6: *Default frequency multipliers: number of defaults from the CR^+ model divided by the corresponding number of defaults from the KMV/CM model. n/a stands for 'not available'.*

Portfolios		Multipliers of Loss Distribution Properties				Value at Risk (VaR)					Expected Shortfall (ES)				
		Mean	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
A	1	0.965	0.966	1.016	1.033	1*	1*	1*	1*	1*	1.012	1.012	1.012	1.012	1.012
A	2	0.983	0.975	1.013	1.031	1*	1*	1*	1*	1*	0.977	0.977	0.977	0.977	0.977
A	3	1.044	1.059	0.979	0.954	1*	1*	1*	1*	1*	1.021	1.021	1.021	1.021	1.021
B	4	0.979	0.974	1.008	1.015	1*	1*	1*	1*	0.976	0.989	0.989	0.989	0.989	0.990
B	5	1.021	1.030	0.992	0.982	1*	1*	1*	1*	1.021	1.009	1.009	1.009	1.009	1.012
B	6	1.000	0.998	1.004	1.010	1*	1*	1*	1*	0.978	0.988	0.988	0.988	0.988	0.999
C	7	1.002	1.006	1.003	1.012	1*	1*	1*	1.011	0.999	0.999	0.999	0.999	1.003	1.001
C	8	1.014	1.017	0.994	0.984	1*	1*	1*	1.004	1.013	1.001	1.001	1.001	1.007	1.007
C	9	1.019	1.022	0.987	0.967	1*	1*	1*	1.020	1.005	1.008	1.008	1.008	1.006	1.003
D	10	1.016	1.024	0.997	0.994	1*	1.020	1.011	1.006	1.004	1.008	1.011	1.009	1.009	1.012
D	11	0.998	1.000	1.007	1.020	1*	0.999	1.002	0.999	0.999	1.000	0.999	1.001	1.004	1.008
D	12	1.006	1.006	0.998	0.997	1*	1.012	1.004	1.000	1.000	1.001	1.003	1.002	1.002	1.004
E	13	0.997	0.995	0.995	0.985	1.000	1.000	0.996	0.993	0.993	0.998	0.996	0.995	0.995	0.995
E	14	1.000	1.006	1.008	1.014	1.000	1.001	1.006	0.998	1.004	1.003	1.005	1.004	1.007	1.011
E	15	1.000	1.001	0.979	0.938	1.003	1.025	1.009	1.001	0.991	1.004	1.004	0.998	0.989	0.984
F	16	0.999	1.001	1.004	1.003	1.001	1.000	1.000	1.003	1.003	1.001	1.001	1.001	1.002	1.000
F	17	0.998	0.994	0.986	0.979	1.000	0.999	0.995	0.994	0.989	0.997	0.995	0.993	0.990	0.988
F	18	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a

Table 6.7: *Multipliers of loss distribution properties: property of the CR^+ loss distribution divided by the corresponding property of the KMV/CM loss distribution. n/a stands for 'not available'. *Two corresponding properties have both value zero. The ratio is zero divided by zero. We then define the multiplier to have value 1.*

Chapter 7

Extension of the KMV/CM methodology

In this chapter we will give a possible generalization of the KMV/CM model. The reason why we do not try to extend the CR^+ model is because none of the assumptions can be relaxed without loosing the analytic derivation of the portfolio loss pfg. Apart from this, the first step in generalizing the KMV/CM methodology is much more obvious. Seeing the standardized asset log-returns $\underline{Z}$ are $\mathcal{N}(\underline{0}, \Sigma)$ (refer to page 7), a possible extension is to assume that $\underline{Z}$ is elliptically distributed. This makes sense because the multivariate normal distribution is a member of that family of distributions.

As with the normal distribution linear correlation is also a canonical dependence measure in the world of elliptical distributions (to be seen later), but it does not anymore contain all information of the dependence structure.

Remember that the correlations among the asset log-returns were estimated using a factor model. We will uphold the factor structure, because there is no other obvious way of producing those correlations otherwise.

First of all we will give an introduction to elliptically distributed random variables.

7.1 Elliptical distributions

Elliptical distributions arise naturally as an extension of $\mathcal{N}(\underline{0}, \Sigma)$, as well as an extension of so-called spherical distributions. To see how all these distributions are related to one another, we first start with the spherical. These provide a family of symmetric distributions for uncorrelated random vectors with mean zero.

Definition 7.1. *A random vector $\underline{X} = (X_1, \dots, X_n)^T$ is said to have a spherical distribution if for every orthogonal map $\Gamma \in \mathbb{R}^{n \times n}$ ($\Gamma\Gamma^T = \mathbf{1}_{n \times n}$)*

$$\Gamma \underline{X} \stackrel{d}{=} \underline{X}. \tag{7.1}$$

$\mathbf{1}_{n \times n}$ stands for the $(n \times n)$-identity matrix.

The characteristic function $\psi(\underline{s}) := \mathbb{E}[\exp(i\underline{s}^T \underline{X})]$ of such distributions takes a particular simple form. There exists a function $\phi : \mathbb{R}^+ \to \mathbb{R}$ such that $\psi(\underline{s}) = \phi(\underline{s}^T \underline{s})$. This function is called the characteristic generator of the spherical distribution and we write

$$\underline{X} \sim S_n(\phi).$$

If $\underline{X}$ has a density $f(\underline{x}) = f(x_1, \dots, x_n)$ then this is equivalent to $f(\underline{x}) = g(\underline{x}^T\underline{x})$ for some function $g : \mathbb{R}^+ \to \mathbb{R}^+$ so that spherical distributions are best interpreted as those distributions whose density is constant on spheres.

An example of an n-dimensional spherical distribution is $\mathcal{N}(\underline{0}, \mathbf{1}_{n\times n})$. Note, that in the class of spherical distributions the multivariate normal is the only distribution with independent components (refer to [9], page 106).

Spherical distributions admit an alternative stochastic representation:

$$\underline{X} \sim S_n(\phi) \quad \Longleftrightarrow \quad \underline{X} \stackrel{d}{=} R\underline{U},$$

where the random vector $\underline{U}$ is uniformly distributed on the unit hypersphere $S_{n-1} := \{\underline{x} \in \mathbb{R}^n | \underline{x}^T\underline{x} = 1\}$ in $\mathbb{R}^n$ and the radial component R is a positive random variable, independent from $\underline{U}$ (refer to [9], page 30).

Spherical distributions can thus be interpreted as mixtures of uniform distributions on spheres of differing radius in $\mathbb{R}^n$. For example if $\underline{X} \sim \mathcal{N}(\underline{0}, \mathbf{1}_{n\times n})$, then $\underline{X}$ can be represented as $\underline{X} \stackrel{d}{=} \sqrt{Y}\underline{U}$ with $Y \sim \mathcal{X}^2_n$, where $\mathcal{X}^2_n$ stands for the Chi-Squared distribution with n degrees of freedom.

Elliptical distributions extend naturally the multivariate normal $\mathcal{N}(\underline{0}, \Sigma)$. Mathematically, they are the affine maps of spherical distributions in $\mathbb{R}^n$.

Definition 7.2. *$\underline{X}$ is said to have an elliptical distribution with parameters $(\underline{\mu}, \Sigma)$ if*

$$\underline{X} \stackrel{d}{=} \mathbf{A}\underline{Y} + \underline{\mu}, \qquad \underline{Y} \sim S_k(\phi), \tag{7.2}$$

where $\mathbf{A}$ is a $(n \times k)$-matrix (the affine map, $k \le n$) and $\mathbf{A}\mathbf{A}^T = \Sigma$, $Rank(\Sigma) = k$.

Since the characteristic function can be written as

$$\begin{aligned} \psi(\underline{s}) &= \mathbb{E}\big[\exp\big((i\underline{s}^T\underline{X})\big] &&= \mathbb{E}\big[\exp\big(i\underline{s}^T(\mathbf{A}\underline{Y} + \underline{\mu})\big)\big] \\ &= \exp\left(i\underline{s}^T\underline{\mu}\right)\exp\big(i(\mathbf{A}^T\underline{s})^T\underline{Y}\big) &&= \exp\left(i\underline{s}^T\underline{\mu}\right)\phi(\underline{s}\Sigma\underline{s}), \end{aligned}$$

we denote the elliptical distribution

$$\underline{X} \sim E_n(\underline{\mu}, \Sigma, \phi). \tag{7.3}$$

If $\underline{Y}$ has a density $f_{\underline{Y}}(\underline{y}) = g(\underline{y}^T\underline{y})$ and if Σ is positive definite, then $\underline{X} = \mathbf{A}\underline{Y} + \underline{\mu}$ has density

$$f_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{\det(\Sigma)}} g\big((\underline{x} - \underline{\mu})^T\Sigma^{-1}(\underline{x} - \underline{\mu})\big),$$

and the contours of equal density now from ellipsoids in $\mathbb{R}^n$. Apart from the multivariate normal also the t-distribution with ν degrees of freedom is a member of the elliptical family. Let $\underline{X}$ be standard t_ν-distributed, $[\Sigma]_{ij} := \mathrm{Corr}[X_i, X_j]$. Then $\underline{X}$ can be represented as

$$\begin{aligned}
\underline{X} &\stackrel{d}{=} \frac{\sqrt{\nu}}{\sqrt{S}}\underline{Z}, \qquad \underline{Z} \sim \mathcal{N}(\underline{0}, \Sigma), \quad S \sim \mathcal{X}_\nu^2 \qquad (7.4)\\
&\stackrel{d}{=} \frac{\sqrt{\nu}}{\sqrt{S}}\mathbf{A}\underline{\tilde{Z}}, \qquad \mathbf{A}\mathbf{A}^T = \Sigma, \quad \underline{\tilde{Z}} \sim \mathcal{N}(\underline{0}, \mathbf{1}_{n\times n})\\
&\stackrel{d}{=} \mathbf{A}\frac{\sqrt{\nu}}{\sqrt{S}}\sqrt{R}\underline{U}, \qquad R \sim \mathcal{X}_n^2, \qquad \underline{U} \text{ uniform on } S_{n-1}\\
&\stackrel{d}{=} \mathbf{A}(\tilde{R}\underline{U}), \qquad \frac{\tilde{R}^2}{n} \sim F(n, \nu),
\end{aligned}$$

and we denote $\underline{X} \sim t_{\nu,\Sigma}$. $F(n, \nu)$ stands for the F-distribution with n and ν degrees of freedom.

