Common Poisson Shock Models:
Applications to Insurance and Credit Risk Modelling

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Abstract

The idea of using common Poisson shock processes to model dependent event frequencies is well known in the reliability literature. In this paper we examine these models in the context of insurance loss modelling and credit risk modelling. To do this we set up a very general common shock framework for losses of a number of different types that allows for both dependence in loss frequencies across types and dependence in loss severities. Our aims are threefold: to demonstrate that the common shock model is a very natural way of approaching the modelling of dependent losses in an insurance or risk management context; to provide a number of analytical results concerning the nature of the dependence implied by the common shock specification; to examine the aggregate loss distribution that results from the model and the sensitivity of its tail to the specification of the model parameters.

1 Introduction

Suppose we are interested in losses of several different *types* and in the numbers of these losses that may occur over a given time horizon. More concretely, we might be interested in insurance losses occurring in several different lines of business or several different countries. In credit risk modelling we might be interested in losses related to the default of various types of counterparty. Further suppose that there are strong a priori reasons for believing that the frequencies of losses of different types are dependent. A natural approach to modelling this dependence is to assume that all losses can be related to a series of underlying and independent *shock* processes. In insurance these shocks might be natural catastrophes; in credit risk modelling they might be a variety of economic events such as local or global recessions; in operational risk modelling they might be the failure

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of various IT systems. When a shock occurs this may cause losses of several different types; the common shock causes the numbers of losses of each type to be dependent.

This kind of construction is very familiar in the *reliability* literature where the failure of different kinds of system components is modelled as being contingent on independent shocks that may affect one or more components. It is commonly assumed that the different varieties of shocks arrive as independent *Poisson processes*, in which case the counting processes for the different loss types are also Poisson and can be easily handled analytically. In reliability such models are known as *fatal shock models*, when the shock always destroys the component, and *non-fatal shock models*, or *not-necessarily-fatal shock models*, when components have a chance of surviving the shock. A good basic reference on such models is Barlow and Proschan (1975) and the ideas go back to Marshall and Olkin (1967). Undoubtedly, one of the practical attractions of such models is the simplicity of simulating from them in Monte Carlo procedures.

In this paper we set up a very general Poisson shock model; the dimension is arbitrary and shocks may be fatal or not-necessarily-fatal. We review and generalise results for the multivariate Poisson process counting numbers of failures of different types. We also consider the modelling of dependent severities. When a loss occurs, whether in insurance or credit risk modelling, a loss size may be assigned to it. It is often natural to assume that losses of different types caused by the same underlying shock also have dependent severities. We set up general multivariate compound Poisson processes to model the losses of each type.

In analysing these multivariate Poisson and compound Poisson processes we address two main issues:

- What can we say about the dependence structure of the multivariate distribution of the cumulative losses of different types at some fixed point in time?
- What can we say about the overall univariate aggregate loss distribution, in particular its tail and the sensitivity of this tail to the exact specification of the shock model?

The paper is structured as follows. In Section 2 we describe the general not-necessarily-fatal-shock model with dependent loss frequencies and dependent loss severities. In Section 3 we ignore loss severities and examine the multivariate distribution of loss frequencies and the consequences for the aggregate loss frequency distribution of specifying the shock structure in different ways. An important key to analysing the model is to see that it may be written in terms of an *equivalent fatal shock model*. This facilitates the approximation of the aggregate loss frequency distribution using the *Panjer recursion* approach and also makes it very easy to analyse the multivariate exponential distribution of the *times to the first losses* of each type. In section 4 the analysis is generalised by including dependent loss severities. The dependence in severities is created using *copula techniques* and the object of interest is now the *tail* of the overall aggregate loss distribution. Sections 3 and 4 are illustrated with a stylized insurance example; Section 5 consists of an extended example of how the model might be applied to the modelling of portfolio credit risk.

2 The Model

2.1 Loss Frequencies

Suppose there are m different types of shock or event and, for $e = 1, \ldots, m$, let

$$\{N^{(e)}(t), t \geq 0\}$$

be a Poisson process with intensity $\lambda^{(e)}$ recording the number of events of type e occurring in $(0, t]$. Assume further that these shock counting processes are independent. Consider losses of n different types and, for $j = 1, \ldots, n$, let

$$\{N_j(t), t \geq 0\}$$

be a counting process that records the *frequency* of losses of the jth type occurring in $(0, t]$.

At the rth occurrence of an event of type e the Bernoulli variable $I_{j,r}^{(e)}$ indicates whether a loss of type j occurs. The vectors

$$\mathbf{I}_r^{(e)} = (I_{1,r}^{(e)}, \ldots, I_{n,r}^{(e)})'$$

for $r = 1, \ldots, N^{(e)}(t)$ are considered to be independent and identically distributed with a multivariate Bernoulli distribution. In other words, each new event represents a new independent opportunity to incur a loss but, for a fixed event, the loss trigger variables for losses of different types may be dependent. The form of the dependence depends on the specification of the multivariate Bernoulli distribution and independence is a special case. We use the following notation for p-dimensional marginal probabilities of this distribution (where the subscript r is dropped for simplicity).

$$P(I_{j_1}^{(e)} = i_{j_1}, \ldots, I_{j_p}^{(e)} = i_{j_p}) = p_{j_1,\ldots,j_p}^{(e)}(i_{j_1}, \ldots, i_{j_p}),\ i_{j_1}, \ldots, i_{j_p} \in \{0, 1\}.$$

We also write $p_j^{(e)}(1) = p_j^{(e)}$ for one-dimensional marginal probabilities, so that in the special case of conditional independence we have

$$p_{j_1,\ldots,j_p}^{(e)}(1, \ldots, 1) = \prod_{k=1}^{p} p_{j_k}^{(e)}.$$

The counting processes for events and losses are thus linked by

$$N_j(t) = \sum_{e=1}^{m} \sum_{r=1}^{N^{(e)}(t)} I_{j,r}^{(e)}. \tag{1}$$

Under the Poisson assumption for the event processes and the Bernoulli assumption for the loss indicators, the loss processes $\{N_j(t), t \geq 0\}$ are clearly Poisson themselves, since they are obtained by superpositioning m independent (possibly thinned) Poisson processes generated by the m underlying event processes. $(N_1(t), \ldots, N_n(t))'$ can be thought of as having a *multivariate Poisson* distribution.

However the total number of losses $N(t) = \sum_{j=1}^{n} N_j(t)$ is not Poisson but rather *compound Poisson*. It is the sum of m independent compound Poisson distributed random variables as can be seen by writing

$$N(t) = \sum_{e=1}^{m} \sum_{r=1}^{N^{(e)}(t)} \sum_{j=1}^{n} I_{j,r}^{(e)}. \tag{2}$$

The compounding distribution of the eth compound Poisson process is the distribution of $\sum_{j=1}^{n} I_j^{(e)}$, which in general is a sum of dependent Bernoulli variables. We return to the compound Poisson nature of the process $\{N(t), t \geq 0\}$ after generalising it in the next section.

2.2 Adding Dependent Severities

We can easily add *severities* to our multivariate Poisson model. Suppose that when the rth event of type e occurs a potential loss of type j with severity $X_{j,r}^{(e)}$ can occur. Whether the loss occurs or not is of course determined by the value of the indicator $I_{j,r}^{(e)}$, which we assume is independent of $X_{j,r}^{(e)}$. The potential losses $\{X_{j,r}^{(e)},\ r = 1, \ldots, N^{(e)}(t), e = 1, \ldots, m\}$ are considered to be iid with distribution F_j. Potential losses of different types caused by the same event may however be dependent. We consider that they have a joint distribution function F. That is, for a vector $\mathbf{X}_r^{(e)}$ of potential losses generated by the same event we assume

$$\mathbf{X}_r^{(e)} = (X_{1,r}^{(e)}, \ldots, X_{n,r}^{(e)})' \sim F.$$

In a more general model it would be possible to make the multivariate distribution of losses caused by the same event depend on the nature of the underlying event e. However, in practice it may make sense to assume that there is a single underlying multivariate severity distribution which generates the severities for all event types. This reflects the fact that it is often standard practice in insurance to model losses of the same type type as having an identical claim size distribution, without necessarily differentiating carefully between the events that caused them.

