COHERENT MULTIPERIOD RISK MEASUREMENT

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February 13, 2002

Abstract. We explain why and how to deal with the definition, acceptability, computation and management of risk in a genuinely multitemporal way. Coherence axioms provide a representation of a risk-adjusted valuation. Some special cases of practical interest allowing for easy recursive computations are presented. The multiperiod extension of Tail VaR is discussed.

1. NEW QUESTIONS WITH MULTIPERIOD RISK

RISK EVOLVING OVER SEVERAL PERIODS of uncertainty is different from one-period risk in many ways. An analysis of multiperiod risk requires consideration of new issues, since:

- availability of information may require taking into account *intermediate monitoring* by supervisors or shareholders of a *locked-in* position,
- the possibility of *intermediate actions*, availability of extraneous cash flows, of possible capital in- or outflows require handling *sequences* of unknown future "values".

A PORTFOLIO OR A STRATEGY built over several periods should be analyzed with respect to these issues. We attempt to:

- distinguish models of future worth at the end of a holding period from models in which successive values or cash flows are examined, and are subject to some investment/financing strategy.
- give some information about the necessity and/or availability of remedial funding at some intermediate date either in the case of sudden loss or in the case of insolvency of the firm, as urged for example in [Be] (notice that one-period models considered neither the source of (extra-) capital at the beginning of the holding period nor the actual consequences of a "bad event" at the end of the same period).
- take into account the actual time evolution of risk and of available capital. Study whether a relevant risk-adjusted measurement should consider *more* than the distribution of final net worth of a strategy, to decide upon its acceptability at the initial date.
- distinguish between the opinion of a risk manager on some strategy, and the attitude of a supervisor/regulator who, at any date, considers only the

This research has benefited from support by PriceWaterhouseCoopers and by RiskLab, ETH Zürich.

current portfolio, refusing to take into account future possible changes in the composition of the portfolio (see the example in Section 8).

Remark. With one period of uncertainty, capital appeared both as a buffer at the initial date and as wealth at the final date. Intermediate dates raise the question of the nature of capital (valued in a market or accounting way) at such dates.

2. REVIEW OF ONE PERIOD COHERENT ACCEPTABILITY

COHERENT ONE PERIOD RISK ADJUSTED VALUES' theory is best approached (see [ADEH1], p. 69, [ADEH2], Section 2.2, as well as [He]) by taking the primitive object to be an "acceptance set", that is a set of acceptable future net worths, also called simply "values". This set is supposed to satisfy some "coherence" requirements. If we assume here (as well as in following sections) a zero interest rate for simplicity, the representation result states the following: for any acceptance set, there exists a set $\mathcal{P}$ of probability distributions (called generalised scenarios or test probabilities) on the space Ω of states of nature, such that a given position, with future (random) value denoted by X, is acceptable if and only if:

For each test probability $\mathbb{P} \in \mathcal{P}$, *the expected value of the future net worth under* $\mathbb{P}$, *i.e.* $\mathbf{E}_{\mathbb{P}}[X]$, *is non-negative.*

The *risk-adjusted value* $\pi(X)$ of a future net worth X is defined as follows:

- compute, under each test probability $\mathbb{P} \in \mathcal{P}$, the average of the future net worth X of the position, in formula $\mathbf{E}_{\mathbb{P}}[X]$,
- take the minimum of all numbers found above, which corresponds to the formula $\pi(X) = \inf_{\mathbb{P}\in\mathcal{P}} \mathbf{E}_{\mathbb{P}}[X]$.

The axioms of coherent risk measures, well known by now (see[ADEH1]), translate for coherent risk-adjusted values into:

- monotonicity: if $X \geq Y$ then $\pi(X) \geq \pi(Y)$,
- translation invariance: if a is a constant then $\pi(a \cdot 1 + X) = a + \pi(X)$,
- positive homogeneity: if $\lambda \geq 0$ then $\pi(\lambda \cdot X) = \lambda \cdot \pi(X)$,
- superadditivity: $\pi(X + Y) \geq \pi(X) + \pi(Y)$.

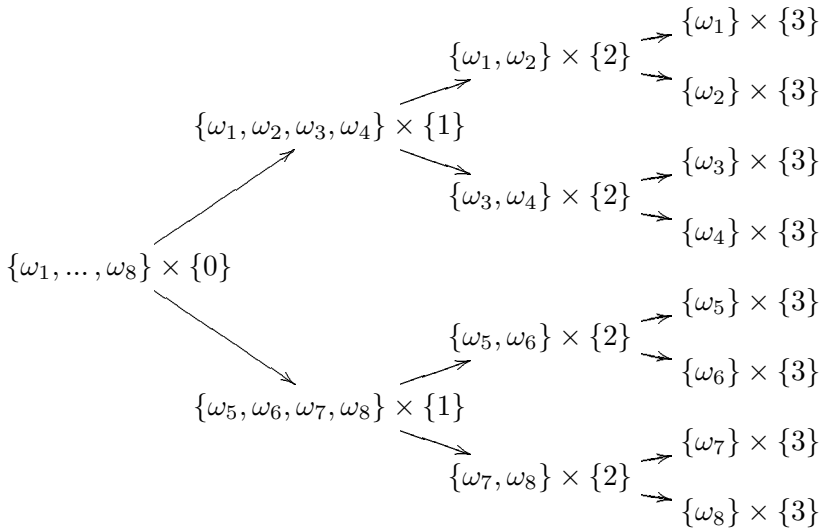
Remark. The risk measure $\rho(X)$ for X studied in [ADEH1] and [ADEH2], is simply the negative of the risk adjusted value $\pi(X)$ for X. The change of sign will simplify the treatment of measures of successive risks.