Note, that (7.3) implies that an elliptical distribution is fully described by its mean, covariance matrix and its characteristic generator. But conversely, knowledge of the distribution of $\underline{X}$ does not completely determine the elliptical representation $E_n(\underline{\mu}, \Sigma, \phi)$; it uniquely determines $\underline{\mu}$ but Σ and ϕ are only fixed up to a positive constant (refer to [9], page 43). Naturally the question arises if it is possible for $\underline{X}$ elliptically distributed to find (Σ, ϕ) such that $\underline{X} \sim E_n(\underline{\mu}, \Sigma, \phi)$ and $Cov[\underline{X}] = \Sigma$? We assume the existence of the second moment of the radial component, i.e. $\mathbb{E}[R^2] < \infty$. Then

$$\begin{aligned}
\text{Cov}[\underline{X}] &= \text{Cov}[\mathbf{A}\underline{Y} + \underline{\mu}]\\
&= \mathbf{A}\mathbf{A}^T\text{Cov}[\underline{Y}]\\
&= \mathbf{A}\mathbf{A}^T\mathbb{E}[R^2]\text{Cov}[\underline{U}].
\end{aligned}$$

Seeing $\underline{U}$ is uniformly distributed on the unit hypersphere in $\mathbb{R}^n$, we have $\text{Cov}[\underline{U}] = \frac{1}{n}\mathbf{1}_{n\times n}$. Hence $\text{Cov}[\underline{X}] = \frac{1}{n}\mathbf{A}\mathbf{A}^T\mathbb{E}[R^2]$. By choosing the characteristic generator $\tilde{\phi}(s) = \phi(\frac{s}{c})$, where $c := \frac{n}{\mathbb{E}[R^2]}$, we get $\text{Cov}[\underline{X}] = \mathbf{A}\mathbf{A}^T = \Sigma$.

The above calculation in conjunction with (7.2) show that linear correlation is a natural dependence measure for elliptical distributions. But unlike for the multivariate normal distribution it does not contain anymore all information of dependence. This fact becomes obvious when we fix a correlation matrix Σ and consider the family of elliptical random vectors given by $\{\underline{X} | S \text{ a random variable on } \mathbb{R}^+,\ \underline{X} \stackrel{d}{=} S\underline{Y},\ Y \sim \mathcal{N}(\underline{0}, \Sigma)\}$, which shares the same correlation matrix Σ for all its members.

7.1.1 Tail dependence

We now introduce an additional measure of dependence, the so called coefficients of upper and lower tail dependence. These coefficients provide information on the amount of dependence in the upper-quadrant tail and lower-quadrant tail of a bivariate distribution. Although this measure will only be defined for bivariate random variables, we will see later how the data of our multivariate simulations can be interpreted using the notion of tail dependence. For the remainder of this section we suppose that X_1 and X_2 are two continuously distributed random variables with distribution functions F_1, F_2 and copula C, i.e. $(X_1, X_2) \sim C(F_1, F_2)$.

Definition 7.3. *The coefficient of upper tail dependence of X and Y is defined as*

$$\lambda_U := \lim_{u \nearrow 1} P[X_2 > F_2^{-1}(u) | X_1 > F_1^{-1}(u)],$$

provided that the limit $\lambda_U \in [0, 1]$ exists. If $\lambda_U \in]0, 1]$, X_1 and X_2 are said to be asymptotically dependent in the upper tail; if $\lambda_U = 0$, X_1 and X_2 are said to be asymptotic independent in the upper tail.

The coefficient of lower tail dependence λ_L is equivalently defined for the lower left quadrant:

$$\lambda_L := \lim_{u \searrow 0} P[X_2 < F_2^{-1}(u) | X_1 < F_1^{-1}(u)],$$

if the limit exists.

By definition we see that the concept of tail dependence is relevant to dependence in extreme values.

We provide an alternative and equivalent definition from which it follows that this concept is a copula property.

Definition 7.4. *If a bivariate copula C is such that*

$$\lambda_U = \lim_{u \nearrow 1} \frac{C(u,u) - 2u + 1}{1 - u}$$

exists, then C has upper tail dependence if $\lambda_U \in]0,1]$ and none if $\lambda_U = 0$.

The coefficient of lower tail dependence λ_L as a function of C is given by

$$\lambda_L = \lim_{u \searrow 0} \frac{C(u,u)}{u}, \tag{7.5}$$

again provided the limit exists.

If we additionally assume the marginals of (X_1, X_2) to be equally distributed $(F := F_1 = F_2)$ and the copula C to be ex-changeable (symmetric), which is true for all bivariate elliptical copulas, we find (see [7], page 18)

$$\begin{aligned} \lambda_U &= 2 \lim_{x \to \infty} P[X_2 > x | X_1 = x] \\ \lambda_L &= 2 \lim_{x \to -\infty} P[X_2 < x | X_1 = x]. \end{aligned}$$

Moreover, it can be shown that for elliptical copulas the upper and lower coefficient of tail dependence are the same. This is due to the symmetry property of elliptical distributions imposed by the radial component. Since we will only focus on elliptical distributions, we refer to λ_U and λ_L as the coefficient of tail dependence, denoted by λ.

As an example we will now calculate $\lambda^{\underline{X}}$ for (X_1, X_2) bivariate standard normal and for (Y_1, Y_2) bivariate standard t-distributed with ν degrees of freedom $\lambda^{\underline{Y}}$ respectively, both with linear correlation coefficient $\rho \in]-1,1[$. Then by elementary calculations we retrieve

$$\begin{gathered} (X_2 | X_1 = x) \sim \mathcal{N}\big(\rho x, (1-\rho^2)\big) \\ \left(\tfrac{\nu+1}{\nu+y^2}\right)^{\frac{1}{2}} \tfrac{Y_2 - \rho y}{\sqrt{1-\rho^2}} \sim t_{\nu+1} \end{gathered}$$

and hence

$$\begin{aligned} \lambda^{\underline{X}} &= 2 \lim_{x \to \infty} \left(1 - \Phi\left(\frac{x(1-\rho)}{\sqrt{1-\rho^2}}\right)\right) \\ &= 0 \\ \lambda^{\underline{Y}} &= 2 \lim_{y \to \infty} \left(1 - t_{\nu+1}\left(\left(\frac{\nu+1}{\frac{\nu}{y^2}+1}\right)^{\frac{1}{2}} \frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)\right) \\ &= 2 t_{\nu+1}\left(-\sqrt{\nu+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right). \end{aligned}$$

We conclude that for linear correlation values $|\rho| \neq 1$ the Gaussian copula has always zero tail dependence. Conversely, the coefficient $\lambda^{\underline{Y}}$ of the t-copula is for all values of ν, ρ strictly positive; increasing in ρ and decreasing in ν. Furthermore letting ν tend to infinity the t-copula converges to the normal copula and hence

$$\lambda^{\underline{Y}} \overset{\nu\to\infty}{\longrightarrow} \lambda^{\underline{X}} = 0.$$

We give a table of calculated tail dependence coefficients λ (expressed as a percentage) for some values of the parameters (ν, ρ)

$\nu \backslash \rho$	-0.5	0	0.3	0.7
3	2.57	11.61	21.61	44.81
5	0.54	4.98	12.24	34.32
10	0.01	0.69	3.32	19.11
20	0.00	0.02	0.29	6.79

and a graphical illustration of tail dependence. Figure 7.1 exhibits that in the lower left and upper right quadrant the t-copula seems to produce more variates along the diagonal than the normal copula, i.e. extreme events tend to show more often in pairs than with the normal copula.

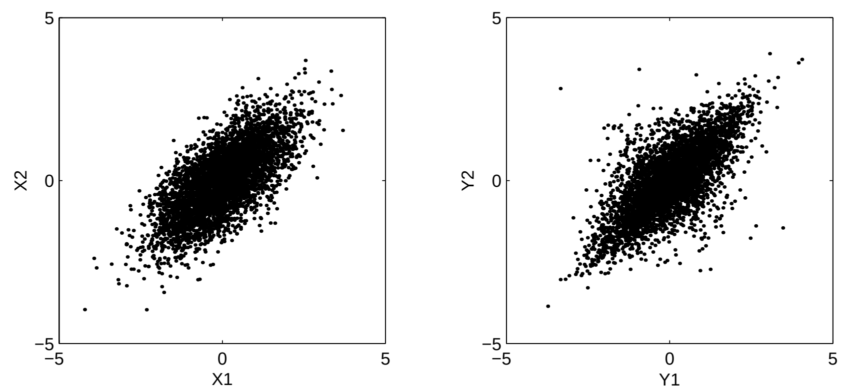


Figure 7.1: *5000 samples from two distribution with equal standard normal marginals, linear correlation coefficient of the copula $\rho = 0.7$ but different dependence structures. $(X1, X2)$ has a Gaussian copula and $(Y1, Y2)$ a t-copula with $\nu = 3$ degrees of freedom.*

Taking another look at (7.4) we conclude that the t-copula is a very useful alternative to the normal copula: simulation from both copulas is straightforward and the parameterisation by the correlation matrix allows for easy interpretation of the parameters. Furthermore, seeing the convergence of the t- to the normal copula is very fast for $\nu \to \infty$ (a t-copula with $\nu = 30$ is almost perfectly a normal copula), by starting at $\nu = 30$ we can move in a 'continuous' manner away from the normal copula and get increasing tail dependence by choosing ν smaller and smaller.

7.2 Mixtures of normal distributions

As we already know, the distribution of the latent variable vector $\underline{Z}$ in the KMV/CM model is multivariate normal. A particular interesting class of elliptical distributions and an obvious extension of the normal distribution is the class of (variance) mixtures of normal distributions. Any random vector $\underline{Z}$ of that category can be written as

$$\underline{Z} \stackrel{d}{=} S\underline{\tilde{Z}} \tag{7.6}$$

with $S > 0$, $\mathbb{E}[S^2] < \infty$, $\underline{\tilde{Z}} \sim \mathcal{N}(\underline{0}, \Sigma)$, $S, \underline{\tilde{Z}}$ independent, $\Sigma_{ii} = 1\ \forall i$.

In the KMV/CM model a factor model was used to produce the correlation matrix of the standardised asset log-returns (refer to page 19). The class of mixtures of normal distributions allows for an easy factor decomposition of the latent vector $\underline{Z}$ as well (to be seen in the next section).