The aggregate loss process for losses of type j is a compound Poisson process given by

$$Z_j(t) = \sum_{e=1}^{m} \sum_{r=1}^{N^{(e)}(t)} I_{j,r}^{(e)} X_{j,r}^{(e)}. \tag{3}$$

The aggregate loss caused by losses of all types can be written as

$$Z(t) = \sum_{e=1}^{m} \sum_{r=1}^{N^{(e)}(t)} \sum_{j=1}^{n} I_{j,r}^{(e)} X_{j,r}^{(e)} = \sum_{e=1}^{m} \sum_{r=1}^{N^{(e)}(t)} \mathbf{I}_r^{(e)\prime} \mathbf{X}_r^{(e)}, \tag{4}$$

and is again seen to be a sum of m independent compound Poisson distributed random variables, and therefore itself compound Poisson distributed. Clearly (2) is a special case of (4) and (1) is a special case of (3). Thus we can understand all of these processes by focusing on (4). The compound Poisson nature of $Z(t)$ can be clarified by rewriting this process as

$$Z(t) = \sum_{s=1}^{S(t)} Y_s, \tag{5}$$

where $\{S(t), t \geq 0\}$ is a Poisson process with intensity $\lambda = \sum_{e=1}^{m} \lambda^{(e)}$, counting all shocks s generated by all event types, and where the random variables $Y_1, \ldots, Y_{S(t)}$ ($\stackrel{\text{d}}{=} Y$) are iid and independent of $S(t)$. Y has the stochastic representation

$$Y \stackrel{\text{d}}{=} \mathbf{I}'\mathbf{X},$$

where $\mathbf{I}$ is a random vector satisfying $P(\mathbf{I} = \mathbf{I}^{(e)}) = \lambda^{(e)}/\lambda$ for $e = 1, \dots, m$, $\mathbf{I}^{(e)}$ is a generic random vector of indicators for shocks of event type e, and $\mathbf{X}$ is a generic random vector of severities caused by the same shock, which is independent of $\mathbf{I}^{(1)}, \dots, \mathbf{I}^{(m)}$.

We consider two examples that fit into the framework of the model we have set up. The first one, an insurance application of the model, we continue to develop throughout the paper. The second one, a credit risk application, is presented separately in Section 5.

2.3 Insurance example: natural catastrophe modelling

Fix $n = 2$, $m = 3$. Let $N_1(t)$ and $N_2(t)$ count windstorm losses in France and Germany respectively. Suppose these are generated by three different kinds of windstorm that occur independently. $N^{(1)}(t)$ counts west European windstorms; these are likely to cause French losses but no German losses. $N^{(2)}(t)$ counts central European windstorms; these are likely to cause German losses but no French losses. $N^{(3)}(t)$ counts pan-European windstorms, which are likely to cause both French and German losses.

3 The Effect of Dependent Loss Frequencies

To begin with we look at the distribution of the random vector $(N_1(t), \dots, N_n(t))'$, particularly with regard to its univariate and bivariate margins as well as the correlation structure.

Proposition 1. *1.* $\{(N_1(t), \dots, N_n(t))', t \geq 0\}$ *is a multivariate Poisson process with*

$$E(N_j(t)) = t \sum_{e=1}^{m} \lambda^{(e)} p_j^{(e)}. \tag{6}$$

2. The two-dimensional marginals are given by

$$P\left(N_j(t) = n_j, N_k(t) = n_k\right) = e^{-\lambda t \left(p_{j,k}(1,1) + p_{j,k}(1,0) + p_{j,k}(0,1)\right)} \times \tag{7}$$
$$\sum_{i=0}^{\min\{n_j, n_k\}} \frac{\left(\lambda t p_{j,k}(1,1)\right)^i \left(\lambda t p_{j,k}(1,0)\right)^{n_j - i} \left(\lambda t p_{j,k}(0,1)\right)^{n_k - i}}{i!(n_j - i)!(n_k - i)!},$$

where $\lambda = \sum_{e=1}^{m} \lambda^{(e)}$ *and*

$$p_{j,k}(i_j, i_k) = \lambda^{-1} \sum_{e=1}^{m} \lambda^{(e)} p_{j,k}^{(e)}(i_j, i_k), \quad i_j, i_k \in \{0, 1\}.$$

3. The covariance and correlation structure is given by

$$\operatorname{cov}\left(N_j(t), N_k(t)\right) = t \sum_{e=1}^{m} \lambda^{(e)} p_{j,k}^{(e)}(1,1) \tag{8}$$

and

$$\rho(N_j(t), N_k(t)) = \frac{\sum_{e=1}^{m} \lambda^{(e)} p_{j,k}^{(e)}(1,1)}{\sqrt{\left(\sum_{e=1}^{m} \lambda^{(e)} p_j^{(e)}\right)\left(\sum_{e=1}^{m} \lambda^{(e)} p_k^{(e)}\right)}}.$$

Proof. 1. obvious using thinning and superposition arguments for independent Poisson processes.

2. is found in Barlow and Proschan (1975).

3. is a special case of Proposition 7 (part 2).

□

Clearly, from Proposition 1 part 3, a necessary condition for $N_j(t)$ and $N_k(t)$ to be independent is that $p^{(e)}_{j,k}(1,1) = 0$ for all e; i.e. it must be impossible for losses of types j and k to be caused by the same event. If for at least one event it is possible that both loss types occur, then we have positive correlation between loss numbers. However Proposition 1, part 2 allows us to make a stronger statement.

Corollary 2. *$N_j(t)$ and $N_k(t)$ are independent if and only if $p^{(e)}_{j,k}(1,1) = 0$ for all e.*

Note that if $p^{(e)}_{j,k}(1,1) = 0$ for all j,k with $j \neq k$, then $P(\mathbf{I}^{(e)}_r = \mathbf{0}) = 1 - \sum_{j=1}^{n} p^{(e)}_j$. Hence

Corollary 3. *If $\sum_{j=1}^{n} p^{(e)}_j > 1$ for some e, then $N_1(t), \ldots, N_n(t)$ are not independent.*

Thus if we begin by specifying univariate conditional loss probabilities $p^{(e)}_j$ it is not always true that a shock model can be constructed which gives independent loss frequencies.

We have already noted that the process of total loss numbers $N(t) = \sum_{j=1}^{n} N_j(t)$ is not Poisson (but rather a sum of independent compound Poissons). If there is positive correlation between components $N_j(t)$ then $\{N(t), t \geq 0\}$ itself cannot be a Poisson process since it is *overdispersed* with respect to Poisson. It can easily be calculated (see Proposition 8 later) that

$$\operatorname{var}(N(t)) = \sum_{j=1}^{n}\sum_{k=1}^{n} \operatorname{cov}(N_j(t), N_k(t)) > E(N(t)) \tag{9}$$

Suppose we define a new vector of mutually independent Poisson distributed loss counters $\hat{N}_j(t)$ such that $\hat{N}_j(t) \stackrel{\mathrm{d}}{=} N_j(t)$. Clearly $\hat{N}(t) = \sum_{j=1}^{n} \hat{N}_j(t)$ is Poisson distributed and

$$\operatorname{var}(\hat{N}(t)) = E(\hat{N}(t)) = E(N(t)).$$

The case where the components $N_j(t)$ are dependent is clearly more dangerous than case with independent components. Although the expected number of total losses is the same in both cases the variance is higher in the dependent case and, using (9) and (8), we can calculate the inflation of the variance that results from dependence.

3.1 Insurance example (continued)

Consider a 5 year period and suppose French losses occur on average 5 times per year and German losses on average 6 times per year; in other words we assume $\lambda_1 = 5$ and $\lambda_2 = 6$. We consider three models for the dependence between these loss frequencies.

- **Case 1: No common shocks.** If there are no common shocks, then $N(5) = N_1(5) + N_2(5)$ has a Poisson distribution with intensity $\lambda = \lambda_1 + \lambda_2 = 5 + 6 = 11$.