3. COHERENT MULTIPERIOD RISK-ADJUSTED VALUE

THE CASE OF T PERIODS OF UNCERTAINTY will be described here in the language of trees. As noted already by one of the authors, they allow for some things "more easily done than said", and we first need to define a few terms. We represent the availability of information over time by the set Ω of "states of nature" at date T and, for each date $t = 0, \ldots, T$, the partition $\mathcal{N}_t$ of Ω consisting of the set of smallest events which by date t are declared to obtain or not. These events are "tagged" by the date t and are called the nodes of the tree at date t. We use for such a node n the notation $(\underline{n}, t(n))$ or $\underline{n} \times \{t(n)\}$.

The partition $\mathcal{N}_{t+1}$ is a refinement of the partition $\mathcal{N}_t$ and this provides the ancestorship relation of $(\underline{m}, t)$ to $(\underline{n}, t+1)$ by means of the inclusion $\underline{n} \subset \underline{m}$.

For example, the "three period (four date) binomial tree" can be described in two ways (see Figure 1) by $\Omega = \mathcal{N}_3 = \{[uuu], [uud], [udu], [udd], [duu], [dud], [ddu], [ddd]\}$, $\mathcal{N}_2 = \{[uu], [ud], [du], [dd]\}$,$\mathcal{N}_1 = \{[u], [d]\}$, $\mathcal{N}_0 = \{[]\}$. The ancestorship relation amounts to suppress the right hand letter in each word based on u and d and the tagging amounts to count the number of letters within the brackets. From now on we shall most of the time neglect to write the brackets [and].



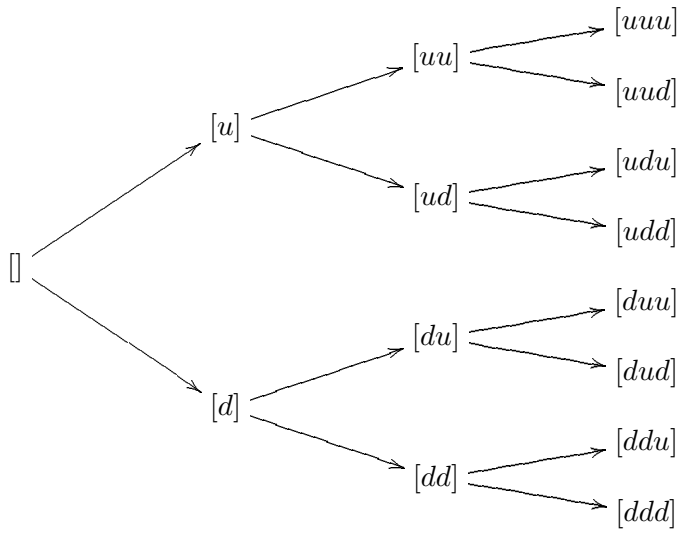


Figure 1.

Remark. The binomial tree is misleadingly simple. It may well happen that some node n of date t stops branching. We then have to distinguish $(\underline{n}, t)$ and $(\underline{n}, t+1)$.

SEQUENCES OF "VALUES" at dates $0, \ldots, T$ will be the object of study. Technically they will be represented as functions on the tree $\mathcal{T}(\Omega)$ (such functions are also called *processes* to emphasize the time dimension). The restriction of such a function X to the set $\mathcal{N}_t$ of nodes at date t is also considered a function on Ω, denoted by X_t. Then $X_t(n)$ denotes (with some redundancy) the "value" at date t in the "node" or event n as well as in any of the states of nature belonging to $\underline{n}$. It is also interesting to view the process $X = (X_t)_{0 \leq t \leq T}$ as a function on the product space $\{0, 1, \ldots, T\} \times \Omega$ which happens for each date to be constant in any node of this date: $X_t(\omega) = X_t(\omega')$ as soon as there exists a node n at date t with both states ω and ω' belonging to $\underline{n}$. For example $X_0(\omega) = X_0(\omega')$ for any pair (ω, ω') .

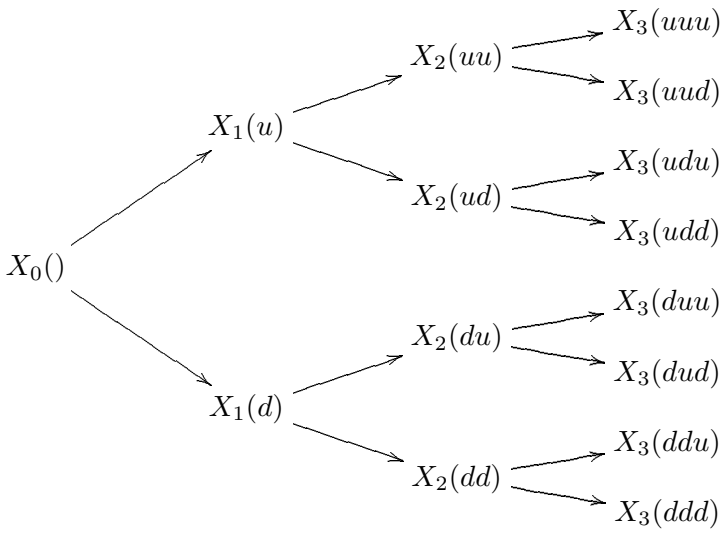


Figure 2.