Surprisingly, another justification of this model extension is given by a result in [14], page 130-132. It says that if we restrict the class of elliptical distributions to those which allow for the same equal marginals for any dimension n, then this category is the class of mixtures of normal distributions given by the representation (7.6). More clearly, if the elliptical distribution of $\underline{Z} = (Z_1, \dots, Z_n)^T$ satisfies the condition

$$\exists F : \forall n \in \mathbb{N} : Z_i \sim F \qquad \forall i = 1, \dots n \tag{7.7}$$

then it can be shown that

$$\exists S > 0 : \underline{Z} \stackrel{d}{=} S\underline{\tilde{Z}},\ \underline{\tilde{Z}} \sim \mathcal{N}(\underline{0}, \Sigma).$$

(The result in [14] is given for spherical distributions, but can easily be relaxed to elliptical ones.)

If we require the elliptical distribution of the latent vector $\underline{Z}$ to be 'compatible' to portfolios of any size n, then this restriction on the marginals given by equation (7.7) is needed! Furthermore, if we use a factor decomposition in the extended model to estimate the covariance matrix of $\underline{Z}$ (coming up in the next section), then one desired property of that factor model is definitely that it can be applied to portfolios of any size n.

7.3 Extended factor model

For the mixture of normal model we also assume that the standardized asset log-returns Z_i, $i = 1, \dots N$ are generated by a factor model. This is by (5.1) equivalent to saying that

$$[\mathrm{Cov}(Z_i, Z_j)]_{ij} = \mathbf{A}\mathbf{A}^T + \mathbf{D}$$

for some $\mathbf{A}, \mathbf{D}$ defined as in (5.1). (7.6) gives $\mathrm{Cov}[Z_i, Z_j] = \mathbb{E}[S^2]\mathrm{Cov}[\tilde{Z}_i, \tilde{Z}_j]$ and we immediately retrieve the factor structure:

$$\begin{aligned} Z_i &= \frac{1}{\sqrt{\mathbb{E}[S^2]}} S \left(\sum_{j=1}^{K} a_{ij} R_j + \mathcal{E}_i \right) \\ &=: \sum_{j=1}^{K} \tilde{a}_{ij} (S R_j) + (S\mathcal{E}_i), \qquad i = 1, \dots N, \end{aligned} \tag{7.8}$$

where $a_{ij}, R_j, \mathcal{E}_i$ are defined as on page 19.

The factor S can be interpreted as a 'global' risk. We see that the former independence among the factors and the idiosyncratic risk has weakened (because of S) to uncorrelatedness only. Since $\mathrm{Cov}[SR_i, SR_j] = \mathrm{Cov}[SR_i, S\mathcal{E}_k] = \mathrm{Cov}[S\mathcal{E}_k, S\mathcal{E}_l] = 0$ ($\forall i, j, k, l$), (7.8) is indeed a factor model according to definition (5.1). For practitioners it is worthwhile mentioning that $\mathrm{Corr}[Z_i, Z_j] = \mathrm{Corr}[\tilde{Z}_i, \tilde{Z}_j]$, which shows that estimation of the correlations has not become more difficult than with the 'older' factor model!

7.4 Possible models for S

For a non-parametric estimation of S a huge amount of asset value time series needed to be available. Since this is hardly ever the case in practice, we will focus on parametric estimation of S. So first a distribution for S needs to be chosen.

7.4.1 t-distribution

In the field of elliptical distributions we could consider the well known multivariate t-distribution with ν degrees of freedom for $\underline{Z}$. This assumption determines S completely (refer to (7.4)):

$$S \stackrel{d}{=} \frac{\sqrt{\nu}}{\sqrt{X}}, \qquad X \sim \mathcal{X}^2_\nu.$$

The only parameter to be estimated is ν, since the correlation matrix is already given by the factor model. A possible method to calibrate ν is maximum likelihood on historical asset data. It is crucial to see that the maximum likelihood estimate needs to be done for the joint distribution of $\underline{Z}$ and not only for the marginal distributions of the $(Z_i)_i$.

7.4.2 Symmetric hyperbolic distributions

In recent history generalized hyperbolic distributions have been very successfully applied to univariate log-returns of financial time series (such as stocks, stock indices). For further information refer to [6]. A possible parameterization of the hyperbolic density $f_{hyp}(x)$ is given by

$$f_{hyp}(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2 - \beta^2})} \exp\big(- \alpha\sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu)\big),$$

where K_1 denotes the modified Bessel function of the third kind with index 1. The parameters α and β with $\alpha > 0$ and $0 \leq |\beta| < \alpha$ determine the shape of the distribution, while the other two, δ and μ, are scale and location parameters.

It was pointed out by Barndorff-Nielsen (see [2]), that the hyperbolic distribution can be represented as normal mean-variance mixture, where the mixing distribution is a generalized inverse Gaussian with density

$$f_{giG}(x) = \frac{\sqrt{\frac{\psi}{\gamma}}}{2K_1(\sqrt{\psi\gamma})} \exp\left(-\frac{1}{2}(\gamma x^{-1} + \psi x)\right) \qquad x > 0.$$

If we set $\gamma = \delta^2$ and $\psi = \alpha^2 - \beta^2$ it follows that

$$X \sim f_{hyp}(x)dx \quad \Longleftrightarrow \quad \begin{array}{l} X|\sigma \sim \mathcal{N}(\mu + \beta\sigma^2, \sigma^2) \\ \sigma^2 \sim f_{giG}(x)dx \end{array}$$

Now if we consider the representation (7.6) for $\underline{Z}$, $\underline{Z} = S\underline{\tilde{Z}}$, and if for a moment we focus on the marginals Z_i only, then obviously each Z_i can be written equivalently as

$$\tilde{Z}_i|S \sim \mathcal{N}(0, S^2), \qquad i = 1, \dots n.$$

Hence, if we choose the distribution of S^2 to follow a generalized inverse Gaussian distribution, then every Z_i has a two parameter (γ, δ) symmetric hyperbolic distribution when the other two parameters μ and β are set to zero. From the parameterization of the density $f_{hyp}(x)$ we deduce that for $\mu = \beta = 0$ the distribution of S^2 is symmetric with respect to the mean and that the mean is equal to zero. For the multivariate distribution of $\underline{Z}$ we then have that

$$\underline{Z}|S \sim \mathcal{N}(\underline{0}, S^2\Sigma), \qquad \Sigma \text{ a correlation matrix.}$$

Chapter 8

t-model versus KMV/CM

8.1 Setup

In the same manner as the KMV/CM and the CR$^+$ model were compared for homogeneous portfolios we now compare the KMV/CM model with the so-called t-model, which is defined as:

$$\begin{aligned} &\underline{Z} \sim t_{\nu,\Sigma}, \qquad \Sigma \text{ a correlation matrix} \\ &X_i := I_{\{Z_i < t_\nu^{-1}(\pi)\}}. \end{aligned}$$

We emphasize that the only difference between the KMV/CM and the t-model is the copula of the latent variable vector $\underline{Z}$. To make this point totally clear, we write down the two models against each other:

$$\begin{aligned} &\underline{Z}^{KMV} \sim \mathcal{N}(\underline{0}, \Sigma) \\ &\underline{Z}^{t} \stackrel{d}{=} \frac{\sqrt{\nu}}{\sqrt{S}}\underline{\tilde{Z}}, \qquad \underline{\tilde{Z}} \sim \mathcal{N}(\underline{0}, \Sigma),\ S \sim \mathcal{X}_\nu^2 \end{aligned} \tag{8.1}$$

and for all $i = 1, \dots N$ we have

$$\begin{aligned} X_i^{KMV} = 1 \quad &\Longleftrightarrow \quad Z_i^{KMV} < -DD_i^{KMV} \\ X_i^{t} = 1 \quad &\Longleftrightarrow \quad Z_i^{t} < -DD_i^{t}. \end{aligned}$$

The distances-to-default $(DD_i^{KMV})_i, (DD_i^t)_i$ are set in each model such that

$$P[X_i^{KMV} = 1] = P[X_i^t = 1] = \pi, \qquad i = 1, \dots N.$$

8.2 Simulation

The following comparison will give an insight of the impact of the choice of copula on the credit risk models developed by KMV and J.P. Morgan. We will evaluate the homogeneous portfolios for the degree of freedom parameter ν at 3,5,10,20 and then compare the results when simulating from the t-model with the former results for the KMV/CM model.

The first table will contain again the default frequencies for the KMV/CM model, followed by the frequencies of the t-model for ν decreasing from 20 to 3.

So the reader can compare very easily the impact of the new t-copula with increasing tail dependence, but keeping linear correlation constant.

We will only present the tables of the default frequencies and the tables containing the multipliers of the loss distribution properties, to make comparison of the KMV/CM model to the t-model easier.

8.3 Results and discussion

8.3.1 Default frequencies

When scrolling through the tables 8.1 to 8.5 the impact of tail dependence incorporated by the t-copula becomes very obvious. The joint default frequencies of higher order of all portfolios just seem to explode as ν decreases.

In this and the next section we will focus on comparison of the KMV/CM and the t-model for $\nu = 3$.

The first column of the tables 8.1 and 8.5 exhibits that the event of incurring no default at all receives even higher probability in the t-model than in KMV/CM. But the remaining columns clearly show that given the event of at least one default, the probability of several defaults at a time increases (more than) substantially! Now it becomes very obvious what is meant by saying 'extreme values tend to appear together'!

Analysis within groups

Two very astonishing observations are that portfolios containing obligors of a very high credibility (A,B) are very much affected by the t-copula and the impact seems to be the highest for portfolio groups C and D (average credibility). Portfolio No. 10 in the KMV/CM model never produced joint defaults of order 4 and higher where as when evaluated with the t-model joint defaults appeared up till order 10! 980 times the event of 4 defaults occurred where as with KMV/CM this event was never observed!

Comparison of the groups

Obviously the t-model produces more joint defaults the lower the credit quality (i.e. the higher π). This follows by definition for all latent variable models and was already observed for the KMV/CM and the CR^+ default frequencies.

8.3.2 Loss distributions

Now we focus on the relative comparison of the loss distribution properties of the KMV/CM and the t-model with degrees of freedom parameter $\nu = 3$.