In reality we believe that there are common shocks, in our case particularly the pan-European windstorms. Suppose west, central and pan-European windstorms occur on average 4, 3 and 3 times per year respectively. In terms of event intensities we have

$$\lambda^{(1)} = 4,\ \lambda^{(2)} = 3 \text{ and } \lambda^{(3)} = 3.$$

In terms of the indicator probabilities we assume that empirical evidence and expert judgement has been used to estimate

$$p_1^{(1)} = 1/2,\ p_2^{(1)} = 1/4,\ p_1^{(2)} = 1/6,\ p_2^{(2)} = 5/6,\ p_1^{(3)} = 5/6 \text{ and } p_2^{(3)} = 5/6$$

which means that, although unlikely, west European windstorms can cause German losses and central European windstorms can cause French losses. Note that these choices provide an example where the assumption of no common shocks is not only unrealistic but also impossible. To see this consider Corollary 3 and note that $p_1^{(3)} + p_2^{(3)} > 1$.

To make sure that our estimates of event frequencies and indicator probabilities tally with our assessment of loss frequencies we must have that

$$\lambda_j = \lambda^{(1)} p_j^{(1)} + \lambda^{(2)} p_j^{(2)} + \lambda^{(3)} p_j^{(3)},\ j = 1, 2.$$

However the specification of the univariate indicator probabilities is insufficient to completely specify the model. We need to fix the dependence structure of the bivariate indicators $(I_1^{(e)}, I_2^{(e)})'$ for $e = 1, 2, 3$. For simplicity we will consider two possibilities.

- **Case 2: Independent indicators.**

$$p_{1,2}^{(e)}(1,1) = p_1^{(e)} p_2^{(e)} \text{ for } e = 1, 2, 3.$$

- **Case 3: Positive dependent indicators.**

$$p_{1,2}^{(e)}(1,1) \geq p_1^{(e)} p_2^{(e)} \text{ for } e = 1, 2, 3.$$

To be specific in Case 3 we will consider $p_{1,2}^{(e)}(1,1) = \min(p_1^{(e)}, p_2^{(e)})$, which is the strongest possible dependence between the indicators, sometimes known as comonotonicity. See Joe (1997) for some discussion of dependence bounds in multivariate Bernoulli models. In terms of interpretation in our application this means:

- if a west European windstorm causes a German loss, then with certainty it also causes a French loss;
- if a central European windstorm causes a French loss, then with certainty it also causes a German loss;
- if a pan-European windstorm causes one kind of loss, then with certainty it causes the other kind of loss.

For cases 1, 2 and 3 we get $\text{var}(N(5)) = 55$, 85 and 95 respectively. Of more interest than the variance as a measure of the riskiness of $N(5)$ are the tail probabilities $P(N(5) > k)$. In this example these probabilities can be calculated analytically using formula (7) for the bivariate frequency function. The left plot in Figure 1 shows exceedence probabilities $P(N(5) > k)$, for $k = 70, 71, \ldots, 90$, for the three cases. The right plot shows by which factor such an exceedence probability is underestimated by case 1 if the correct model would be given by case 2 or 3. Clearly, both the presence of common shocks and then the subsequent addition of dependent indicators have a profound effect on the aggregate frequency distribution of $N(5)$.

3.2 The Equivalent Fatal Shock Model

The not-necessarily-fatal shock model set up in the previous section has the nice property of being easily interpreted. As we will now show this model has an equivalent representation as a fatal shock model. Basically, instead of counting all shocks, we only count loss-causing shocks. From this representation we can draw a number of non-trivial conclusions about our original model.

Let $\mathcal{S}$ be the set of non-empty subsets of $\{1,\dots,n\}$. For $s \in \mathcal{S}$ we introduce a new counting process $\widetilde{N}_s(t)$, which counts shocks in $(0,t]$ resulting in losses of all types in s *only.* Thus if $s = \{1,2,3\}$, then $\widetilde{N}_s(t)$ counts shocks which cause simultaneous losses of types 1, 2 and 3, but not of types 4 to n. We have

$$\widetilde{N}_s(t) = \sum_{e=1}^{m} \sum_{r=1}^{N^{(e)}(t)} \sum_{s':s'\supseteq s} (-1)^{|s'|-|s|} \prod_{k\in s'} I_{k,r}^{(e)},$$

where $\sum_{s':s'\supseteq s}(-1)^{|s'|-|s|}\prod_{k\in s'} I_{k,r}^{(e)}$ is an indicator random variable which takes the value 1 if the rth shock of type e causes losses of all type in s only, and the value 0 otherwise. Furthermore let $\widetilde{N}(t)$ count all shocks in $(0,t]$ which result in losses of any kind. Clearly we have

$$\widetilde{N}(t) = \sum_{s\in\mathcal{S}} \widetilde{N}_s(t).$$

The key to a fatal shock representation is the following result.

Proposition 4. *$\{\widetilde{N}_s(t), t \geq 0\}$ for $s \in \mathcal{S}$ are independent Poisson processes.*

Proof. Let $J_{s,r}^{(e)} = \sum_{s':s'\supseteq s}(-1)^{|s'|-|s|}\prod_{k\in s'} I_{k,r}^{(e)}$. First note that the random variable $J_{s,r}^{(e)}$ takes values in $\{0,1\}$, and that $P(J_{s,r}^{(e)} = 1) = \sum_{s':s'\supseteq s}(-1)^{|s'|-|s|} p_{s'}^{(e)}$, where $p_{s'}^{(e)} = P(\prod_{k\in s'} I_{k,r}^{(e)} = 1)$, does not depend on r. Hence $\{\sum_{r=1}^{N^{(e)}(t)} J_{s,r}^{(e)}, t \geq 0\}$ is obtained by thinning the Poisson process $\{N^{(e)}(t), t \geq 0\}$, and is therefore a Poisson process with intensity $\lambda^{(e)}\sum_{s':s'\supseteq s}(-1)^{|s'|-|s|} p_{s'}^{(e)}$. $\{\widetilde{N}_s(t), t \geq 0\}$ is obtained by superpositioning the independent Poisson processes $\{\sum_{r=1}^{N^{(e)}(t)} J_{s,r}^{(e)}, t \geq 0\}$ for $e = 1,\dots,m$, and is therefore a Poisson process with intensity $\lambda_s = \sum_{e=1}^{m}\lambda^{(e)}\sum_{s':s'\supseteq s}(-1)^{|s'|-|s|} p_{s'}^{(e)}$. Since $P(\sum_{s\in\mathcal{S}}\sum_{s':s'\supseteq s}(-1)^{|s'|-|s|}\prod_{k\in s'} I_{k,r}^{(e)} = 1) = P(1 - \prod_{k=1}^{n}(1 - I_{k,r}^{(e)}) = 1)$ does not depend on r, thinning and superpositioning arguments give that $\{\widetilde{N}(t), t \geq 0\}$ is a Poisson process with intensity $\widetilde{\lambda} = \sum_{s\in\mathcal{S}}\lambda_s = \sum_{e=1}^{m}\lambda^{(e)}(1 - P(I_{1,r}^{(e)} = 0,\dots,I_{1,r}^{(e)} = 0))$. Each jump in the process $\{\widetilde{N}(t), t \geq 0\}$ corresponds to a jump in exactly one of the processes $\{\widetilde{N}_s(t), t \geq 0\}$ for $s \in \mathcal{S}$. Given a jump in $\{\widetilde{N}(t), t \geq 0\}$, the probability of the jump being in $\{\widetilde{N}_s(t), t \geq 0\}$ is given by $q_s = \lambda_s/\widetilde{\lambda}$ for $s \in \mathcal{S}$. Order the $l = |\mathcal{S}| = 2^n - 1$ non-empty subsets of $\{1,\dots,n\}$ in some arbitrary way. Then

$$P(\widetilde{N}_{s_1}(t) = n_1,\dots,\widetilde{N}_{s_l}(t) = n_l | \widetilde{N}(t) = \widetilde{n}) = \begin{cases} \widetilde{n}!\prod_{j=1}^{l}(q_{s_j}^{n_j}/n_j!) & ,\widetilde{n} = \sum_{j=1}^{l} n_j \\ 0 & ,\widetilde{n} \neq \sum_{j=1}^{l} n_j \end{cases}$$

and hence

$$\begin{aligned} P(\widetilde{N}_{s_1}(t) = n_1, \ldots, \widetilde{N}_{s_l}(t) = n_l) &= P\big(\widetilde{N}(t) = \sum_{j=1}^{l} n_j\big)\big(\sum_{j=1}^{l} n_j\big)! \prod_{j=1}^{l} \frac{q_{s_j}^{n_j}}{n_j!} \\ &= \prod_{j=1}^{l} e^{-\lambda_{s_j} t} \frac{(\lambda_{s_j} t)^{n_j}}{n_j!} = \prod_{j=1}^{l} P(\widetilde{N}_{s_j}(t) = n_j) \end{aligned}$$

It follows that the processes $\{\widetilde{N}_s(t), t \geq 0\}$ for $s \in \mathcal{S}$ are independent Poisson processes. □

Since the Poisson processes $\{\widetilde{N}_s(t), t \geq 0\}$ for $s \in \mathcal{S}$ are independent and since the loss counting processes may be written as

$$N_j(t) = \sum_{s: j \in s} \widetilde{N}_s(t),$$

it also follows that we have obtained a fatal shock model representation for the original not-necessarily-fatal set-up.