Remark. Probabilists prefer the language of filtrations and optional processes to that of trees. We shall later write a companion paper dealing with these more mathematical issues, and will instead concentrate here on the ideas.

We obtain from any probability $\mathbb{P}$ on Ω (with $\mathbb{P}[\{\omega\}] > 0$ for each $\omega \in \Omega$ for simplicity) a probability $\mathbb{P}_{\mathcal{T}}$ on $\mathcal{T}(\Omega)$ by the equality relative to each node n:

$$\mathbb{P}_{\mathcal{T}}[\{n\}] = \frac{1}{T+1} \sum_{\omega \in \underline{n}} \mathbb{P}[\{\omega\}] .$$

For each function Y on $\mathcal{T}(\Omega)$ we have the formula:

$$\mathbf{E}_{\mathbb{P}_{\mathcal{T}}}[Y] = \frac{1}{T+1} \sum_{0 \leq t \leq T} \mathbf{E}_{\mathbb{P}}[Y_t]$$

For each probability $\mathbb{P}$ on Ω, each date t and each random variable X_T the conditional expectation at date t of X_T is the function on $\mathcal{N}_t$ (or equivalently the function on Ω which is constant on every node of $\mathcal{N}_t$) defined by:

$$\mathbf{E}_{\mathbb{P}}[X_T \mid \mathcal{N}_t](n) = \mathbf{E}_{\mathbb{P}}[X_T \mid \underline{n}] .$$

Figure 3 shows the case of the reference probability $\mathbb{P}_0$ on the three period binomial tree (for readability, several { and } signs as well as dates have been neglected).

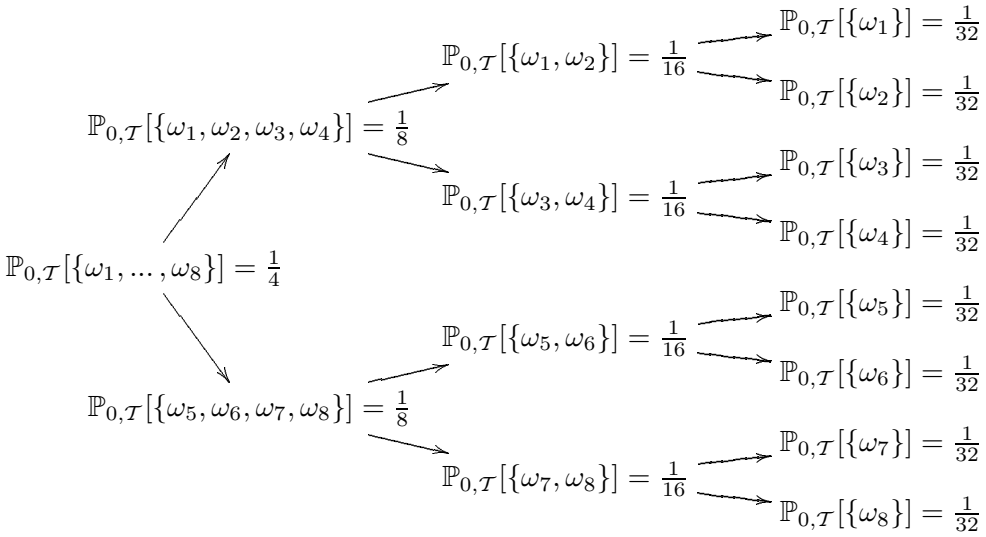


Figure 3.

A SUPERVISOR, risk manager or regulator will, as in the one-period case, decide at date 0 upon a set of acceptable "Values", a subset of the set $\mathcal{G}_{\mathcal{T}}$ of all value processes. There are many interpretations of the meaning of "value":

- market values of equity,
- accounting values of equity, (see for example the notion of default mode paradigm in [Ba], Section II.2.B),
- *cumulative* cash flows,
- liquidation values (see by contrast the notion of untradable positions, like loans mentioned in [GT], Ch. 12, p. 594),
- surplus,
- actuarial values.

We therefore deal with both prospective and retrospective notions of value.

SOLVENCY is an important concern. For the "value" $(X_t)_{0\leq t\leq T}$ of a portfolio or of a strategy, one defines formally the "insolvency time" $\sigma = \inf\{t | X_t < 0,\ 1 \leq t \leq T\}$, and the stopped process X^σ equal to X_t before the time σ and to X_σ from time σ on. When X is a market value, one may say that risk measurement balances the costs of insolvency with the benefits of risk-taking. With a liquidation value, one can imagine that after "insolvency" time, there may be more favorable dates and events where to close the business, and the risk measurement should try to incorporate this possibility.

