Coherent allocation of risk capital*

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January 2001
Original version: September 1999

Abstract

The allocation problem stems from the diversification effect observed in risk measurements of financial portfolios: the sum of the "risks" of many portfolios is larger than the "risk" of the sum of the portfolios. The allocation problem is to apportion this diversification advantage to the portfolios in a fair manner, yielding, for each portfolio, a risk appraisal that accounts for diversification.

Our approach is axiomatic, in the sense that we first argue for the necessary properties of an allocation principle, and then consider principles that fulfill the properties. Important results from the area of game theory find a direct application. Our main result is that the Aumann-Shapley value is both a coherent and practical approach to financial risk allocation.

Keywords: allocation of capital, coherent risk measure, risk-adjusted performance measure; game theory, fuzzy games, Shapley value, Aumann-Shapley prices.

*The author expresses special thanks to F. Delbaen, who provided both the initial inspiration for this work and generous subsequent ideas and advice, and to Ph. Artzner for drawing his attention to Aubin's literature on fuzzy games. Discussions with P. Embrechts, H.-J. Lüthi, D. Straumann, and S. Bernegger have been most fruitful. Finally, he gratefully acknowledges the financial support of both RiskLab (Switzerland) and the S.S.H.R.C. (Canada)

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1 Introduction

The theme of this paper is the sharing of costs between the constituents of a firm. We call this sharing "allocation", as it is assumed that a higher authority exists within the firm, which has an interest in unilaterally dividing the firm's costs between the constituents. We will refer to the constituents as portfolios, but business units could just as well be understood.

As an insurance against the uncertainty of the net worths (or equivalently, the profits) of the portfolios, the firm could well, and would often be regulated to, hold an amount of riskless investments. We will call this buffer, the *risk capital* of the firm. From a financial perspective, holding an amount of money dormant, i.e. in extremely low risk, low return money instruments, is seen as a burden. It is therefore natural to look for a fair allocation of that burden between the constituents, especially when the allocation provides a basis for performance comparisons of the constituents between themselves (for example in a RORAC approach).

The problem of allocation is interesting and non-trivial, because the sum of the risk capitals of each constituent, is usually larger than the risk capital of the firm taken as a whole. That is, there is a decline in total costs to be expected by pooling the activities of the firm, and this advantage needs to be shared fairly between the constituents. We stress fairness, as all constituents are from the same firm, and none should receive preferential treatment for the purpose of this allocation exercise. In that sense, the risk capital of a constituent, minus its allocated share of the diversification advantage, is effectively a *firm-internal risk measure*.

The allocation exercise is basically performed for comparison purposes: knowing the profit generated *and* the risk taken by the components of the firm, allows for a much wiser comparison than knowing only of profits. This idea of a richer information set underlies the popular concepts of risk-adjusted performance measures (RAPM) and return on risk-adjusted capital (RORAC).

Our approach of the allocation problem is axiomatic, in a sense that is very similar to the approach taken by Artzner, Delbaen, Eber and Heath [3]. Just as they defined a set of necessary "good qualities" of a risk measure, we suggest a set of properties to be fulfilled by a fair risk capital allocation principle. Their set of axioms defines the *coherence of risk measures,* our set of axioms defines the *coherence of risk capital allocation principles.* (Incidentally, the starting point of our development, the risk capitals of the firm and its constituents, is a coherent risk measure)

We make, throughout this article, liberal use of the concepts and results of *game theory.* As we hope to convince the reader, game theory provides an excellent framework on which to cast the allocation problem, and a eloquent language to discuss it. There is an impressive amount of literature on the allocation problem within the area of game theory, with applications ranging from telephone billing to airport landing fees and to water treatment costs. The main sources for this article are the seminal articles of Shapley [28] and [30] on one hand; and the book of Aubin [5], the articles of Billera and Heath [9]), and Mirman and Tauman [18], on the other hand.

At a more general level, the interested reader may consult a game theory reference as the nice Osborne and Rubinstein [21], the edited book of Roth [24] (including the survey of Tauman [32]), or the survey article of Young [33], which contain legions of references on the subject.