Furthermore, since $\lambda_s = 0$ for all s with $|s| \geq 2$ if and only if $p_{j,k}^{(e)}(1,1) = 0$ for all e and all j, k with $j \neq k$, Corollary 2 can be strengthened.

Corollary 5. *$N_1(t), \ldots, N_n(t)$ are mutually independent if and only if $p_{j,k}^{(e)}(1,1) = 0$ for all e and all j, k with $j \neq k$.*

Further consequences of this construction are contained in the following sections.

3.3 Panjer Recursion

If there are common shocks, then $N(t) = \sum_{j=1}^{n} N_j(t)$ does not have a Poisson distribution. In our insurance example we have considered only two loss types and it is thus easy to calculate the distribution of $N(t)$ directly using convolution and the bivariate frequency function in (7).

A more general method of calculating the probability distribution function of $N(t)$, which will also work in higher dimensional examples, is Panjer recursion. We use the notation of the preceding section. In addition, let W_i denote the number of losses due to the ith loss-causing shock. The total number of losses, $N(t)$, has the stochastic representation

$$N(t) \stackrel{\mathrm{d}}{=} \sum_{i=1}^{\widetilde{N}(t)} W_i,$$

where $W_1, \ldots, W_{\widetilde{N}(t)}$ $(\stackrel{\mathrm{d}}{=} W)$ are iid and independent of $\widetilde{N}(t)$. The probability $P(N(t) = r)$ can now easily be calculated using Panjer recursion:

$$P(N(t) = r) = \sum_{i=1}^{r} \frac{\widetilde{\lambda} t i}{r} P(W = i) P(N(t) = r - i), \; r \geq 1,$$

where $P(N(t) = 0) = \exp(-\widetilde{\lambda} t)$. Since $P(W = k) = 0$ for $k > n$, the above equation simplifies to

$$P(N(t) = r) = \sum_{i=1}^{\min(r,n)} \frac{\widetilde{\lambda} t i}{r} P(W = i) P(N(t) = r - i), \; r \geq 1.$$

The probability distribution of W and a discussion of under which model assumptions the Panjer recursion scheme is computationally most efficient is given in Appendix A. It should be noted that other techniques for calculating the probability distribution of $N(t)$ might prove even more efficient.

3.4 Multivariate times to first losses

Let $T_j = \inf\{t : N_j(t) > 0\}$ denote the time to the first loss of type j. We now consider briefly the distribution of $(T_1, \ldots, T_n)'$ whose dependence structure is well understood.

Proposition 6. *Let $T_j = \inf\{t : N_j(t) > 0\}$ for $j = 1, \ldots, n$. Then $(T_1, \ldots, T_n)'$ has a multivariate exponential distribution whose survival copula is a Marshall-Olkin copula.*

Proof. Let $Z_s = \inf\{t : \widetilde{N}_s(t) > 0\}$. Z_s, for $s \in \mathcal{S}$, are independent exponential distributed random variables with parameters λ_s.

$$T_j = \inf\{t : N_j(t) > 0\} = \inf\{t : \sum_{s:j\in s} \widetilde{N}_s(t) > 0\} = \min_{s:j\in s} Z_s.$$

Hence T_j is exponentially distributed. Furthermore the survival copula of

$$(T_1, \ldots, T_n)' = (\min_{s:1\in s} Z_s, \ldots, \min_{s:n\in s} Z_s)'$$

is by construction an n-dimensional Marshall-Olkin copula. See Marshall and Olkin (1967) or Joe (1997) for details. □

The survival distributions of the bivariate margins of $(T_1, \ldots, T_n)'$ can be derived quite easily. From the Marshall-Olkin construction it follows that

$$\begin{aligned}
P(T_i > t_i, T_j > t_j) &= P(\min_{s:i\in s} Z_s > t_i, \min_{s:j\in s} Z_s > t_j) \\
&= P(\min_{s:i\in s, j\notin s} Z_s > t_i) P(\min_{s:j\in s, i\notin s} Z_s > t_j) P(\min_{s:i,j\in s} Z_s > \max(t_i, t_j)) \\
&= \exp\Big\{ - t_i \sum_{e=1}^{m} \lambda^{(e)} (p_i^{(e)} - p_{i,j}^{(e)}(1,1)) - t_j \sum_{e=1}^{m} \lambda^{(e)} (p_j^{(e)} - p_{i,j}^{(e)}(1,1)) \\
&\qquad - \max(t_i, t_j) \sum_{e=1}^{m} \lambda^{(e)} p_{i,j}^{(e)}(1,1) \Big\} \\
&= \exp\Big\{ - t_i \sum_{e=1}^{m} \lambda^{(e)} p_i^{(e)} - t_j \sum_{e=1}^{m} \lambda^{(e)} p_j^{(e)} + \min(t_i, t_j) \sum_{e=1}^{m} \lambda^{(e)} p_{i,j}^{(e)}(1,1) \Big\}.
\end{aligned}$$

Since $\overline{F}_k(t) = P(T_k > t) = \exp\big\{ - t \sum_{e=1}^{m} \lambda^{(e)} p_k^{(e)} \big\}$ for $k = i, j$,

$$P(T_i > t_i, T_j > t_j) = \overline{F}_i(t_i)\overline{F}_j(t_j) \min\Big(\exp\big\{t_i \sum_{e=1}^{m} \lambda^{(e)} p_{i,j}^{(e)}(1,1)\big\}, \exp\big\{t_j \sum_{e=1}^{m} \lambda^{(e)} p_{i,j}^{(e)}(1,1)\big\}\Big).$$

Let

$$\alpha_i = \frac{\sum_{e=1}^{m} \lambda^{(e)} p_{i,j}^{(e)}(1,1)}{\sum_{e=1}^{m} \lambda^{(e)} p_i^{(e)}} \text{ and } \alpha_j = \frac{\sum_{e=1}^{m} \lambda^{(e)} p_{i,j}^{(e)}(1,1)}{\sum_{e=1}^{m} \lambda^{(e)} p_j^{(e)}}.$$

Then

$$\exp\left\{t_i \sum_{e=1}^{m} \lambda^{(e)} p_{i,j}^{(e)}(1,1)\right\} = \overline{F}_i(t_i)^{-\alpha_i}, \ \exp\left\{t_j \sum_{e=1}^{m} \lambda^{(e)} p_{i,j}^{(e)}(1,1)\right\} = \overline{F}_j(t_j)^{-\alpha_j}$$

and hence

$$P(T_i > t_i, T_j > t_j) = C_{\alpha_i,\alpha_j}(\overline{F}_i(t_i), \overline{F}_j(t_j)),$$

where

$$C_{\alpha_i,\alpha_j}(u,v) = \min(u^{1-\alpha_i}v, uv^{1-\alpha_j}) = \begin{cases} u^{1-\alpha_i}v, & u^{\alpha_i} \geq v^{\alpha_j}, \\ uv^{1-\alpha_j}, & u^{\alpha_i} \leq v^{\alpha_j}. \end{cases}$$

This is the bivariate Marshall-Olkin copula.