A COHERENT ACCEPTANCE SET of "values" is a closed convex cone $\mathcal{A}_{cc}$ in $\mathcal{G}_{\mathcal{T}}$, with vertex at the origin, containing the positive orthant and intersecting the negative orthant only at the origin. As in the framework of one-period risk we define the *risk adjusted valuation* associated with the cone $\mathcal{A}_{cc}$ by computing for each "value" process X the number $\pi(X) = \sup\{m \mid X - m \in \mathcal{A}_{cc}\}$. This reflects the fact that risk adjusted value is the largest amount of capital which can be subtracted from the project X and still leave it acceptable. The assumptions on

$\mathcal{A}_{cc}$ ensure that the the risk adjusted valuation is coherent, i.e. satisfies the four conditions listed at the end of Section 2.

The incorporation of time in the set $\mathcal{T}(\Omega)$ allows us to *directly* deduce from the study of the one-period analysis that there exists a set $\mathcal{P}_{\mathcal{T}}$ of probabilities on $\mathcal{T}(\Omega)$ such that:

$$\text{for each } X \in \mathcal{G}_{\mathcal{T}},\ \pi(X) = \inf_{\mathbb{P}_{\mathcal{T}} \in \mathcal{P}_{\mathcal{T}}} \mathbf{E}_{\mathbb{P}_{\mathcal{T}}}[X].$$

Each "test probability" $\mathbb{P}_{\mathcal{T}} \in \mathcal{P}_{\mathcal{T}}$ can be described by its density $f_{\mathcal{T}} = \frac{d\mathbb{P}_{\mathcal{T}}}{d\mathbb{P}_{0,\mathcal{T}}}$ with respect to $\mathbb{P}_{0,\mathcal{T}}$, where $\frac{d\mathbb{P}_{\mathcal{T}}}{d\mathbb{P}_{0,\mathcal{T}}}(n) = \frac{\mathbb{P}_{\mathcal{T}}(n)}{\mathbb{P}_{0,\mathcal{T}}(n)}$, for each node n in $\mathcal{T}(\Omega)$. This density has to be a function $f_{\mathcal{T}}$ on the tree $\mathcal{T}(\Omega)$, and we represent it as $f_{\mathcal{T}} = (f_t)_{0 \le t \le T}$ where each f_t is a positive function on $\mathcal{N}_t$, such that $\sum_{0 \le t \le T} \frac{1}{T+1} \mathbf{E}_{\mathbb{P}_0}[f_t] = 1$.

We then have by definition for each X, $\mathbf{E}_{\mathbb{P}_{\mathcal{T}}}[X] = \sum_{0 \le t \le T} \frac{1}{T+1} \mathbf{E}_{\mathbb{P}_0}[f_t X_t]$. Defining the increasing process A by $A_t = A_{t-1} + \frac{1}{T+1} f_t$, with $A_{-1} = 0$, we get that $\mathbf{E}_{\mathbb{P}_0}[A_T] = 1$ and we obtain the:

REPRESENTATION RESULT: *For each coherent risk-adjusted valuation, there is a set $\mathcal{A}$ of positive increasing processes A with $\mathbf{E}_{\mathbb{P}_0}[A_T] = 1$ such that for each value process X its date 0 risk-adjusted value $\pi(X)$ is given by:*

$$\pi(X) = \inf_{A \in \mathcal{A}} \mathbf{E}_{\mathbb{P}_0} \left[\sum_{0 \le t \le T} X_t \cdot (A_t - A_{t-1}) \right].$$

Example 1. If for each $t < T$ all the A_t are zero, we simply obtain the risk-adjusted value formula:

$$\pi(X) = \inf_{A \in \mathcal{A}} \mathbf{E}_{\mathbb{P}_0}[A_T X_T] = \inf_{\mathbb{P} \in \mathcal{P}} \mathbf{E}_{\mathbb{P}}[X_T],$$

where $\mathcal{P}$ is the set of all probabilities on Ω having as density with respect to the reference probability $\mathbb{P}_0$ any random variable A_T such that $\mathbf{E}_{\mathbb{P}_0}[A_T] = 1$. Thus our earlier one-period analysis is a special case of this more general theory.

Example 2. We can define a process $A \in \mathcal{A}$ which allows for stopping times "at which the value process is being considered" (a stopping time being any map σ from Ω into the set of dates such that for each date t, $0 \le t \le T$, the event $\{\sigma = t\}$ is a union of nodes from $\mathcal{N}_t$). A stopping time σ defines (if $\mathbb{P}_0[\sigma \le T] > 0$) an increasing process A^σ by $A^\sigma_0 = 0$ and by $A^\sigma_t = \frac{1}{\mathbb{P}_0[\sigma \le T]} \mathbf{1}_{\{\sigma \le t\}}$ for $0 \le t \le T$, where for any event E, $\mathbf{1}_E(\omega) = 1$ or 0 depending on whether or not $\omega \in E$. The coherent risk-adjusted value given by $\pi(X) = \mathbf{E}_{\mathbb{P}_0}[X_\sigma]$ is also $\mathbf{E}_{\mathbb{P}_0}\left[\sum_{0 \le t \le T} X_t (A^\sigma_t - A^\sigma_{t-1})\right]$. As an example, energy risk might involve the occurrence of time intervals having length at least L, of temperature θ in a given range I. Such feature can be handled by the set of all the stopping times $\sigma_k = \inf\{t,\ \theta_s \in I \text{ for } t - k < s \le t\}$, $k \ge L$.