The article is divided as follows. We recall the concept of coherent risk measure in the next section. Section 3 presents the idea of the coherence in allocation. Game theory concepts are introduced in section 4, where the risk capital allocation problem is modelled as a game between portfolios. We turn in section 5 to fuzzy games, and the coherence of allocation is extended to that setting. This is where the Aumann-Shapley value emerges as a most attractive allocation principle. We treat the question of the non-negativity of allocations in section 6. The final section is devoted to a "toy example" of a coherent risk measure based on the margin rules of the SEC, and to allocations that arise while using that measure.

Remark: Beware that two concepts of coherence are discussed in this paper: the coherence of *risk measures* was introduced in [3], but is used it here as well; the coherence of *allocations* is introduced here.

2 Risk measure and risk capital

In this paper, we follow Artzner, Delbaen, Eber and Heath [3] in relating the risk of a firm to the uncertainty of its future worth. The danger, inherent to the idea of risk, is that the firm's worth reach such a low net worth at a point in the future, that it must stop its activities. Risk is then defined as a random variable X representing a firm's net worth at a specified point of the future.

A risk *measure* ρ quantifies the level of risk. Specifically, it is a mapping from a set of random variables (risks) to the real numbers: $\rho(X)$ is the amount of a numéraire (e.g. cash dollars) which, added to the firm's assets, ensures that its future worth be acceptable to the regulator, the chief risk officer or others. (For a discussion of acceptable worths, see [3]) Clearly, the heftier the required safety net is, the riskier the firm is perceived. We call $\rho(X)$ the **risk capital** of the firm. The **risk capital allocation problem** is to allocate the amount of risk $\rho(X)$ between the portfolios of the firm.

We will assume that all random variables are defined on a fixed probability space $(\Omega, \mathcal{A}, \mathbb{P})$. By $L^\infty(\Omega, \mathcal{A}, \mathbb{P})$, we mean the space of bounded random variables; we assume that ρ is only defined on that space. The reader who wishes to do so can generalize the results along the lines of [12].

In their papers, Artzner, Delbaen, Eber and Heath ([3], [2]) have suggested a set of properties that risk measures should satisfy, thus defining the concept of *coherent measures of risk*[1]:

Definition 1 *A risk measure* $\rho : L^\infty \rightarrow \mathbb{R}$ *is* **coherent** *if it satisfies the following properties:*

Subadditivity *For all bounded random variables* X *and* Y*,*

$$\rho(X + Y) \leq \rho(X) + \rho(Y)$$

Monotonicity *For all bounded random variables* X, Y *such that* $X \leq Y$[2]*,*

$$\rho(X) \geq \rho(Y)$$

Positive homogeneity *For all* $\lambda \geq 0$ *and bounded random variable* X*,*

$$\rho(\lambda X) = \lambda \rho(X)$$

[1] On the topic, see also Artzner's [1], and Delbaen's [12] and [13]

[2] The relation $X \leq Y$ between two random variables is taken to mean $X(\omega) \leq Y(\omega)$ for almost all $\omega \in \Omega$, in a probability space $(\Omega, \mathcal{F}, P)$.

Translation invariance *For all* $\alpha \in \mathbb{R}$ *and bounded random variable* X*,*

$$\rho(X + \alpha r_f) = \rho(X) - \alpha$$

where r_f *is the price, at some point in the future, of a reference, riskless investment whose price is* 1 *today.*

The properties that define coherent risk measures are to be understood as necessary conditions for a risk measure to be *reasonable*. Let us briefly justify them. Subadditivity reflects the diversification of portfolios, or that "a merger does not create extra risk" [3, p.209]. Monotonicity says that if a portfolio Y is *always* worth more than X, then Y cannot be riskier than X. Homogeneity is a limit case of subadditivity, representing what happens when there is precisely *no* diversification effect. Translation invariance is a natural requirement, given the meaning of the risk measure given above and its relation to the numéraire.