3.5 Time to kth loss

Recall that for a Poisson process with intensity μ, the time to the kth jump is $\Gamma(k, 1/\mu)$-distributed, where $\Gamma(\cdot,\cdot)$ denotes the Gamma distribution. Hence the time to the kth loss-causing shock is $\Gamma(k, 1/\widetilde{\lambda})$-distributed, where $\widetilde{\lambda} = \sum_{e=1}^{m} \lambda^{(e)}(1 - P(I_{1,r}^{(e)} = 0, \ldots, I_{n,r}^{(e)} = 0))$. The time to the kth loss is $\inf\{t : N(t) \geq k\}$, where

$$N(t) = \sum_{i=1}^{n} i \sum_{s:|s|=i} \widetilde{N}_s(t).$$

$\{N(t), t \geq 0\}$ is not a Poisson process but rather a compound Poisson process, the time to the kth jump is still $\Gamma(k, 1/\widetilde{\lambda})$-distributed but there are non unit jump sizes. By noting that the probability that the time to the kth loss is less than or equal to t can be expressed as $P(N(t) \geq k)$, it is clear that the distribution of the time to the kth loss can be fully understood from the distribution of $N(t)$ for $t \geq 0$, and this distribution can be evaluated using Panjer recursion or other methods. See the appendix for a detailed discussion of how Panjer recursion can be applied to our set-up.

4 The Effect of Dependent Severities

We now consider adding severities to our shock model and study the multivariate distribution of $(Z_1(t), \ldots, Z_n(t))'$. Again we can calculate first and second moments of the marginal distributions and correlations between the components.

Proposition 7. *1. $\{(Z_1(t), \ldots, Z_n(t))', t \geq 0\}$ is a multivariate compound Poisson process. If $E(|X_j|) < \infty$, then*

$$E(Z_j(t)) = E(X_j)E(N_j(t)).$$

2. If $E(|X_j|), E(|X_k|) < \infty$, then the covariance and correlation structure is given by

$$\operatorname{cov}(Z_j(t), Z_k(t)) = E(X_j X_k) \operatorname{cov}(N_j(t), N_k(t))$$

and

$$\rho(Z_j(t), Z_k(t)) = \frac{E(X_j X_k)}{\sqrt{E(X_j^2)E(X_k^2)}} \rho(N_j(t), N_k(t)).$$

Proof. 1. is easily established from formula (3).

2. We observe that $\forall j,k \in \{1,\ldots,n\}$,

$$\begin{aligned}
\operatorname{cov}(Z_j(t), Z_k(t)) &= \sum_{e=1}^{m} \operatorname{cov}\left(\sum_{r=1}^{N^{(e)}(t)} I_{j,r}^{(e)} X_{j,r}^{(e)}, \sum_{r=1}^{N^{(e)}(t)} I_{k,r}^{(e)} X_{k,r}^{(e)} \right) \\
&\quad + \sum_{e=1}^{m} \sum_{f \neq e} \operatorname{cov}\left(\sum_{r=1}^{N^{(e)}(t)} I_{j,r}^{(e)} X_{j,r}^{(e)}, \sum_{r=1}^{N^{(f)}(t)} I_{k,r}^{(f)} X_{k,r}^{(f)} \right) \\
&= \sum_{e=1}^{m} \operatorname{cov}\left(\sum_{r=1}^{N^{(e)}(t)} I_{j,r}^{(e)} X_{j,r}^{(e)}, \sum_{r=1}^{N^{(e)}(t)} I_{k,r}^{(e)} X_{k,r}^{(e)} \right) \\
&= E(X_j X_k) \sum_{e=1}^{m} E\left(N^{(e)}(t)\right) E\left(I_j^{(e)} I_k^{(e)}\right) \\
&= E(X_j X_k) t \sum_{e=1}^{m} \lambda^{(e)} p_{j,k}^{(e)}(1,1) \\
&= E(X_j X_k) \operatorname{cov}(N_j(t), N_k(t))
\end{aligned}$$

□

Now consider the distribution of the total loss $Z(t) = \sum_{j=1}^{n} Z_j(t)$. The expected total loss is easily calculated to be

$$E(Z(t)) = \sum_{j=1}^{n} E(X_j) E(N_j(t)),$$

and higher moments of $Z(t)$ can also be calculated, by exploiting the compound Poisson nature of this process as shown in (5). Since $Z(t)$ is the most general aggregate loss process that we study in this paper we collect some useful moment results for this process.

Proposition 8. *1. If they exist, the 2nd and 3rd order central moment of $Z(t)$ are given by*

$$E\left(Z(t) - E(Z(t))\right)^p = \lambda t E\left(Y^p\right), \quad p = 2, 3, \tag{10}$$

where $\lambda = \sum_{e=1}^{m} \lambda^{(e)}$ and

$$E(Y^p) = \frac{1}{\lambda} \sum_{j_1=1}^{n} \cdots \sum_{j_p=1}^{n} E(X_{j_1} \ldots X_{j_p}) \sum_{e=1}^{m} \lambda^{(e)} p_{j_1,\ldots,j_p}^{(e)}(1,\ldots,1). \tag{11}$$

2. Whenever they exist, the non-central moments of $Z(t)$ are given recursively by

$$E\left(Z(t)^p\right) = \lambda t \sum_{k=0}^{p-1} \binom{p-1}{k} E\left(Y^{k+1}\right) E\left(Z(t)^{p-k-1}\right),$$

with $E\left(Y^{k+1}\right)$ given by (11).

Proof. 1. For a compound Poisson process of the form (5) the formula (10) is well known. We can calculate that for all p

$$\begin{aligned} E\left(Y^p\right) &= \sum_{e=1}^{m} \frac{\lambda^{(e)}}{\lambda} E\left(\mathbf{I}^{(e)\prime}\mathbf{X}\right)^p \\ &= \sum_{e=1}^{m} \frac{\lambda^{(e)}}{\lambda} E\left(\sum_{j_1=1}^{n} \cdots \sum_{j_p=1}^{n} I_{j_1}^{(e)} \ldots I_{j_p}^{(e)} X_{j_1} \ldots X_{j_p}\right) \\ &= \lambda^{-1} \sum_{e=1}^{m} \lambda^{(e)} \sum_{j_1=1}^{n} \cdots \sum_{j_p=1}^{n} E(X_{j_1} \ldots X_{j_p}) p_{j_1,\ldots,j_p}^{(e)}(1,\ldots,1) \\ &= \lambda^{-1} \sum_{j_1=1}^{n} \cdots \sum_{j_p=1}^{n} E(X_{j_1} \ldots X_{j_p}) \sum_{e=1}^{m} \lambda^{(e)} p_{j_1,\ldots,j_p}^{(e)}(1,\ldots,1). \end{aligned}$$

2. It may be proved by induction that the pth derivative of the moment generating function of $Z(t)$ satisfies

$$M_{Z(t)}^{(p)}(x) = \lambda t \sum_{k=0}^{p-1} \binom{p-1}{k} M_Y^{(k+1)}(x) M_{Z(t)}^{(p-k-1)}(x), \tag{12}$$

where $M_Y(x)$ denotes the mgf of Y.

□

We are particularly interested in the effect of different levels of dependence between both loss frequencies and loss severities on the tail of the distribution of $Z(t)$, and on higher quantiles of this distribution. The distribution of $Z(t)$ is generally not available analytically but, given the ease of simulating from our Poisson common shock model, it is possible to estimate quantiles empirically to a high enough degree of accuracy that differences between different dependence specifications become apparent.

It is also possible, given the ease of calculating moments of $Z(t)$, to use a moment fitting approach to approximate the distribution of $Z(t)$ with various parametric distributions, and we implement this approach in the following example.

4.1 Insurance example (continued)

Assume that French and German severities are Pareto$(4, 3)$ distributed, i.e.

$$F_i(x) = P\left(X_i \le x\right) = 1 - \left(\frac{3}{3+x}\right)^4, \; E(X_i) = 1, \; E(X_i^2) = 3, \; E(X_i^3) = 27, \quad i = 1, 2.$$

We have to fix the dependence structure of potential losses $(X_1, X_2)'$ at the same shock. We do this using the copula approach. The copula C of $(X_1, X_2)'$ is the distribution function of $(F_1(X_1), F_2(X_2))'$. The distribution function of $(X_1, X_2)'$ can be expressed in terms of C as

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)).$$

For more on copulas see Embrechts, McNeil, and Straumann (1999), Nelsen (1999) or Joe (1997). We consider three cases.