One property, which remains unchanged relatively to KMV/CM is the mean (refer to table 8.9). For all 18 portfolios the multiplier is very close to 1. For the variance, we have on average more than double the amount for the t- than for the KMV/CM model. Relative skewness is always above one and on average close to 2. Hence the t loss distribution is always more asymmetric than the one produced by KMV/CM. Very interesting is also the comparison of relative variance with relative kurtosis. Relative kurtosis is on average way above 1. A relative increase in variance goes always with a relative increase in kurtosis, which shows again that the mass of the loss distributions, which lies to the right of the mean, gets pushed far out in the tail by the t-copula. For example portfolio No. 1 produces double the variance and even 6 times higher kurtosis for the t-model than for KMV/CM!

For a closer look at the tail of the loss distributions we again analyze the results within groups and compare different groups separately.

Analysis within groups

Within each group, all multipliers of each property are very close to one another; the higher π, the closer. This exhibits that for increasing default correlation, keeping the marginal default probability π constant, has approximately the same relative effect on the KMV/CM as on the t-model! Hence, all multiplier values different from 1 are 'only' due to the additional tail dependence property of the t-copula!

Comparison of the groups

Surprisingly, as already noticed when evaluating the frequency tables, the t-copula seems to have the highest impact on the group C and D, i.e. on portfolios with an average credit rating grade and not as one may intuitively think in the group F, the group of worst credit quality. Let's take a closer look at group C. The 99% VaR is on average 1/3 smaller for the t-copula than for the normal one, but the corresponding ES is on average 45% higher! Similarly for the 99.5% level. An increase of 10% VaR for the t-model yields in an increase of 90% in the corresponding ES. For the portfolio group E and F the VaR's percentage increase goes with an approximate equal increase in ES.

Furthermore, if we take a look at ES at the level 99.5% we see by scrolling down the column that the multiplier reaches its maximum in portfolio group D!

To understand this phenomenon, we take another quick look at the setup at the beginning of this chapter. The factor $\tilde{S} := \sqrt{\nu}/\sqrt{S}$ is the only cause of the tail dependence property of the t-copula. Say if $\hat{\underline{\tilde{Z}}}$ is a variate of $\underline{\tilde{Z}}$ containing a lot of negative marginal variates $\hat{\underline{\tilde{Z}}}_i$ which lie in the interval $]-DD_i^{KMV}, 0[$, then a possible large variate of the factor $\tilde{S}$ can push these variates below the default points $-DD_i^t$ and hence the event of many simultaneous defaults occurs in the t-model, but not in the KMV/CM model. The results of the simulation reveal that the impact of the factor $\tilde{S}$ on the very outer tail of the loss distribution (compare the values of $\text{ES}_{99.5\%}$) is high for groups of a high and average credit rating (highest for group D) and weakens as credit quality decreases. This is due to the fact that the normal distribution decays extremely fast when moving away from its mean. But when credit quality deteriorates the corresponding default points $-DD_i^{KMV}$ move closer to the mean zero and for the groups E,F they turn out to be close enough to let the normal distribution produce many simultaneous defaults. So if a variate $\underline{\tilde{Z}}$ already contains many marginal variates which lie below the default point, a possible high variate of $\tilde{S}$ does not change the picture. Note that for $\pi = 50\%$ (corresponds to a default point of zero in both models) the impact of $\tilde{S}$ vanishes completely.

8.3.3 Specific portfolios

The last table compares the models for 3 specific portfolios, namely No. 5, 11 and 17. No. 5 represents a portfolio of high credit quality ($\pi = 0.06\%$), the second portfolio has an average credit grade ($\pi = 0.5\%$) and portfolio No. 17 contains obligors of very low credit quality ($\pi = 7.5\%$).

Table 8.10 exhibits once again very clearly (leaving the VaR-thresholds out of consideration) that independent which of the 3 portfolios considered, the t-model with increasing tail dependence always aggravates the portfolio loss distribution (from a portfolio manager perspective). For portfolio No. 5 and 11 we see why stand-alone VaR should not be considered as a risk measure. The empirical VaR's of portfolio No. 5 would lead us to choose the t-model, $\nu = 3$, from a VaR-point of view!

This result also holds for some of the quantiles for the average credit grade portfolio No. 11. The values of $\mathrm{VaR}_{90\%}$, $\mathrm{VaR}_{95\%}$ and $\mathrm{VaR}_{97\%}$ are declining as ν increases! Since expected shortfall is a coherent risk measure, (see [1]) for comparison of the models ES should anyway be taken stronger into account. Those values reveal that the portfolio manager would prefer that his portfolio loss distribution followed the KMV/CM model!

		KMV/CM Model: Number of Defaults														
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Portfolios																
A	1	499267	733	0	0	0	0	0	0	0	0	0	0	0	0	0
A	2	499304	696	0	0	0	0	0	0	0	0	0	0	0	0	0
A	3	499316	682	2	0	0	0	0	0	0	0	0	0	0	0	0
B	4	495774	4204	22	0	0	0	0	0	0	0	0	0	0	0	0
B	5	495871	4107	22	0	0	0	0	0	0	0	0	0	0	0	0
B	6	495883	4082	35	0	0	0	0	0	0	0	0	0	0	0	0
C	7	489640	10250	109	1	0	0	0	0	0	0	0	0	0	0	0
C	8	489769	10099	131	1	0	0	0	0	0	0	0	0	0	0	0
C	9	489810	9980	204	6	0	0	0	0	0	0	0	0	0	0	0
D	10	466419	32492	1068	21	0	0	0	0	0	0	0	0	0	0	0
D	11	466432	32176	1337	54	1	0	0	0	0	0	0	0	0	0	0
D	12	467313	30676	1886	122	3	0	0	0	0	0	0	0	0	0	0
E	13	351323	124820	21191	2438	208	18	2	0	0	0	0	0	0	0	0
E	14	357637	114798	23157	3842	501	55	8	2	0	0	0	0	0	0	0
E	15	367730	99664	24586	6053	1501	345	89	23	9	0	0	0	0	0	0
F	16	171719	186205	98976	33258	8165	1426	220	27	4	0	0	0	0	0	0
F	17	198713	161221	84535	35837	13357	4503	1294	421	87	27	4	1	0	0	0
F	18	237758	129718	66144	33216	16797	8465	4236	2097	883	427	172	66	18	3	0

Table 8.1: *Number of defaults produced by Monte Carlo simulation from the KMV/CM model.*

		t-Model, ν=20: Number of Defaults														
Portfolios		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A	1	499288	705	6	1	0	0	0	0	0	0	0	0	0	0	0
A	2	499294	690	15	1	0	0	0	0	0	0	0	0	0	0	0
A	3	499364	619	16	1	0	0	0	0	0	0	0	0	0	0	0
B	4	496043	3833	114	10	0	0	0	0	0	0	0	0	0	0	0
B	5	495798	4057	134	9	2	0	0	0	0	0	0	0	0	0	0
B	6	495949	3860	176	13	2	0	0	0	0	0	0	0	0	0	0
C	7	489841	9645	483	28	3	0	0	0	0	0	0	0	0	0	0
C	8	490082	9409	474	32	3	0	0	0	0	0	0	0	0	0	0
C	9	490339	9006	585	61	8	0	0	1	0	0	0	0	0	0	0
D	10	468178	29042	2499	244	33	4	0	0	0	0	0	0	0	0	0
D	11	468493	28326	2767	356	49	9	0	0	0	0	0	0	0	0	0
D	12	469629	26793	3008	458	91	16	3	1	1	0	0	0	0	0	0
E	13	362933	107268	23879	4857	898	137	24	3	1	0	0	0	0	0	0
E	14	368626	98745	24519	6057	1559	384	88	18	4	0	0	0	0	0	0
E	15	377462	86874	24243	7597	2487	904	287	106	26	12	1	1	0	0	0
F	16	189302	169671	89607	35749	11596	3145	746	148	31	2	3	0	0	0	0
F	17	212905	149238	77534	35767	15115	6019	2223	806	281	86	23	2	1	0	0
F	18	248339	121538	61325	32263	17269	9252	5037	2627	1283	611	281	115	42	15	3

Table 8.2: *Number of defaults produced by Monte Carlo simulation from the t-model with degrees of freedom parameter $\nu = 20$.*

			t-Model, v=10: Number of Defaults														
			0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Portfolios	A	1	499335	629	29	6	0	1	0	0	0	0	0	0	0	0	0
		2	499340	607	46	5	1	1	0	0	0	0	0	0	0	0	0
		3	499378	579	36	6	1	0	0	0	0	0	0	0	0	0	0
	B	4	496236	3448	273	35	7	1	0	0	0	0	0	0	0	0	0
		5	496299	3358	288	48	6	1	0	0	0	0	0	0	0	0	0
		6	496174	3383	342	77	21	2	1	0	0	0	0	0	0	0	0
	C	7	490695	8335	797	141	29	3	0	0	0	0	0	0	0	0	0
		8	490871	8050	888	162	24	4	0	0	0	0	1	0	0	0	0
		9	490820	7993	948	180	42	15	1	0	1	0	0	0	0	0	0
	D	10	470584	25112	3417	692	165	22	7	0	0	0	1	0	0	0	0
		11	471277	24122	3553	806	165	57	18	2	0	0	0	0	0	0	0
		12	471867	22977	3808	981	266	75	18	5	2	0	1	0	0	0	0
	E	13	372029	93575	24993	6792	1939	483	137	40	9	2	1	0	0	0	0
		14	377586	86407	24631	7612	2563	837	255	76	26	5	2	0	0	0	0
		15	386833	75719	23319	8368	3458	1380	551	225	93	34	10	8	2	0	0
	F	16	204428	155936	82114	36210	14106	5018	1619	424	111	29	5	0	0	0	0
		17	225992	137849	71744	35291	16445	7393	3172	1349	498	185	58	19	4	1	0
		18	259422	113109	57263	31016	17358	9766	5554	3048	1740	923	499	182	86	28	6

Table 8.3: *Number of defaults produced by Monte Carlo simulation from the t-model with degrees of freedom parameter* $\nu = 10$.