Example 3. Taking $\mathcal{A}$ to be the set of all the processes A^t, $0 \le t \le T$, with $A^t(s, \omega) = \mathbf{1}_{\{s \ge t\}}$, we find for each X that $\pi(X) = \min_{0 \le t \le T} \mathbf{E}_{\mathbb{P}_0}[X_t]$.

Example 4. Let τ be any random time, i.e. any map from Ω into the set of dates. Consider the test probability $\mathbb{P}^\tau_{\mathcal{T}}$ associated to the process A^τ defined by $A^\tau_t = A^\tau_{t-1} + \mathbf{E}_{\mathbb{P}_0}[\mathbf{1}_{C_t} \mid \mathcal{N}_t]$ with $C_t = \{\tau = t\}$. For each process X we

have $\mathbf{E}_{\mathbb{P}_0}\left[\sum_{0\leq t\leq T} X_t \cdot \mathbf{1}_{C_t}\right] = \mathbf{E}_{\mathbb{P}_0}\left[\sum_{0\leq t\leq T} X_t \cdot \left(A_t^\tau - A_{t-1}^\tau\right)\right]$. Using the random time $\bar{\tau}(\omega) = argmin_t\left(X_t\left(\omega\right)\right)$ we find for the risk-adjusted valuation $\pi(X) = \mathbf{E}_{\mathbb{P}_0}\left[\inf_{0\leq t\leq T} X_t\right]$ that

$$\pi(X) = \inf_{\tau} \mathbf{E}_{\mathbb{P}_0}\left[\sum_{0\leq t\leq T} X_t \cdot \left(A_t^\tau - A_{t-1}^\tau\right)\right].$$

Remark 1. It is the elaborate formulation of the general representation of risk-adjusted measurement which allowed us to deal with Example 4 where time and states of nature mix in an interesting way. In particular it would have not been possible to deal directly there (as we did in Example 2) with the variable $\mathbf{1}_{\{\bar{\tau}\leq t\}}$ since it is not constant on the nodes at date t,

Remark 2. If the value process X is a $\mathbb{P}_0-$martingale, meaning that for each $t < T$, $X_t = \mathbf{E}_{\mathbb{P}_0}\left[X_{t+1} \mid \mathcal{N}_t\right]$, the general expression for $\pi(X)$ simplifies to the expression $\inf_{A\in\mathcal{A}} \mathbf{E}_{\mathbb{P}_0}\left[X_T A_T\right]$. In words, for our risk measurement method, "only the final value matters" when dealing with martingales. Since our method handles more general processes it therefore deals with more general business conditions than "marking to market".

Remark 3. One could consider more general acceptance sets than convex cones, as was done in [He] to represent the risk measurement constraints imposed by the shareholders of a firm.

4. TWO MULTIPERIOD RISK MEASUREMENTS OF A FINAL VALUE

THE ABSENCE OF INTERMEDIATE MARKETS or any other form of "locked-in" position provides a situation which is actually more simple than the one studied in Section 3. The model is a sequence $(\mathcal{N}_t)_{0\leq t\leq T}$ of the sets of nodes and one "final value" X_T, i.e. a mere function on $\mathcal{N}_T = \Omega$. No change can be made to the position but information is revealed over time and the risk manager anticipates this fact in the acceptance decision at date 0. The one-period analysis of [ADEH 2] would consist, starting from a set $\mathcal{P}$ of test probabilities on Ω, in defining the number $\phi_0(X_T) = \inf_{\mathbb{P}\in\mathcal{P}} \mathbf{E}_{\mathbb{P}}\left[X_T\right]$. The same analysis applied at a later date t, would, at that date, define the "date t risk-adjusted value" $\phi_t(X_T)$ as $\inf_{\mathbb{P}\in\mathcal{P}} \mathbf{E}_{\mathbb{P}}\left[X_T|\mathcal{N}_t\right]$, defining therefore a risk-adjusted value *process* $\left(\phi_t(X_T)\right)_{0\leq t\leq T}$.

There is another construction of a risk-adjusted value, built by backward induction from the same set $\mathcal{P}$ of test probabilities on Ω. For any final value X_T, let us define the process $\psi(X_T)$ by the equality $\psi_T(X_T) = X_T$ and by the recurrence relation

$$\psi_t(X_T) = \inf_{\mathbb{P}\in\mathcal{P}} \mathbf{E}_{\mathbb{P}}\left[\psi_{t+1}(X_T) \mid \mathcal{N}_t\right], \; 0 \leq t < T.$$

The reader may check easily that ψ_0 is indeed a coherent risk adjusted value. Section 5 provides conditions on the set $\mathcal{P}$ of test probabilities on Ω under which this recursive approach is equivalent to the one-period calculation mentioned at the beginning of this section, i.e. the processes ϕ and ψ are equal. Such an equality implies a good relation between the test probabilities and the time evolution of uncertainty, as well as a link between $\phi_0(X_T)$ and the result of successive refinement of the time grid.

The recursive construction can be extended to value *processes*.

5. RECURSIVITY OF RISK MEASUREMENT AND STABILITY OF THE SET OF TEST PROBABILITIES

It can be shown that for a set $\mathcal{P}$ of test probabilities the following two properties "stability" by pasting and "recursivity", are equivalent.