- Independent severities:

$$F(x_1, x_2) = F_1(x_1)F_2(x_2).$$

- Positively dependent severities with Gaussian dependence:

$$F(x_1, x_2) = C_\rho^{\text{Ga}}(F_1(x_1), F_2(x_2)),$$

where

$$C_\rho^{\text{Ga}}(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left\{\frac{-(s^2 - 2\rho st + t^2)}{2(1-\rho^2)}\right\} dsdt.$$

and $\rho > 0$.

- Positively dependent severities with Gumbel dependence:

$$F(x_1, x_2) = C_\theta^{\text{Gu}}(F_1(x_1), F_2(x_2)),$$

where

$$C_\theta^{\text{Gu}}(u, v) = \exp\left(-\left\{(-\log u)^\theta + (-\log v)^\theta\right\}^{1/\theta}\right),$$

and $\theta > 1$.

For both of the positive dependence models we will parameterize the copulas such that Kendall's rank correlation (τ) (see e.g. Embrechts, McNeil, and Straumann (1999) for details) between X_1 and X_2 is 0.5. This is achieved by setting

$$\rho = \sin\left(\frac{\pi}{2}\tau\right) \text{ and } \theta = \frac{1}{1-\tau}.$$

As we have discussed there are several possibilities for modelling the tail of $Z(5)$. One approach is to fit a heavy-tailed generalised F-distribution (referred to as a generalised Pareto distribution in Hogg and Klugman (1984)) to $Z(5)$ using moment fitting with the first three moments. The distribution function is given by

$$G\left(2k, 2\alpha, \frac{\alpha}{k\lambda}x\right) \text{ for } \alpha > 0, \lambda > 0, k > 0,$$

where $G(\nu_1, \nu_2, \cdot)$ is the distribution function for the F-distribution with ν_1 and ν_2 degrees of freedom. The nth moment exists if $\alpha > n$ and is then given by

$$\lambda^n\left(\prod_{i=0}^{n-1}(k+i)\right)\Big/\left(\prod_{i=1}^{n}(\alpha - i)\right).$$

By calculating the first three moments of $Z(5)$ for different frequency and severity dependencies we fit generalised F-distributions and study the difference in tail behaviour. Figure 2 shows quantiles of generalised F-distributions determined by fitting the first three moments to $Z(5)$ for case 1, 2 and 3 and for different dependence structures between the severities. It clearly shows the effect of common shocks on the tail of $Z(5)$ and perhaps even more the drastic change in tail behaviour when adding moderate dependence between the severities.

It should be noted that the quantile estimates of $Z(5)$ given by moment fitted generalised F-distributions are slight overestimates of the true quantiles for $\alpha \in [0.900, 0.995]$. However the accuracy is sufficient to show the major differences between the quantile curves of $Z(5)$ for our different copula choices.

5 Applying the Methology to Credit

The modelling of defaults (or more generally credit events such as downgrades) as jumps in Poisson processes or other more general processes such as Cox-processes has received much attention during recent years. See Duffie and Singleton (1999) for a discussion of such approaches with a view towards efficient simulation. In this section we will use our model to study the effect of different parameterizations on the tail of the distribution of the total number of defaults.

Consider a loan portfolio consisting of $n = 100000$ obligors. Suppose the counterparties can be divided into four sectors and two rating categories (e.g. A and B). The sectors may be geographical, for example Europe, North America, South America and Japan, or defined by the type of company, for example banking, manufacturing etc. Although it would be possible to consider defaults in the four sectors as defining losses of four different types, we consider a more general model where the default of each individual counterparty defines a loss type; thus there are 100000 types of loss.

A variety of shocks are possible. We will consider a global shock which could be thought of as a recession in the world economy and sector shocks which represent adverse economic conditions for specific industries or geographical sectors. We also consider that each counterparty is subject to its own idiosyncratic shock process; these might be termed "bad management" shocks. In total their will be $m = n + 5$ shock event processes.

Suppose that the jth obligor belongs to sector $k = k(j)$ and rating class $l = l(j)$ where $k = 1, \ldots, 4, l = 1, 2$. From formula (6) we know that $N_j(t)$, the number of defaults of obligor j in $(0, t]$ is Poisson with intensity given by

$$\lambda_j = \lambda^{(j)} + p_j^{(n+k(j))} \lambda^{(n+k(j))} + p_j^{(n+5)} \lambda^{(n+5)},$$

where the three terms represent the contributions to the default intensity of idiosyncratic, sector and global events respectively. Note that in general this intensity will be set so low that the probability of a firm defaulting more than once in the period of interest can be considered negligible.

We will reduce the number of parameters and simplify the notation in a number of ways to create a model where all companies with the same rating l have the same overall default intensity λ_{total}^l where

$$\lambda_j = \lambda_{\text{total}}^{l(j)}. \tag{13}$$

This reflects the fact that it is common to base the assessment of default intensities for individual companies on information about default intensities for similarly rated companies, where this information comes either from an internal rating system and database or from an external rating agency.

To achieve (13) we first assume that the rate of occurrence of idiosyncratic shocks depends only on the rating class of the company and we adopt the following more obvious notation

$$\lambda^{(j)} = \lambda_{\text{idio}}^{l(j)}, \quad \lambda_j^{(n+k(j))} = \lambda_{\text{sector}}^{k(j)}, \quad \lambda^{(n+5)} = \lambda_{\text{global}}.$$

Thus we have a total of 7 shock intensities to set.

We assume that the conditional default probabilities given the occurrence of sector shocks only depend on the rating class of the company and write

$$p_j^{(n+k(j))} = s_{l(j)}^{k(j)}.$$

We assume moreover that the default indicators for several companies in the same sector are conditionally independent given the occurrence of an event in that sector.

Analogously, we assume that the conditional default probabilities given the occurrence of global shocks depend on both rating class and sector of the company and write

$$p_j^{(n+5)} = g_{l(j)}^{k(j)}.$$

We assume that the default indicators for any group of companies are conditionally independent given the occurrence of a global event.

In total we have 16 conditional default probabilities to set and we have the system of equations

$$\lambda_{\text{total}}^l = \lambda_{\text{idio}}^l + s_l^k \lambda_{\text{sector}}^k + g_l^k \lambda_{\text{global}}, \quad k = 1, \dots, 4, \ l = 1, 2,$$

subject to the constraint, imposed by (13), that

$$s_l^k \lambda_{\text{sector}}^k + g_l^k \lambda_{\text{global}} = s_l^{k'} \lambda_{\text{sector}}^{k'} + g_l^{k'} \lambda_{\text{global}}, \ \forall k \neq k'.$$

Again we are interested in $N(t)$, the total number of defaults and its tail in particular. Of course $N(t)$ may differ slightly from the number of defaulting obligors since the Poisson assumption allows obligors to default more than once. However, given realistic default intensities the number of obligors that default more than once is an essentially negligible portion of the total number of defaulting obligors.

In our examples we take $t = 1$ year, $\lambda_{\text{total}}^1 = 0.005$ and $\lambda_{\text{total}}^2 = 0.02$. Let $n_{l,k}$ denote the number of companies in rating class l and sector k. We set

$$n_{1,1} = 10000, n_{1,2} = 20000, n_{1,3} = 15000, n_{1,4} = 5000,$$
$$n_{2,1} = 10000, n_{2,2} = 25000, n_{2,3} = 10000, n_{2,4} = 5000.$$

In the following two cases we will investigate by means of a simulation experiment the sensitivity of the tail of $N(1)$ to the finer details of the specification of model parameters. In all cases our results are based on 10000 simulated realizations of $N(1)$.