Portfolios			*t*-Model, ν=5: Number of Defaults														
			0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	A	1	499448	435	83	26	6	2	0	0	0	0	0	0	0	0	0
		2	499497	402	73	19	5	1	3	0	0	0	0	0	0	0	0
		3	499510	371	79	18	14	7	0	1	0	0	0	0	0	0	0
	B	4	496827	2445	513	138	54	16	5	1	1	0	0	0	0	0	0
		5	496873	2419	471	151	59	20	4	2	1	0	0	0	0	0	0
		6	496983	2301	479	141	44	27	20	3	2	0	0	0	0	0	0
	C	7	492379	5839	1199	348	157	52	16	7	3	0	0	0	0	0	0
		8	492251	5849	1228	420	154	61	25	6	2	2	1	1	0	0	0
		9	492594	5497	1226	397	162	60	30	22	9	1	1	1	0	0	0
	D	10	475120	18258	4397	1417	511	195	72	21	8	1	0	0	0	0	0
		11	475443	17701	4427	1474	560	220	110	41	15	6	2	1	0	0	0
		12	476572	16486	4229	1554	647	267	154	58	22	8	2	0	1	0	0
	E	13	389601	71271	23633	9363	3787	1484	574	198	63	21	4	0	1	0	0
		14	394212	66225	22793	9378	4200	1823	832	341	135	42	16	1	2	0	0
		15	401931	58769	20791	9350	4512	2381	1180	558	304	128	61	24	10	1	0
	F	16	231569	131951	70072	35741	17319	8099	3380	1267	418	128	44	11	1	0	0
		17	249017	118664	62069	33566	17815	9594	4993	2394	1138	450	202	74	22	2	0
		18	277437	99111	50496	28946	17362	10751	6492	4036	2481	1447	787	392	175	69	18

Table 8.4: *Number of defaults produced by Monte Carlo simulation from the t-model with degrees of freedom parameter $\nu = 5$.*

Portfolios		t-Model, ν=3: Number of Defaults														
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A	1	499613	252	76	32	13	6	4	2	1	1	0	0	0	0	0
A	2	499601	260	73	38	15	8	5	0	0	0	0	0	0	0	0
A	3	499570	272	83	34	19	9	10	0	0	1	2	0	0	0	0
B	4	497569	1546	445	229	110	60	28	9	4	0	0	0	0	0	0
B	5	497651	1469	479	190	118	48	28	11	3	2	1	0	0	0	0
B	6	497577	1425	530	236	106	58	35	19	10	4	0	0	0	0	0
C	7	493831	3840	1262	546	261	132	74	33	15	3	3	0	0	0	0
C	8	493951	3698	1208	585	273	134	86	37	19	8	0	1	0	0	0
C	9	494304	3470	1086	516	316	143	69	47	31	12	3	2	1	0	0
D	10	480140	11964	4114	1899	980	502	238	98	46	16	3	0	0	0	0
D	11	480860	11312	3997	1849	991	512	253	120	72	24	6	2	2	0	0
D	12	481443	10817	3801	1835	976	527	289	161	79	40	20	10	2	0	0
E	13	407249	51218	20607	10423	5329	2810	1341	613	281	92	28	8	1	0	0
E	14	411142	47775	19447	10051	5430	2986	1591	902	378	197	68	26	7	0	0
E	15	416481	43403	17769	9398	5384	3192	1884	1141	670	368	176	87	34	11	2
F	16	261602	105772	58254	33689	19452	11044	5707	2807	1094	410	130	31	8	0	0
F	17	274999	97399	52329	30875	18773	11396	6651	3763	2130	982	438	198	44	22	1
F	18	298279	83544	43221	26258	17112	11232	7422	5112	3229	2022	1304	720	366	146	33

Table 8.5: *Number of defaults produced by Monte Carlo simulation from the t-model with degrees of freedom parameter* $\nu = 3$.

KMV/CM versus *t*-Model (v=20): Multipliers of Loss Distribution Properties															
						Value at Risk (VaR)					Expected Shortfall (ES)				
Portfolios		*Mean*	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
A	1	0.980	1.006	1.035	1.086	1*	1*	1*	1*	1*	1.009	1.009	1.009	1.009	1.009
A	2	1.021	1.054	1.052	1.212	1*	1*	1*	1*	1*	1.007	1.007	1.007	1.007	1.007
A	3	0.976	1.038	1.076	1.225	1*	1*	1*	1*	1*	1.049	1.049	1.049	1.049	1.049
B	4	0.959	1.010	1.090	1.260	1*	1*	1*	1*	0.950	1.024	1.024	1.024	1.024	1.011
B	5	1.060	1.116	1.032	1.132	1*	1*	1*	1*	1.050	1.042	1.042	1.042	1.042	1.044
B	6	1.027	1.100	1.076	1.254	1*	1*	1*	1*	1.006	1.043	1.043	1.043	1.043	1.042
C	7	1.017	1.096	1.092	1.303	1*	1*	1*	1.006	1.018	1.037	1.037	1.037	1.049	1.077
C	8	1.016	1.093	1.081	1.258	1*	1*	1*	1.004	1.022	1.048	1.048	1.048	1.047	1.071
C	9	0.998	1.095	1.118	1.353	1*	1*	1*	0.999	1.013	1.053	1.053	1.053	1.052	1.089
D	10	1.011	1.131	1.140	1.422	1*	0.936	1.009	1.028	1.035	1.067	1.044	1.069	1.146	1.243
D	11	1.002	1.129	1.153	1.436	1*	0.907	1.000	1.026	1.036	1.067	1.040	1.071	1.155	1.257
D	12	0.990	1.124	1.186	1.551	1*	0.865	0.982	1.023	1.100	1.066	1.029	1.065	1.152	1.243
E	13	0.993	1.150	1.182	1.330	1.010	1.029	1.159	1.097	1.127	1.083	1.125	1.128	1.143	1.175
E	14	0.999	1.178	1.209	1.421	1.010	1.047	1.120	1.110	1.193	1.090	1.134	1.135	1.180	1.207
E	15	1.002	1.168	1.164	1.316	1.005	1.123	1.084	1.140	1.128	1.079	1.110	1.117	1.152	1.159
F	16	0.997	1.159	1.181	1.164	1.039	1.074	1.079	1.096	1.108	1.072	1.089	1.092	1.109	1.113
F	17	0.997	1.125	1.119	1.133	1.029	1.057	1.060	1.074	1.084	1.059	1.069	1.075	1.085	1.091
F	18	1.000	1.094	1.062	1.080	1.024	1.042	1.048	1.057	1.057	1.046	1.051	1.055	1.059	1.060

Table 8.6: *Multipliers of loss distribution properties: property of the t loss distribution ($\nu = 20$) divided by the corresponding property of the KMV/CM loss distribution. *Two corresponding properties have value zero for both models. The ratio is zero divided by zero. We then define the multiplier to have value 1.*

KMV/CM versus *t*-Model (v=10): Multipliers of Loss Distribution Properties																
						Value at Risk (VaR)					Expected Shortfall (ES)				
		Mean	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
Portfolios	A	1	0.983	1.129	1.218	1.779	1*	1*	1*	1*	1*	1.083	1.083	1.083	1.083	1.083
	A	2	1.023	1.206	1.266	2.039	1*	1*	1*	1*	1*	1.078	1.078	1.078	1.078	1.078
	A	3	0.989	1.143	1.201	1.668	1*	1*	1*	1*	1*	1.088	1.088	1.088	1.088	1.088
	B	4	0.963	1.123	1.242	1.814	1*	1*	1*	1*	0.921	1.081	1.081	1.081	1.081	1.053
	B	5	1.008	1.198	1.216	1.731	1*	1*	1*	1*	0.974	1.124	1.124	1.124	1.124	1.082
	B	6	1.057	1.342	1.308	2.115	1*	1*	1*	1*	0.985	1.138	1.138	1.138	1.138	1.126
	C	7	0.993	1.196	1.255	1.838	1*	1*	1*	0.983	1.026	1.106	1.106	1.106	1.102	1.176
	C	8	1.007	1.246	1.282	2.038	1*	1*	1*	0.988	1.046	1.129	1.129	1.129	1.126	1.210
	C	9	1.034	1.315	1.314	2.102	1*	1*	1*	1.005	1.042	1.148	1.148	1.148	1.141	1.232
	D	10	1.000	1.281	1.341	2.084	1*	0.767	0.988	1.054	1.267	1.142	1.074	1.143	1.324	1.518
	D	11	0.990	1.293	1.374	2.196	1*	0.713	0.984	1.053	1.328	1.157	1.072	1.147	1.337	1.512
	D	12	1.009	1.347	1.373	2.135	1*	0.691	0.986	1.061	1.410	1.173	1.084	1.163	1.359	1.494
	E	13	1.001	1.345	1.381	1.781	1.021	1.130	1.276	1.215	1.313	1.173	1.260	1.261	1.324	1.373
	E	14	1.001	1.360	1.384	1.810	1.013	1.175	1.207	1.251	1.344	1.171	1.251	1.257	1.343	1.378
	E	15	0.993	1.331	1.315	1.629	0.999	1.198	1.144	1.263	1.261	1.146	1.206	1.226	1.287	1.303
	F	16	0.997	1.327	1.345	1.327	1.075	1.147	1.152	1.198	1.216	1.145	1.176	1.187	1.219	1.226
	F	17	0.997	1.262	1.227	1.265	1.060	1.111	1.124	1.154	1.167	1.120	1.141	1.154	1.171	1.180
	F	18	0.998	1.198	1.127	1.162	1.048	1.083	1.099	1.117	1.126	1.093	1.107	1.114	1.126	1.130

Table 8.7: *Multipliers of loss distribution properties: property of the t loss distribution ($\nu = 10$) divided by the corresponding property of the KMV/CM loss distribution. *Two corresponding properties have value zero for both models. The ratio is zero divided by zero. We then define the multiplier to have value 1.*