In this paper, we will not be concerned with specific risk measures, until our example of section 7; we however *assume all risk measures to be coherent.*

3 Coherence of the allocation principle

An allocation principle is a solution to the risk capital allocation problem. We suggest in this section a set of axioms, which we argue are necessary properties of a "reasonable" allocation principle. We will call *coherent* an allocation principle that satisfies the set of axioms. The following definitions are used:

- X_i, $i \in \{1, 2, \ldots, n\}$, is a bounded random variable representing the net worth at time T of the i^{th} portfolio of a firm. We assume that the n^{th} portfolio is a riskless instrument with net worth at time T equal to $X_n = \alpha r_f$, where r_f the time T price of a riskless instrument with price 1 today.
- X, the bounded random variable representing the firm's net worth at some point in the future T, is defined as $X \triangleq \sum_{i=1}^{n} X_i$.
- N is the set of all portfolios of the firm.
- A is the set of risk capital allocation problems: pairs (N, ρ) composed of a set of n portfolios and a coherent risk measure ρ.
- $K = \rho(X)$ is the risk capital of the firm.

We can now define:

Definition 2 *An* **allocation principle** *is a function* $\Pi : A \rightarrow \mathbb{R}^n$ *that maps each allocation problem* (N, ρ) *into a unique* **allocation***:*

$$\Pi : (N,\rho) \longmapsto \begin{bmatrix} \Pi_1(N,\rho) \\ \Pi_2(N,\rho) \\ \vdots \\ \Pi_n(N,\rho) \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{bmatrix} \text{ such that } \sum_{i \in N} K_i = \rho(X).$$

The condition ensures that the risk capital is fully allocated. The K_i–notation is used when the arguments are clear from the context.

Definition 3 *An allocation principle* Π *is* **coherent** *if for every allocation problem* (N, ρ)*, the allocation* $\Pi(N, \rho)$*satisfies the three properties:*

1) No undercut

$$\forall\, M \subseteq N, \quad \sum_{i \in M} K_i \;\leq\; \rho\left(\sum_{i \in M} X_i\right)$$

2) Symmetry *If by joining any subset* $M \subseteq N \backslash \{i, j\}$*, portfolios* i *and* j *both make the same contribution to the risk capital, then* $K_i = K_j$*.*

3) Riskless allocation

$$K_n = \rho(\alpha r_f) = -\alpha$$

Recall that the n^{th} *portfolio is a riskless instrument.*

Furthermore, we call **non-negative coherent allocation** a coherent allocation which satisfies $K_i \geq 0, \;\; \forall i \in N$.

It is our proposition that the three axioms of Definition 3 are necessary conditions of the fairness, and thus credibility, of allocation principles. In that sense, coherence is a yardstick by which allocation principles can be evaluated.

The properties can be justified as follows. The "no undercut" property ensures that no portfolio can undercut the proposed allocation: an undercut occurs when a portfolio's allocation is higher than the amount of risk capital it would face as an entity separate from the firm. Given subadditivity, the rationale is simple. Upon a portfolio joining the firm (or any subset thereof), the total risk capital increases by no more than the portfolio's own risk capital: in all fairness, that portfolio cannot justifiably be allocated *more* risk capital than it can possibly have brought to the firm. The property also ensures that *coalitions* of portfolios cannot undercut, with the same rationale. The symmetry property ensures that a portfolio's allocation depends only on its contribution to risk within the firm, and nothing else. According to the riskless allocation axiom, a riskless portfolio should be allocated exactly its risk measure, which incidentally will be negative. It also means that, all other things being equal, a portfolio that increases its cash position, should see its allocated capital decrease by the same amount.

4 Game theory and allocation to atomic players

Game theory is the study of situations where players adopt various strategies to best attain their individual goals. For now, players will be atomic, meaning that fractions of players are considered senseless. We will focus here on coalitional games:

Definition 4 *A coalitional game (N, c) consists of:*

- *a finite set N of n* **players,** *and*
- *a* **cost** *function c that associates a real number $c(S)$ to each subset S of N (called a* **coalition***).*

We denote by G the set of games with n players.

The goal of each player is to minimize the cost she incurs, and her strategies consist of accepting or not to take part in coalitions (including the coalition of all players).