- **Case 1**

  Suppose we attribute 40% of defaults for companies in both ratings classes to idiosyncratic shocks and 60% to common shocks. That is we assume

  $$(\lambda_{\text{idio}}^1, \lambda_{\text{idio}}^2) = (0.002, 0.008).$$

  Suppose, for both rating classes, we attribute to sector specific causes, 20% of defaults of sector 1 companies, 50% of defaults of sector 2 companies, 10% of defaults of sector 3 companies and 40% of defaults of sector 4 companies. Moreover we believe that the frequencies of sector and global shocks are in the ratio

  $$\lambda_{\text{sector}}^1 : \lambda_{\text{sector}}^2 : \lambda_{\text{sector}}^3 : \lambda_{\text{sector}}^4 : \lambda_{\text{global}} = 1 : 5 : 2 : 4 : 1$$

  We have now specified the model up to a single factor f. For any $f \geq 0.05$ the following choices of model parameters would satisfy our requirements

  $$(\lambda_{\text{sector}}^1, \lambda_{\text{sector}}^2, \lambda_{\text{sector}}^3, \lambda_{\text{sector}}^4, \lambda_{\text{global}}) = f(0.2, 1.0, 0.4, 0.8, 0.2)$$
  $$(s_1^1, s_1^2, s_1^3, s_1^4, s_2^1, s_2^2, s_2^3, s_2^4) = \frac{1}{f}(0.5, 0.25, 0.125, 0.25, 2, 1, 0.5, 1)10^{-2}$$
  $$(g_1^1, g_1^2, g_1^3, g_1^4, g_2^1, g_2^2, g_2^3, g_2^4) = \frac{1}{f}(1, 0.25, 1.25, 0.5, 4, 1, 5, 2)10^{-2}.$$

	Case 1				Case 2			
	$f=1$	$f=2$	$f=4$	$f=8$	$f=1$	$f=2$	$f=4$	$f=8$
$\alpha=0.95$	2742	2307	1957	1734	1308	1769	2106	2346
$\alpha=0.99$	3898	2889	2381	1972	1331	2180	2622	2948

Table 1: Empirical quantiles of $N(1)$ corresponding to the samples of size 10000 shown in Figures 3 and 4.

The condition $f \geq 0.05$ is to ensure that $s_1^1, \ldots, s_2^4, g_1^1, \ldots, g_2^4 \leq 1$. When f is increased by a factor Δf the intensities of the common shocks are increased by a factor Δf and the univariate conditional default probabilities are decreased by a factor $1/\Delta f$. The effect of increasing f on the distribution of $N(1)$ is seen in figure 3, where histograms are plotted by row for $f = 1, 2, 4, 8$. The key message is that low shock intensities and high conditional default probabilities are more "risky" than the other way around. Values for the empirical 95th and 99th percentiles of the distribution of $N(1)$ are given in Table 1.

- **Case 2**

  Now we study the effects of increasing the intensity of the common shocks and decreasing the intensity of the idiosyncratic shocks when the univariate conditional default probabilities are held constant.

  Suppose we are able to quantify the probabilities with which sector or global shocks cause the default of individual companies. We set the values

$$
\begin{aligned}
(s_1^1, s_1^2, s_1^3, s_1^4, s_2^1, s_2^2, s_2^3, s_2^4) &= (0.25, 0.08, 0.05, 0.1, 1, 0.3, 0.25, 0.25)10^{-2} \\
(g_1^1, g_1^2, g_1^3, g_1^4, g_2^1, g_2^2, g_2^3, g_2^4) &= (0.25, 0.1, 0.4, 0.1, 1, 0.5, 1.5, 1)10^{-2}.
\end{aligned}
$$

  We have some flexibility in choosing the intensities

$$
(\lambda_{\text{idio}}^1, \lambda_{\text{idio}}^2, \lambda_{\text{sector}}^1, \lambda_{\text{sector}}^2, \lambda_{\text{sector}}^3, \lambda_{\text{sector}}^4, \lambda_{\text{global}})
$$

  and we vary them progressively in the following way

$$
\begin{aligned}
&(0.005, 0.02, 0.0, 0.0, 0.0, 0.0, 0.0) \rightarrow (0.004, 0.016, 0.2, 1.0, 0.4, 0.8, 0.2) \rightarrow \\
&(0.002, 0.008, 0.6, 3.0, 1.2, 2.4, 0.6) \rightarrow (0.0, 0.0, 1.0, 5.0, 2.0, 4.0, 1.0).
\end{aligned}
$$

  Hence we start with the special case of no common shocks and a situation where the individual default processes $N_j(t)$, for $j = 1, \ldots, n$, are independent Poisson and the total number of defaults $N(t)$ is Poisson. In the second model we still attribute 80% of the default intensities λ_j to idiosyncratic shocks, but we now have 20% in common shocks. In the third model we have 60% in common shocks and in the final model we have only common shocks. The effect of the increasing portion of defaults due to common shocks on the distribution of $N(1)$ is seen in Figure 4 and quantiles of $N(1)$ are given in Table 1.

Clearly both these cases show that the exact specification of the common shock model is critical to the nature of the tail of the aggregate loss distribution. In large loan portfolio situations in credit risk management there is often a lack of relevant historical data to allow the estimation of the parameters of a portfolio risk model; calibration using a large element of judgement is inevitable. Our analyses give some insight into the model risk and uncertainty implicit in such a calibration.

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A Panjer Recursion for the Distribution of $N(t)$

If there are common shocks, then $N(t) = \sum_{j=1}^{n} N_j(t)$ does not have a Poisson distribution. However, the probability distribution function of $N(t)$ can still be calculated using Panjer recursion. As before let $\widetilde{N}(t)$ count the number of loss-causing shocks in $(0, t]$. In addition, let W_i denote the number of losses due to the ith loss-causing shock. The total number of losses, $N(t)$, has the stochastic representation

$$N(t) \stackrel{\mathrm{d}}{=} \sum_{i=1}^{\widetilde{N}(t)} W_i,$$

where $W_1, \ldots, W_{\widetilde{N}(t)}$ ($\stackrel{\mathrm{d}}{=} W$) are iid and independent of $\widetilde{N}(t)$. The probability that a loss-causing shock causes exactly k losses is given by

$$P(W = k) = \frac{1}{\widetilde{\lambda}} \sum_{s:|s|=k} \lambda_s,$$

for $k \in \{1, \ldots, n\}$. Clearly

$$\sum_{k=1}^{n} P(W = k) = \frac{1}{\widetilde{\lambda}} \sum_{k=1}^{n} \sum_{s:|s|=k} \lambda_s = \frac{1}{\widetilde{\lambda}} \sum_{s \in \mathcal{S}} \lambda_s = 1.$$

Recall that

$$\sum_{s:|s|=k} \lambda_s = \sum_{e=1}^{m} \lambda^{(e)} \sum_{s:|s|=k} \sum_{s':s' \supseteq s} (-1)^{|s'|-|s|} p_{s'}^{(e)}.$$

The expression of the probability that a loss-causing shock causes n losses can be simplified to

$$P(W=n)=\frac{1}{\widetilde{\lambda}}\sum_{e=1}^{m}\lambda^{(e)}p^{(e)}_{\{1,\dots,n\}}=\frac{1}{\widetilde{\lambda}}\sum_{e=1}^{m}\lambda^{(e)}p^{(e)}_{1,\dots,n}(1,\dots,1).$$

For $k<n$ we note that there are $\binom{n}{k}$ sets s with $|s|=k$, and for each such s there are $\binom{n-k}{i}$ sets of size $k+i$ ($i\in\{1,\dots,n-k\}$) which contains s as a proper subset. Hence

$$\sum_{s:|s|=k}\sum_{s':s'\supseteq s}(-1)^{|s'|-|s|}p^{(e)}_{s'} \tag{14}$$

consists of $\binom{n}{k}\binom{n-k}{i}$ terms $(-1)^{|s'|-|s|}p^{(e)}_{s'}$ for which $|s'|=k+i$ and $s\subset s'$. Since there are $\binom{n}{k+i}$ sets s' with $|s'|=k+i$ it follows that (14) is equal to

$$\sum_{s:|s|=k}p^{(e)}_s+\sum_{i=1}^{n-k}(-1)^i\frac{\binom{n}{k}\binom{n-k}{i}}{\binom{n}{k+i}}\sum_{s:|s|=k+i}p^{(e)}_s$$

and hence

$$P(W=k)=\frac{1}{\widetilde{\lambda}}\sum_{e=1}^{m}\lambda^{(e)}\left(\sum_{s:|s|=k}p^{(e)}_s+\sum_{i=1}^{n-k}(-1)^i\frac{\binom{n}{k}\binom{n-k}{i}}{\binom{n}{k+i}}\sum_{s:|s|=k+i}p^{(e)}_s\right),\ k<n.$$

The probability $P(N(t)=r)$ can now easily be calculated using Panjer recursion:

$$P(N(t)=r)=\sum_{i=1}^{r}\frac{\widetilde{\lambda}ti}{r}P(W=i)P(N(t)=r-i),\ r\geq 1,$$

where $P(N(t)=0)=\exp(-\widetilde{\lambda}t)$. Since $P(W=k)=0$ for $k>n$,

$$P(N(t)=r)=\sum_{i=1}^{\min(r,n)}\frac{\widetilde{\lambda}ti}{r}P(W=i)P(N(t)=r-i),\ r\geq 1.$$

For large n, say $n>100$, the usefulness of the Panjer recursion scheme relies heavily on the calculation of $\sum_{s:|s|=k}p^{(e)}_s$ for $k\in\{1,\dots,n\}$. We now look at two specific assumptions on the multivariate Bernoulli distribution of $\mathbf{I}^{(e)}$ conditional on a shock of type e.