STABILITY BY PASTING means that if for any date t we are given for each node n in $\mathcal{N}_t$ a probability $\mathbb{P}_n$ in $\mathcal{P}$, conditioned by this node n, the pasting of all these conditional probabilities with any probability $\mathbb{P}_0$ in $\mathcal{P}$ restricted up to time t, still provides an element of $\mathcal{P}$.

Figure 4 shows the simple binomial example of pasting, with $T = 2$, $t = 1$, $n_1 = (u, 1)$, $n_2 = (d, 1)$, $\mathbb{P}_0 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, $\mathbb{P}_1 = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\mathbb{P}_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$.

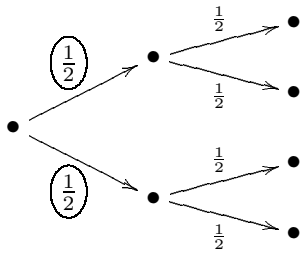


Figure 4.1: probability $\mathbb{P}_0$

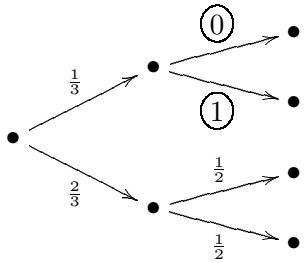


Figure 4.2: probability $\mathbb{P}_1$

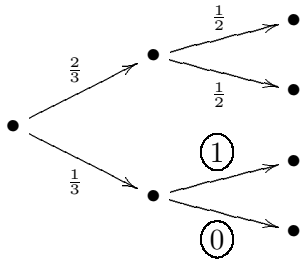


Figure 4.3: probability $\mathbb{P}_2$

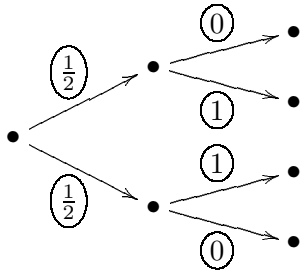


Figure 4.4: pasted probability

The pasting of probabilities amounts to looking over successive time intervals or, at the same date, over disjoint events, at the risk attitudes of various agents.

RECURSIVITY means that for each random variable X_T on Ω and for each $0 \leq t \leq T-1$:

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbf{E}_{\mathbb{P}}\left[X_T \mid \mathcal{N}_t\right] = \inf_{\mathbb{P}\in\mathcal{P}} \mathbf{E}_{\mathbb{P}}\left[\inf_{\mathbb{R}\in\mathcal{P}} \mathbf{E}_{\mathbb{R}}\left[X_T \mid \mathcal{N}_{t+1}\right] \mid \mathcal{N}_t\right].$$

Using this equality, we obtain for the risk-measurement *process* ϕ introduced in Section 4 out of a set of test probabilities on Ω:

$$\phi_t(X_T) = \inf_{\mathbb{P}\in\mathcal{P}} \mathbf{E}_{\mathbb{P}}\left[X_T \mid \mathcal{N}_t\right],$$

the recurrence relation:

$$\phi_t(X_T) = \inf_{\mathbb{P}\in\mathcal{P}} \mathbf{E}_{\mathbb{P}}\left[\phi_{t+1}(X_T) \mid \mathcal{N}_t\right],$$

and therefore the equality of the two risk-adjusted value processes $\phi(X_T)$ and $\psi(X_T)$.

The recursivity property of a set of test probabilities ensures that they deal with the timely resolution of uncertainty and the associated intermediate risk measurements. In particular, if two final values have the same risk-adjusted value at date 1 in every node, they will have the same risk-adjusted value at date 0 (a question asked in [Ha]) and if a final value is acceptable in any node of date 1 it will be acceptable at date 0.

6. DISCUSSION OF TAIL VALUE AT RISK IN THE MULTIPERIOD CASE

TAIL VALUE-AT-RISK (see [ADEH1], [ADEH2]) has become popular, in particular in the Credit Risk field. This comes from its ability to take into account extreme events (see [JZ]). The best construction is given in terms of a set of test probabilities, but this set does *not* fulfill the "stability by pasting" property.

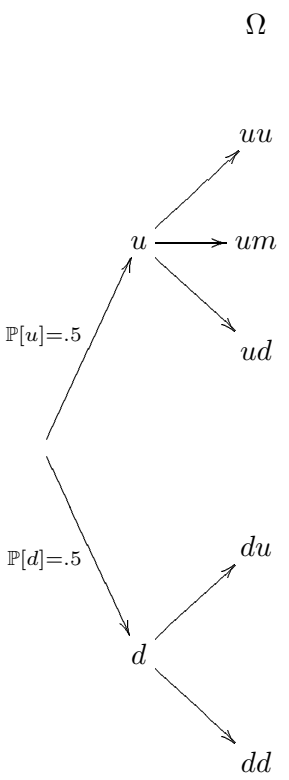
In the example of Section 5. the set $\mathcal{P}$ generating the date 0 tail value-at-risk ("TailVaR") *with level* α *equal to* $\frac{3}{4}$, is the set of probabilities with densities (with respect to $\mathbb{P}_0$) *bounded by* $\frac{1}{\alpha} = \frac{4}{3}$. The three probabilities used in the example have densities with respect to $\mathbb{P}_0$ respectively $\frac{1}{3}(0,4,4,4)$, $\frac{1}{3}(4,4,4,0)$, and $(1,1,1,1)$. The pasted probability has the density $(0,2,2,0)$ which is not bounded by $\frac{4}{3}$.