		KMV/CM versus *t*-Model (v=5): Multipliers of Loss Distribution Properties													
						Value at Risk (VaR)					Expected Shortfall (ES)				
Portfolios		*Mean*	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
A	1	0.984	1.481	1.464	2.506	1*	1*	1*	1*	1*	1.307	1.307	1.307	1.307	1.307
A	2	0.901	1.376	1.735	3.967	1*	1*	1*	1*	1*	1.246	1.246	1.246	1.246	1.246
A	3	1.032	1.914	1.846	4.414	1*	1*	1*	1*	1*	1.440	1.440	1.440	1.440	1.440
B	4	0.992	1.603	1.646	3.424	1*	1*	1*	1*	0.770	1.322	1.322	1.322	1.322	1.205
B	5	1.014	1.652	1.634	3.353	1*	1*	1*	1*	0.779	1.339	1.339	1.339	1.339	1.211
B	6	0.993	1.715	1.790	4.124	1*	1*	1*	1*	0.690	1.355	1.355	1.355	1.355	1.210
C	7	0.974	1.607	1.719	3.726	1*	1*	1*	0.886	1.054	1.324	1.324	1.324	1.254	1.472
C	8	1.032	1.795	1.776	4.138	1*	1*	1*	0.911	1.083	1.362	1.362	1.362	1.311	1.548
C	9	1.013	1.846	1.868	4.520	1*	1*	1*	0.878	1.065	1.394	1.394	1.394	1.311	1.561
D	10	1.011	1.723	1.693	3.339	1*	0**	0.957	1.183	1.647	1.365	1.141	1.313	1.719	1.972
D	11	1.010	1.773	1.762	3.703	1*	0**	0.941	1.221	1.671	1.381	1.153	1.322	1.736	1.969
D	12	1.006	1.823	1.767	3.527	1*	0**	0.911	1.256	1.714	1.403	1.185	1.322	1.742	1.936
E	13	0.995	1.730	1.689	2.467	1.023	1.315	1.447	1.489	1.589	1.329	1.484	1.505	1.621	1.693
E	14	0.997	1.740	1.656	2.404	1.009	1.328	1.351	1.498	1.627	1.317	1.457	1.487	1.627	1.684
E	15	0.991	1.683	1.519	2.017	0.985	1.308	1.271	1.481	1.512	1.274	1.385	1.435	1.534	1.558
F	16	0.996	1.646	1.580	1.572	1.144	1.264	1.287	1.358	1.377	1.271	1.322	1.345	1.389	1.404
F	17	1.000	1.535	1.380	1.427	1.125	1.210	1.249	1.299	1.318	1.234	1.272	1.294	1.319	1.328
F	18	1.002	1.397	1.205	1.241	1.102	1.168	1.192	1.226	1.232	1.180	1.206	1.218	1.228	1.226

Table 8.8: *Multipliers of loss distribution properties: property of the t loss distribution ($\nu = 5$) divided by the corresponding property of the KMV/CM loss distribution. *Two corresponding properties have value zero for both models. The ratio is zero divided by zero. We then define the multiplier to have value 1.**Value of KMV/CM property is zero and corresponding value of t-property is greater than zero.*

KMV/CM versus *t*-Model (v=3): Multipliers of Loss Distribution Properties															
						Value at Risk (VaR)					Expected Shortfall (ES)				
		Mean	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
Portfolios A	1	0.897	2.055	2.194	6.022	1*	1*	1*	1*	1*	1.699	1.699	1.699	1.699	1.699
A	2	0.916	1.903	1.964	4.680	1*	1*	1*	1*	1*	1.598	1.598	1.598	1.598	1.598
A	3	1.096	2.744	2.226	6.622	1*	1*	1*	1*	1*	1.744	1.744	1.744	1.744	1.744
B	4	0.968	2.221	2.027	4.887	1*	1*	1*	1*	0**	1.683	1.683	1.683	1.683	1.302
B	5	0.974	2.234	2.047	5.102	1*	1*	1*	1*	0**	1.712	1.712	1.712	1.712	1.325
B	6	1.038	2.535	2.064	5.141	1*	1*	1*	1*	0**	1.763	1.763	1.763	1.763	1.371
C	7	1.010	2.350	2.059	5.134	1*	1*	1*	0.720	1.100	1.696	1.696	1.696	1.453	1.857
C	8	1.024	2.455	2.050	4.946	1*	1*	1*	0.680	1.108	1.732	1.732	1.732	1.475	1.903
C	9	0.990	2.471	2.171	5.598	1*	1*	1*	0.573	1.069	1.770	1.770	1.770	1.438	1.872
D	10	1.013	2.385	2.029	4.567	1*	0**	0.792	1.499	2.017	1.712	1.431	1.497	2.168	2.537
D	11	0.985	2.345	2.052	4.606	1*	0**	0.741	1.472	2.037	1.728	1.443	1.475	2.137	2.469
D	12	0.995	2.451	2.090	4.730	1*	0**	0.702	1.467	2.052	1.753	1.480	1.471	2.122	2.420
E	13	0.997	2.252	1.942	3.011	1.003	1.472	1.640	1.783	1.935	1.504	1.737	1.792	1.955	2.034
E	14	0.999	2.256	1.882	2.893	0.979	1.463	1.535	1.787	1.953	1.482	1.696	1.762	1.943	2.006
E	15	1.004	2.188	1.699	2.368	0.943	1.430	1.443	1.764	1.811	1.425	1.604	1.690	1.830	1.856
F	16	0.996	2.064	1.770	1.728	1.240	1.412	1.457	1.543	1.571	1.425	1.494	1.523	1.569	1.577
F	17	1.000	1.862	1.502	1.538	1.210	1.336	1.388	1.447	1.468	1.360	1.412	1.438	1.461	1.463
F	18	1.001	1.633	1.273	1.301	1.167	1.267	1.304	1.341	1.345	1.277	1.314	1.326	1.332	1.323

Table 8.9: *Multipliers of loss distribution properties: property of the t loss distribution ($\nu = 3$) divided by the corresponding property of the KMV/CM loss distribution. *Two corresponding properties have value zero for both models. The ratio is zero divided by zero. We then define the multiplier to have value 1.**Value of KMV/CM property is zero and corresponding value of t-property is greater than zero.*

Portfolio No. 5					Value at Risk (VaR)					Expected Shortfall (ES)				
	Mean	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
KMV/CM	4.96E+02	3.51E+07	1.30E+01	1.83E+02	0	0	0	0	5.38E+04	6.00E+04	6.00E+04	6.00E+04	6.00E+04	7.76E+04
t, v=20	5.26E+02	3.91E+07	1.35E+01	2.07E+02	0	0	0	0	5.65E+04	6.26E+04	6.26E+04	6.26E+04	6.26E+04	8.10E+04
t, v=10	5.00E+02	4.20E+07	1.59E+01	3.17E+02	0	0	0	0	5.24E+04	6.75E+04	6.75E+04	6.75E+04	6.75E+04	8.40E+04
t, v=5	5.03E+02	5.79E+07	2.13E+01	6.14E+02	0	0	0	0	4.19E+04	8.04E+04	8.04E+04	8.04E+04	8.04E+04	9.40E+04
t, v=3	4.83E+02	7.84E+07	2.67E+01	9.34E+02	0	0	0	0	0	1.03E+05	1.03E+05	1.03E+05	1.03E+05	1.03E+05

Portfolio No. 11					Value at Risk (VaR)					Expected Shortfall (ES)				
	Mean	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
KMV/CM	4.21E+03	3.01E+08	4.52E+00	2.47E+01	0	4.24E+04	6.82E+04	9.06E+04	9.62E+04	6.27E+04	7.51E+04	8.77E+04	1.04E+05	1.15E+05
t, v=20	4.22E+03	3.39E+08	5.21E+00	3.55E+01	0	3.84E+04	6.82E+04	9.29E+04	9.96E+04	6.69E+04	7.80E+04	9.39E+04	1.20E+05	1.45E+05
t, v=10	4.17E+03	3.89E+08	6.21E+00	5.43E+01	0	3.03E+04	6.71E+04	9.54E+04	1.28E+05	7.25E+04	8.05E+04	1.01E+05	1.39E+05	1.74E+05
t, v=5	4.25E+03	5.33E+08	7.96E+00	9.16E+01	0	0	6.41E+04	1.11E+05	1.61E+05	8.66E+04	8.66E+04	1.16E+05	1.81E+05	2.26E+05
t, v=3	4.14E+03	7.05E+08	9.27E+00	1.14E+02	0	0	5.05E+04	1.33E+05	1.96E+05	1.08E+05	1.08E+05	1.29E+05	2.23E+05	2.84E+05

Portfolio No. 17					Value at Risk (VaR)					Expected Shortfall (ES)				
	Mean	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
KMV/CM	6.31E+04	5.51E+09	1.45E+00	5.54E+00	1.66E+05	2.10E+05	2.42E+05	3.06E+05	3.44E+05	2.27E+05	2.69E+05	2.99E+05	3.60E+05	3.97E+05
t, v=20	6.29E+04	6.20E+09	1.62E+00	6.28E+00	1.71E+05	2.22E+05	2.56E+05	3.29E+05	3.73E+05	2.40E+05	2.88E+05	3.21E+05	3.90E+05	4.33E+05
t, v=10	6.30E+04	6.95E+09	1.77E+00	7.01E+00	1.76E+05	2.33E+05	2.72E+05	3.53E+05	4.01E+05	2.54E+05	3.07E+05	3.45E+05	4.21E+05	4.68E+05
t, v=5	6.31E+04	8.46E+09	2.00E+00	7.91E+00	1.87E+05	2.54E+05	3.02E+05	3.98E+05	4.53E+05	2.80E+05	3.42E+05	3.86E+05	4.75E+05	5.27E+05
t, v=3	6.31E+04	1.03E+10	2.17E+00	8.53E+00	2.01E+05	2.80E+05	3.36E+05	4.43E+05	5.05E+05	3.08E+05	3.80E+05	4.29E+05	5.26E+05	5.80E+05

Table 8.10: *Loss distribution properties of KMV/CM and t-model for 3 specific portfolios.*

Chapter 9

Alternative risk transfer

9.1 Definition

Suppose we want to insure some of our potential risk of incurring high losses on our credit portfolio (denote by L the portfolio losses). Say we want to hand over all losses which are above a certain threshold U to a third party, such that our maximal loss never exceeds the limit U. Hence the amount of money V the third party is obliged to pay at time T is

$$V = \max(L - U, 0).$$

It is clear that the price of such a contract highly depends on the way the portfolio loss distribution (at our fixed time horizon T) is modeled and hence what kind of copula is incorporated in the latent variable model. If all our loans were tradeable at an exchange and if interest rates were taken to be constant, then we could consider to price the contract under the 'no-arbitrage' assumption. Thus there would exist a risk neutral probability measure Q such that the price p of our contract would yield

$$p = \frac{1}{1+r}\mathbb{E}_Q[V]$$

with

$$r \quad = \quad \text{non-stochastic interest rate over time period } [0, T].$$

Since the tradeability assumption is violated for most portfolios and the task to find the 'proper' Q out of infinitely many is practically impossible for credit portfolios, we will use as a pricing tool the discounted expected payoff of our contract under the true probability measure given by the model, i.e.

$$p := \frac{1}{1+r}\mathbb{E}[V].$$

We will consider a contract with a threshold U equal to \$100'000. In practice this threshold is often a certain quantile of the portfolio loss distribution. Since we compare different models, a relative threshold (such as a quantile), would not make much sense.