The assumption of conditional independence is attractive for computations since in this case

$$\sum_{s:|s|=k}p^{(e)}_s=\sum_{j_1=1}^{n}\sum_{j_2>j_1}\cdots\sum_{j_k>j_{k-1}}p^{(e)}_{j_1}p^{(e)}_{j_2}\dots p^{(e)}_{j_k}.$$

Under the assumption of conditional comonotonicity

$$\sum_{s:|s|=k}p^{(e)}_s=\sum_{j_1=1}^{n}\sum_{j_2>j_1}\cdots\sum_{j_k>j_{k-1}}\min(p^{(e)}_{j_1},p^{(e)}_{j_2},\dots,p^{(e)}_{j_k}).$$

The latter assumption leads to very efficient computations of $\sum_{s:|s|=k}p^{(e)}_s$. Let

$$(p^{(e)}_{\pi_1},p^{(e)}_{\pi_2},\dots,p^{(e)}_{\pi_n})'$$

denote the sorted vector of univariate conditional indicator probabilities, such that $p_{\pi_1}^{(e)} \leq p_{\pi_2}^{(e)} \leq \cdots \leq p_{\pi_n}^{(e)}$. Then

$$\sum_{s:|s|=k} p_s^{(e)} = \sum_{i=1}^{n} \binom{n-i}{k-1} p_{\pi_i}^{(e)},$$

where $\binom{n-i}{k-1}$ is the number of subsets of size k of $\{1, \ldots, n\}$ with i as smallest element.

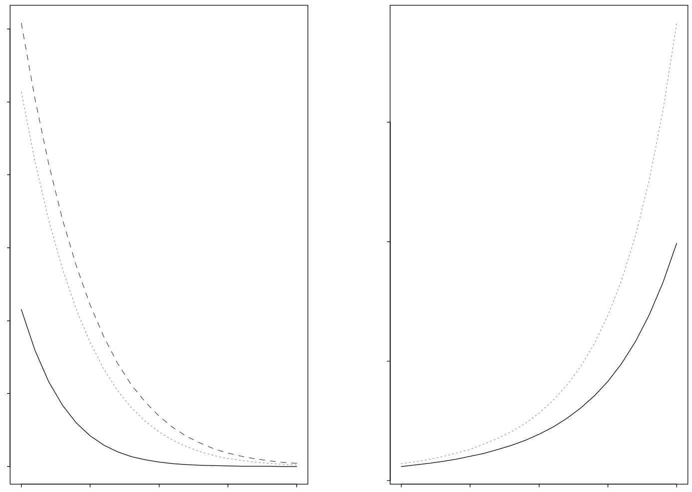

Figure 1: Left: Exceedence probabilities $P(N(5) > k)$ for $k = 70, 71, \ldots, 90$, for case 1, 2 and 3, from lower to upper (linear interpolation between the points). Right: Ratios of such exceedence probabilities for cases $1-2$ and $1-3$ from lower to upper (linear interpolation between the points).

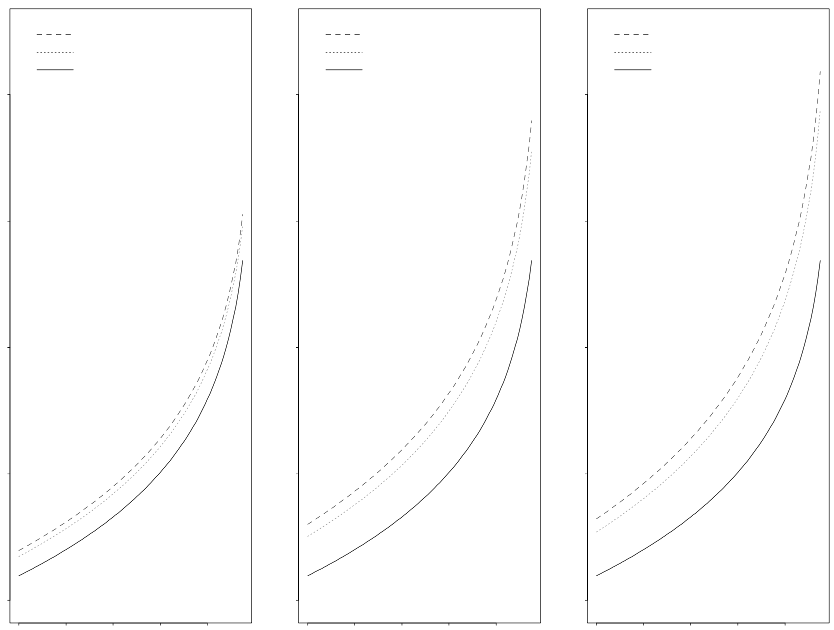

Figure 2: The curves from lower to upper show the quantiles of moment fitted generalised F-distributions for case 1, 2 and 3 and $\alpha \in [0.900, 0.995]$. The first three moments coincide with those of $Z(5)$.

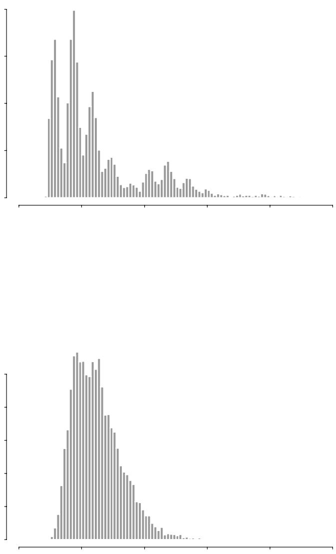

Figure 3: Histograms of 10000 independent simulations of $N(1)$ for $f = 1, 2, 4$ and 8.

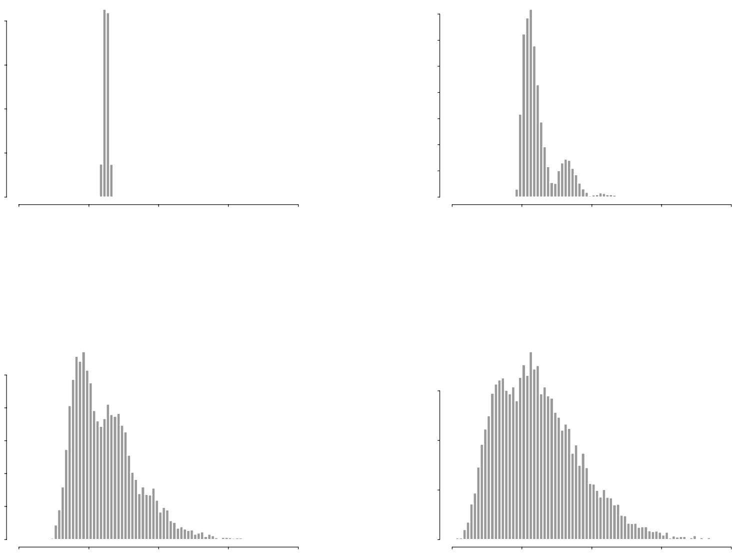

Figure 4: Histograms of 10000 independent simulations of $N(1)$ when increasing the intensities of the common shocks and decreasing the intensities of the idiosyncratic shocks while holding the univariate conditional default probabilities fixed.