At intermediate (date,event) nodes the similar, direct, computation of TailVaR follows neither the line of construction of the ϕ_t nor the one of the ψ_t of Section 4. There is also non-recursivity: here is an example of two date 2 future values with different Tail-VaR (at the 1% level) at date 0 and the same Tail-VaR (as random variable) at date 1:

$$\Omega = \{[uu],[um],[ud],[du],[dd]\},\ \mathbb{P} = \{0.487,\ 0.01,\ 0.003,\ 0.4955,\ 0.0045\},$$

$$\mathcal{N}_1 = \{[u],[d]\},\ \mathcal{N}_0 = \{[]\}$$

$$X_2 = 1, Y_2([uu]) = Y_2([du]) = 10, Y_2([um]) = 2.5, Y_2([ud]) = Y_2([dd]) = 0.$$



Ω	$\mathbb{P}$	$\mathbb{P}[\cdot \mid u]$	$\mathbb{P}[\cdot \mid d]$	X_2	Y_2
uu	0.487	0.974	0	1	10
um	0.01	0.02	0	1	2.5
ud	0.003	0.006	0	1	0
du	0.4955	0	0.991	1	10
dd	0.0045	0	0.009	1	0

Figure 5.

We find for Y_2 the following TailVaR values at date 0 and at date 1:

$$TailVaR(Y_2)([]) = \frac{1}{0.01} \cdot ((0.0045 + 0.0030) \cdot 0 + 0.0025 \cdot 2.5) = 0.625$$

$$TailVaR(Y_2)([u]) = \frac{1}{0.01} \cdot (0.006 \cdot 0 + 0.004 \cdot 2.5) = 1$$

$$TailVaR(Y_2)([d]) = \frac{1}{0.01} \cdot (0.009 \cdot 0 + 0.001 \cdot 10) = 1.$$

Another potential weakness of Tail-VaR is the fact that $TailVaR(X_T)$ depends only on the *distribution* of X_T. This explains the feature below: on the three period binomial tree of Section 3 we can find final values X_3 and Y_3 with the same initial Tail-Var (level $\frac{3}{8}$) of 1 and *different* (random) Tail-VaR at date 1. It suffices to take:

$$X_3([uuu]) = X_3([uud]) = Y_3([uuu]) = Y_3([ddd]) = -5,$$

all others X_3 and Y_3 values being equal to 13, to find

$$TailVaR(X_3)([u]) = \frac{1}{\frac{1}{4}+\frac{1}{8}}\Big(-\frac{1}{4} \cdot 5 - \frac{1}{8} \cdot 5\Big) = -5 \text{ and}$$

$$TailVaR(Y_3)([u]) = \frac{1}{\frac{1}{4}+\frac{1}{8}}\Big(-\frac{1}{4} \cdot 5 + \frac{1}{8} \cdot 13\Big) = 1.$$

7. REPRESENTATION OF STABLE SETS OF TEST PROBABILITIES

The set of test probabilities allowing for recursive computations will be characterized in a second way when the information structure is given by a binomial tree and a random walk $W_t = U_1 + ... + U_t$, $0 \leq t \leq T$, the $(U_t)_{0 \leq t \leq T}$ being ± 1 valued independent variables with symmetric distribution. Let $\mathbb{P}_0$ be the resulting measure on Ω.

REPRESENTABILITY of a set $\mathcal{P}$ of test probabilities is defined here as follows. There exists for each t, $0 \leq t \leq T$ a random closed convex set $\mathcal{Q}_t$, $0 \leq t \leq T$ of $[-1, +1]$ depending in a $\mathcal{N}_{t-1}$-measurable way, (i.e. $\mathcal{Q}_{t+1}((n,u)) = \mathcal{Q}_{t+1}((n,d))$ for each pair $((n,u),(n,d))$ of nodes at date $t+1$, having the same ancestor n) such that:

The random variable $Z = \prod_{0 \leq t \leq T}(1 + q_t U_t)$ *is the density with respect to* $\mathbb{P}_0$ *of an element of* $\mathcal{P}$ *if and only if each* q_t *belongs to* $\mathcal{Q}_t$.

This property is equivalent to the stability property of $\mathcal{P}$ as well as to the recursivity property for the computations.

For example the set of test probabilities may be the set of probabilities $\mathbb{Q}$ such that $\frac{d\mathbb{Q}}{d\mathbb{P}_0} = Z_T$ satisfies $Z_t = \mathbf{E}_{\mathbb{P}_0}[Z_T \mid \mathcal{N}_t] = (1 + q_1 U_1)...(1 + q_t U_t)$ where q is a "predictable" process with $\delta_1 \leq q \leq \delta_2$, with $-1 \leq \delta_1 \leq \delta_2 \leq 1$ two given numbers. Predictability in the simple binomial tree framework means that for each pair $((n,u),(n,d))$ of nodes at date $t+1$ having the same ancestor n, the function q takes the same value, denoted by $q^n_{t+1} = q_{t+1}((n,u)) = q_{t+1}((n,d))$.