Now the contract will be priced for our various homogeneous portfolios under the KMV/CM and under the t-model for the same parameter values $\nu = 3, 5, 10, 20$ as used before and under the empirical loss distribution produced by the Monte Carlo simulation. We choose our time horizon T equal to one year and the risk free interest rate r equal to 5%.

9.2 Results and discussion

The prices of the contract for the various portfolios calculated under the different models are produced in the following table:

			Contract Prices (in $)				
			Models				
			KMV/CM	t, v=20	t, v=10	t, v=5	t, v=3
Portfolios	A	1	0	0.38	2.97	10.26	19.47
		2	0	0.86	3.95	9.30	18.29
		3	0.03	1.04	3.18	14.77	24.75
	B	4	1.26	7.63	21.45	71.86	133.48
		5	1.41	7.94	24.12	74.02	128.76
		6	1.75	10.84	36.08	80.30	151.23
	C	7	5.34	30.09	68.23	185.00	345.19
		8	6.46	29.16	78.89	214.59	364.42
		9	12.06	43.29	96.85	233.70	369.26
	D	10	54.58	173.82	332.11	704.77	1190.59
		11	76.94	211.97	380.89	775.33	1218.42
		12	116.44	258.69	460.81	864.23	1300.81
	E	13	1513.24	2264.02	3121.51	4679.93	6482.11
		14	1926.67	2811.61	3638.30	5180.14	6981.33
		15	2771.01	3646.24	4428.74	5921.41	7742.30
	F	16	13012.75	14819.31	16590.90	19669.88	23322.56
		17	16120.79	17643.18	19228.09	22202.99	25424.16
		18	20832.37	22146.74	23444.50	25934.69	28630.67

Table 9.1: *Prices of the contract for the KMV/CM and the t-models. The loss threshold U is set at $100'000.*

These prices provide some additional information on the amount of mass in the tails of the various loss distributions. In each row of table 9.1 we see again that the more tail dependence incorporated in the copula of the latent variables, the higher the price of the contract turns out to be. Scrolling down each column we conclude that the lower the credit quality, the higher the price (as expected).

As in the previous comparison of the KMV/CM and the t-model we notice that the portfolios of high rating grades react much more sensitive to an increase in tail dependence. The price of the contract for portfolio No. 4 under the t-model (with degrees of freedom parameter $\nu = 3$) is more than 100 times as large as for the KMV/CM model, where as the relative increase in price for portfolio No. 18 is not even 50%! We must admit that we didn't expect the results to be so drastic!

Another surprising empirical fact is that prices for the KMV/CM model react much faster to changes in default correlation ρ^X for portfolios of all rating classes! Consider portfolio No. 10 and 12: the price of the contract under the KMV/CM model for portfolio No. 12 is more than double the price of portfolio 10, where as the relative increase under the t-model ($\nu = 3$) is not even 10%.

Chapter 10

Increasing portfolio size

It would be extremely interesting to see how the KMV/CM and the t-model behave for homogeneous portfolios of larger size. Intuitively we would assume that by increasing the number of obligors the tail dependence property of the t-copula would shift even more mass out in the tail, i.e. our loss distribution would 'worsen' with increasing portfolio size relative to the KMV/CM model.

We will investigate on this topic by letting the size of our homogeneous portfolios grow from 14 to 200, with intermediate sizes of 50 and 100. Because we didn't have enough time to simulate all 18 portfolios (processing time is not linear in portfolio size n), we chose for comparison 3 portfolios out of the 18, again No. 5, 11 and 17 (refer to table 6.1, page 31). For these portfolios we compared their loss distribution for the KMV/CM and the t-model for $\nu = 3, 5, 10, 20$.

For comparison we will take a look at the ratio of the t-loss distribution properties and the KMV/CM properties, i.e. the multipliers, for each of the 3 portfolios. Again we use Monte Carlo simulation with 0.5 million runs to produce the empirical loss distributions of the portfolios of different sizes. The results are again produced in the following three tables.

10.1 Results and discussion

The three tables 10.1 - 10.3 exhibit that the relative mean is independent of portfolio size. We have already noticed in the previous comparison of the KMV/CM and the t-model for portfolio size 14 that the relative mean was independent of default correlation ρ^X and of marginal default probability π aswell.

10.1.1 Analysis of each portfolio

For all three portfolios we see that the relative values for the variance, skewness, kurtosis and expected shortfall (at all levels) is increasing as portfolio size mounts. Hence our intuition was correct that the impact of the t-copula grows for increasing portfolio size and that the corresponding loss distribution worsens substantially relative to the KMV/CM model; the higher ν, the worse.

Consider the multipliers (t-model $\nu = 20$ versus KMV/CM) for portfolio No. 11. For the portfolio of size 14 we can hardly see any difference in the loss distributions of the two models. All multiplier values are very close to 1. But for the portfolio of size 200 the difference is tremendous: double the variance, 6 times the amount of kurtosis, double the amount of expected shortfall at the level 99.5%! Not to mention the impact if we let portfolio size increase for the t-model with very high tail dependence. Consider again portfolio No. 10 in the last sub-table of table 10.1.

The t-model against KMV/CM gives 20 times higher variance and 50 times higher kurtosis for a portfolio of size 200! The values of expected shortfall show the same effect. We can only imagine what an impact this change in copula would have on a 'typical' credit portfolio containing a few thousand obligors!

10.1.2 Comparison of the portfolios

When scrolling through the tables 10.1 to 10.3 we see straight away that the relative impact of the t-copula is much weaker on portfolio No. 17 than on the other two portfolios of higher rating grade. Consider the relative values of ES at the level 99.5% ($\nu = 3$), i.e. the lowest right cell in each of the three tables. The multiplier for the first two portfolios with high and average credit quality takes a value above 7, where as the one for the lowest rated portfolio is below the value of 2!

10.1.3 Loss histograms

At the end of our investigations on the various models we show a graphical comparison of the simulated loss distribution for portfolio No. 11, size 200, for the KMV/CM and the t-models. The histograms are produced from the loss data retrieved from the Monte Carlo simulation. These (unscaled) distributions were cut off at the level of 100 observations to show more clearly the mass contained in the tails of the distributions. The histogram is plotted for each of the 5 models (KMV/CM, t-model with $\nu = 20, 10, 5, 3$).

Figure 10.1 confirms all previous results!

Portfolio No. 5		KMV/CM versus t-Model (v=20): Multipliers of Loss Distribution Properties													
						Value at Risk (VaR)					Expected Shortfall (ES)				
		Mean	Var	Skew	Kurt	90%	95%	97%	99%	99.5%	90%	95%	97%	99%	99.5%
Size	14	1.060	1.116	1.032	1.132	1*	1*	1*	1*	1.050	1.042	1.042	1.042	1.042	1.044
	50	0.967	1.191	1.364	2.412	1*	1*	1*	0.999	1.029	1.107	1.107	1.107	1.136	1.240
	100	1.002	1.486	1.820	6.092	1*	0**	0.935	1.058	1.309	1.213	1.090	1.173	1.417	1.670
	200	1.000	1.896	2.217	6.521	0**	0.913	1.033	1.537	1.725	1.249	1.293	1.448	1.858	1.975

Portfolio No. 5		KMV/CM versus t-Model (v=10): Multipliers of Loss Distribution Properties													
						Value at Risk (VaR)					Expected Shortfall (ES)				
		Mean	Var	Skew	Kurt	90%	95%	97%	99%	99.5%	90%	95%	97%	99%	99.5%
Size	14	1.008	1.198	1.216	1.731	1*	1*	1*	1*	0.974	1.124	1.124	1.124	1.124	1.082
	50	0.979	1.750	2.146	7.613	1*	1*	1*	0.992	1.075	1.334	1.334	1.334	1.374	1.656
	100	1.011	2.619	2.899	12.027	1*	0**	0.682	1.109	1.709	1.585	1.424	1.372	2.006	2.512
	200	1.003	4.019	4.165	22.056	0**	0.552	0.968	1.821	2.426	1.769	1.532	1.897	2.836	3.271

Portfolio No. 5		KMV/CM versus t-Model (v=5): Multipliers of Loss Distribution Properties													
						Value at Risk (VaR)					Expected Shortfall (ES)				
		Mean	Var	Skew	Kurt	90%	95%	97%	99%	99.5%	90%	95%	97%	99%	99.5%
Size	14	1.014	1.652	1.634	3.353	1*	1*	1*	1*	0.779	1.339	1.339	1.339	1.339	1.211
	50	0.976	3.456	3.242	14.151	1*	1*	1*	0.823	1.158	1.998	1.998	1.998	1.791	2.464
	100	0.970	5.810	4.628	26.493	1*	0**	0**	1.066	1.978	2.591	2.328	1.915	2.779	3.867
	200	1.017	11.550	6.315	44.113	0**	0**	0.106	1.848	3.304	3.440	2.639	2.393	4.520	5.812

Portfolio No. 5		KMV/CM versus t-Model (v=3): Multipliers of Loss Distribution Properties													
						Value at Risk (VaR)					Expected Shortfall (ES)				
		Mean	Var	Skew	Kurt	90%	95%	97%	99%	99.5%	90%	95%	97%	99%	99.5%
Size	14	0.974	2.234	2.047	5.102	1*	1*	1*	1*	0**	1.712	1.712	1.712	1.712	1.325
	50	0.983	5.974	3.677	15.510	1*	1*	1*	0**	1.074	3.157	3.157	3.157	2.159	3.173
	100	1.036	12.207	5.063	27.073	1*	0**	0**	0.702	1.941	4.825	4.336	3.567	3.593	5.588
	200	0.988	20.560	6.992	47.227	0**	0**	0**	0.996	3.207	6.515	4.997	4.494	5.390	7.668

Table 10.1: *Increasing portfolio size for portfolio No. 5.**Two corresponding properties have both value zero. The ratio is zero divided by zero. We then define the multiplier to have value 1.***Value of KMV/CM property is zero and corresponding value of t-property is greater than zero.*

Portfolio No. 11	KMV/CM versus *t*-Model (ν=20): Multipliers of Loss Distribution Properties													
					Value at Risk (VaR)					Expected Shortfall (ES)				
Size	*Mean*	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
14	1.002	1.129	1.153	1.436	1*	0.907	1.000	1.026	1.036	1.067	1.040	1.071	1.155	1.257
50	0.996	1.426	1.530	2.434	0.960	1.038	1.248	1.249	1.374	1.160	1.264	1.344	1.415	1.507
100	0.995	1.828	1.965	3.497	1.024	1.190	1.265	1.519	1.619	1.319	1.423	1.522	1.700	1.811
200	0.995	2.453	2.468	4.782	1.112	1.324	1.463	1.752	1.911	1.474	1.641	1.765	2.013	2.161

Portfolio No. 11	KMV/CM versus *t*-Model (ν=10): Multipliers of Loss Distribution Properties													
					Value at Risk (VaR)					Expected Shortfall (ES)				
Size	*Mean*	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
14	0.990	1.293	1.374	2.196	1*	0.713	0.984	1.053	1.328	1.157	1.072	1.147	1.337	1.512
50	1.002	2.099	2.159	4.753	0.868	1.068	1.427	1.606	1.831	1.316	1.544	1.700	1.934	2.120
100	0.993	3.011	2.858	6.852	0.996	1.288	1.610	2.006	2.292	1.579	1.823	2.035	2.450	2.700
200	1.005	4.626	3.498	8.288	1.105	1.547	1.854	2.483	2.889	1.876	2.244	2.516	3.067	3.383

Portfolio No. 11	KMV/CM versus *t*-Model (ν=5): Multipliers of Loss Distribution Properties													
					Value at Risk (VaR)					Expected Shortfall (ES)				
Size	*Mean*	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
14	1.010	1.773	1.762	3.703	1*	0**	0.941	1.221	1.671	1.381	1.153	1.322	1.736	1.969
50	0.995	3.663	2.954	7.953	0.345	1.033	1.596	2.161	2.653	1.490	1.968	2.287	2.836	3.202
100	0.995	5.995	3.879	11.103	0.813	1.302	1.789	2.936	3.535	1.930	2.471	2.928	3.810	4.284
200	0.999	10.072	4.728	13.006	0.829	1.600	2.248	3.769	4.668	2.360	3.148	3.760	4.998	5.638

Portfolio No. 11	KMV/CM versus *t*-Model (ν=3): Multipliers of Loss Distribution Properties													
					Value at Risk (VaR)					Expected Shortfall (ES)				
Size	*Mean*	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
14	0.985	2.345	2.052	4.606	1*	0**	0.741	1.472	2.037	1.728	1.443	1.475	2.137	2.469
50	0.986	5.857	3.461	9.870	0**	0.846	1.541	2.770	3.569	2.059	2.309	2.859	3.824	4.363
100	1.022	10.729	4.353	12.413	0**	1.066	1.879	4.054	5.116	2.280	3.097	3.907	5.419	6.102
200	0.998	17.471	5.310	14.880	0.374	1.284	2.358	5.056	6.542	2.625	3.886	4.923	6.939	7.872

Table 10.2: *Increasing portfolio size for portfolio No. 11.*Two corresponding properties have both value zero. The ratio is zero divided by zero. We then define the multiplier to have value 1.**Value of KMV/CM property is zero and corresponding value of t-property is greater than zero.*

Portfolio No. 17		KMV/CM versus *t*-Model (v=20): Multipliers of Loss Distribution Properties													
						Value at Risk (VaR)					Expected Shortfall (ES)				
		Mean	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
Size	*14*	0.997	1.125	1.119	1.133	1.029	1.057	1.060	1.074	1.084	1.059	1.069	1.075	1.085	1.091
	50	1.000	1.308	1.211	1.201	1.075	1.103	1.117	1.143	1.156	1.113	1.131	1.142	1.163	1.175
	100	1.002	1.398	1.218	1.195	1.089	1.119	1.143	1.167	1.190	1.134	1.156	1.169	1.191	1.201
	200	1.001	1.464	1.209	1.186	1.097	1.133	1.153	1.186	1.196	1.146	1.168	1.182	1.203	1.213

Portfolio No. 17		KMV/CM versus *t*-Model (v=10): Multipliers of Loss Distribution Properties													
						Value at Risk (VaR)					Expected Shortfall (ES)				
		Mean	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
Size	*14*	0.997	1.262	1.227	1.265	1.060	1.111	1.124	1.154	1.167	1.120	1.141	1.154	1.171	1.180
	50	0.997	1.604	1.361	1.343	1.136	1.193	1.221	1.266	1.288	1.211	1.245	1.265	1.297	1.314
	100	0.998	1.798	1.364	1.309	1.164	1.231	1.270	1.318	1.346	1.252	1.292	1.315	1.346	1.357
	200	0.999	1.927	1.351	1.304	1.178	1.247	1.287	1.349	1.370	1.271	1.315	1.341	1.377	1.390

Portfolio No. 17		KMV/CM versus *t*-Model (v=5): Multipliers of Loss Distribution Properties													
						Value at Risk (VaR)					Expected Shortfall (ES)				
		Mean	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
Size	*14*	1.000	1.535	1.380	1.427	1.125	1.210	1.249	1.299	1.318	1.234	1.272	1.294	1.319	1.328
	50	0.999	2.215	1.530	1.457	1.260	1.366	1.421	1.491	1.515	1.392	1.450	1.480	1.518	1.530
	100	0.999	2.588	1.543	1.451	1.301	1.422	1.486	1.573	1.611	1.454	1.523	1.560	1.606	1.617
	200	1.000	2.851	1.500	1.400	1.330	1.468	1.522	1.615	1.642	1.489	1.560	1.597	1.642	1.652

Portfolio No. 17		KMV/CM versus *t*-Model (v=3): Multipliers of Loss Distribution Properties													
						Value at Risk (VaR)					Expected Shortfall (ES)				
		Mean	*Var*	*Skew*	*Kurt*	*90%*	*95%*	*97%*	*99%*	*99.5%*	*90%*	*95%*	*97%*	*99%*	*99.5%*
Size	*14*	1.000	1.862	1.502	1.538	1.210	1.336	1.388	1.447	1.468	1.360	1.412	1.438	1.461	1.463
	50	1.000	2.966	1.658	1.535	1.398	1.569	1.639	1.717	1.736	1.587	1.665	1.700	1.733	1.740
	100	0.999	3.561	1.645	1.478	1.459	1.640	1.734	1.819	1.854	1.673	1.762	1.803	1.837	1.836
	200	1.000	4.000	1.599	1.427	1.495	1.689	1.784	1.882	1.911	1.722	1.817	1.858	1.898	1.896

Table 10.3: *Increasing portfolio size for portfolio No. 17.*

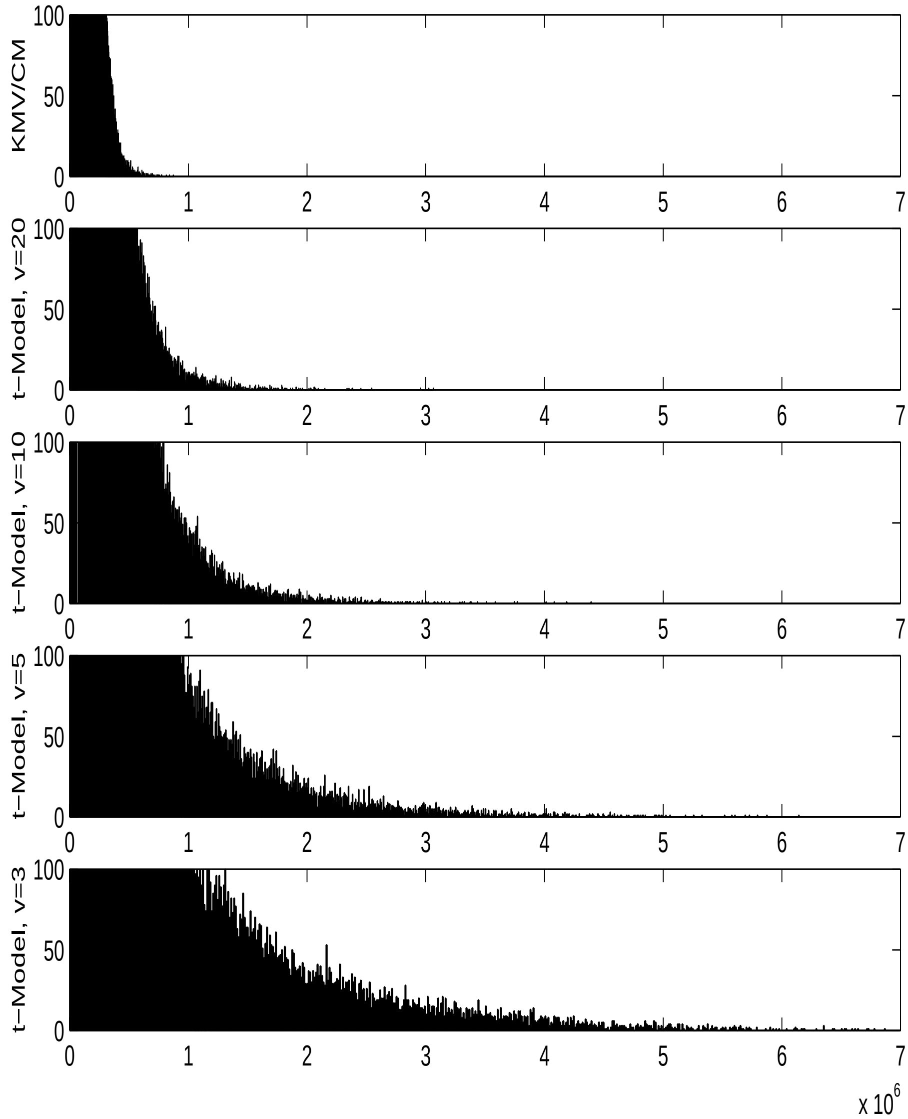


Figure 10.1: *Unscaled loss histograms of portfolio No. 11, size 200, for the various models.*

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