THE NOTION OF PRICE OF RISK provides an interesting interpretation of the stability/representation of the set of test probabilities. Suppose that the probability described above is the subjective probability for some investor and that the market uses the numbers $(1+q)/2$, $(1-q)/2$ as one-step conditional pricing probabilities. The investor pays therefore $1 + rq$ at date 0 to get the random return $1 + r$, $1 - r$ at date 1, r a number known at date 0. Under his subjective probability he has an excess expected return of $1 - (1 + rq)$ and a standard deviation of r. Therefore q (or rather $-q$) can be called price of risk for him.

THE RECURRENCE RELATION becomes when $\delta_1 = -\delta_2 = \delta \geq 0$:

$$\psi_t(X_T) = \inf_{q_{t+1}, |q_{t+1}| \leq \delta} \mathbf{E}_{\mathbb{P}_0}\left[(1 + q_{t+1}\, U_{t+1})\psi_{t+1}(X_T) \mid \mathcal{N}_t\right],$$

for $0 \leq t \leq T-1$, an easy form not requiring much storage at the nodes. In node n of date t, $\psi^n_t(X_T)$ will be computed as:

$$\psi^n_t(X_T) = 0.5 \cdot \inf_{q^n_{t+1}, |q^n_{t+1}| \leq \delta} \left((1 + q^n_{t+1}) \cdot \psi^{(n,u)}_{t+1}(X_T) + (1 - q^n_{t+1}) \cdot \psi^{(n,d)}_{t+1}(X_T)\right),$$

which requires a very simple optimisation and reduces to

$$0.5 \cdot (1+\delta) \min\{\psi^{(n,u)}_{t+1}(X_T), \psi^{(n,d)}_{t+1}(X_T)\} + 0.5 \cdot (1-\delta) \max\{\psi^{(n,u)}_{t+1}(X_T), \psi^{(n,d)}_{t+1}(X_T)\}.$$

8. EXAMPLE OF APPLICATION

The example below assumes the information structure described at the beginning of Section 7 with the set of test probabilities of the paragraph "Application to simple examples" in that Section.

A REGULATOR may not accept or even consider the intended strategy of a firm. He may choose to make his acceptance decision at date 0 on the sole basis of the consequences between date 0 and date T of holding the portfolio decided upon *initially* by the regulated firm. This provides the regulator with a value process $(X_t^{reg,0})_{0\le t\le T}$. At date 1 the regulator will again consider only the firm's portfolio as it stands (after possible trade) at date 1 and *not* the future possible effect of the firm's strategy. This attitude will provide him with *another* value process $(X_t^{reg,1})_{1\le t\le T}$.

Even if the regulator uses the same set of test probabilities as the risk manager, he may have required extra capital out of consideration of $\phi_0(X_T^{reg,0})$ at date 0, of $\phi_1(X_T^{reg,1})$ at date 1 and so on. This capital may very well differ from the extra capital decided upon internally. Such a difference raises the challenge of managing risk under both external and internal constraints.

As an illustration suppose that a firm declares a strategy of holding, at date 0 *and* at date 1, half of its assets in a money market account (with 10% interest rate per period) and the other half in a stock with i.i.d. risk neutral $(1/2, 1/2)$ returns of $\pm 20\%$. Suppose the initial value of the stock is 10 and that the firm's initial value is 20, next to a debt of 18.5 (at zero interest rate). The regulator therefore sees, at date 0, 10 in cash and 10 in stock and expects date-2 assets made of 12.1 in cash and of one stock.

This is different from the actual date 2 assets resulting from the strategy, namely:

12.65 in cash and $\frac{11.5}{12}$ stock if the stock was valued at 12 in date 1,
10.45 in cash and $\frac{9.5}{8}$ stock if the stock was valued at 8 in date 1.

The explicit risk measure built out of the upper limit for price of risk described in Section 8, differ on X_2 and $X_2^{reg,0}$. With the (admittedly very high) limit value $\delta = 0.9$ for the price of risk, the regulator accepts at date 0 since

$$\phi_1^u(X_2^{reg,1}) = 0.5 \cdot 1.9 \cdot 3.2 + 0.5 \cdot 0.1 \cdot 8 = 3.44$$

$$\phi_1^d(X_2^{reg,1}) = 0.5 \cdot 1.9 \cdot 0 + 0.5 \cdot 0.1 \cdot 3.2 = 0.16$$

$$\phi_0(X_2^{reg,0}) = 0.5 \cdot 1.9 \cdot 0.16 + 0.5 \cdot 0.1 \cdot 3.44 = 0.324\,,$$

while the risk manager refuses since

$$\phi_1^u(X_2) = 0.5 \cdot 1.9 \cdot 3.35 + 0.5 \cdot 0.1 \cdot 7.95 = 3.58$$

$$\phi_1^d(X_2) = -0.5 \cdot 1.9 \cdot 0.45 + 0.5 \cdot 0.1 \cdot 3.35 = -0.26$$

$$\phi_0(X_2) = -0.5 \cdot 1.9 \cdot 0.26 + 0.5 \cdot 0.1 \cdot 3.58 = -0.068\,.$$

CONCLUSION

The test probabilities method extends naturally from the one-period risk case to the multi-period risk case, providing a probabilisation over time. It allows analysis of the evolution of risk adjusted value over time, and of the related risk capital.

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