In the literature, the cost function is usually assumed to be subadditive: $c(S \cup T) \leq c(S) + c(T)$ for all subsets S and T of N with empty intersection; an assumption which we make as well.

One of the main questions tackled in coalitional games, is the allocation of the cost $c(N)$ between all players; this question is formalized by the concept of value:

Definition 5 *A* **value** *is a function* $\Phi : G \to \mathbb{R}^n$ *that maps each game* (N, c) *into a unique* **allocation***:*

$$\Phi : (N,c) \longmapsto \begin{bmatrix} \Phi_1(N,c) \\ \Phi_2(N,c) \\ \vdots \\ \Phi_n(N,c) \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{bmatrix} \text{ where } \sum_{i \in N} K_i = c(N)$$

Again, the K_i–notation can be used when the arguments are clear from the context, and when it is also clear whether we mean $\Pi_i(N, \rho)$ or $\Phi_i(N, c)$.

4.1 The core of a game

Given the subadditivity of c, the players of a game have an incentive to form the largest coalition N, since this brings an improvement of the total cost, when compared with the sum of their individual costs. They need only find a way to allocate the cost $c(N)$ of the full coalition N, between themselves; but in doing so, players still try to minimize their own share of the burden. Player i will even threaten to leave the coalition N if she is allocated a share K_i of the total cost that is higher that her own individual cost $c(\{i\})$. Similar threats may come from coalitions $S \subseteq N$: if $\sum_{i \in S} K_i$ exceeds $c(S)$ then *every* player i in S could carry an allocated cost lower than his current K_i, if S separated from N.

The set of allocations that do not allow such threat from any player nor coalition is called the core:

Definition 6 *The* core of a coalitional game (N, c) *is the set of allocations* $K \in \mathbb{R}^n$ *for which* $\sum_{i \in S} K_i \leq c(S)$ *for all coalitions* $S \subseteq N$*.*

A condition for the core to be non-empty is the *Bondareva-Shapley theorem.* Let $\mathcal{C}$ be the set of all coalitions of N, let us denote by $1_S \in \mathbb{R}^n$ the characteristic vector of the coalition S:

$$(1_S)_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

A *balanced collection of weights* is a collection of $|\mathcal{C}|$ numbers λ_S in $[0, 1]$ such that $\sum_{S \in \mathcal{C}} \lambda_S 1_S = 1_N$. A game is *balanced* if $\sum_{S \in \mathcal{C}} \lambda_S \, c(S) \geq c(N)$ for all balanced collections of weights. Then:

Theorem 1 (Bondareva-Shapley, [11], [29]) *A coalitional game has a non-empty core if and only if it is balanced.*

Proof: see e.g. [21].

4.2 The Shapley value

The Shapley value was introduced by L. Shapley [28] and has ever since received a considerable amount of interest (see [24]).

We use the abbreviation $\Delta_i(S) = c(S \cup i) - c(S)$ for any set $S \subset N, i \notin S$. Two players i and j are *interchangeable in* (N, c) if either one makes the same contribution to any coalition S it may join, that contains neither i nor j: $\Delta_i(S) = \Delta_j(S)$ for each $S \subset N$ and $i, j \notin S$. A player i is a *dummy* if it brings the contribution $c(i)$ to any coalition S that does not contain it already: $\Delta_i(S) = c(i)$. We need to define the three properties:

Symmetry If players i and j are interchangeable, then $\Phi(N, c)_i = \Phi(N, c)_j$

Dummy player For a dummy player, $\Phi(N, c)_i = c(i)$

Additivity over games For two games (N, c_1) and (N, c_2), $\Phi(N, c_1 + c_2) = \Phi(N, c_1) + \Phi(N, c_2)$, where the game $(N, c_1 + c_2)$ is defined by $(c_1 + c_2)(S) = c_1(S) + c_2(S)$ for all $S \subseteq N$.

The rationale of these properties will be discussed in the next section. The axiomatic definition of the Shapley value is then:

Definition 7 ([28]) *The* **Shapley value** *is the only value that satisfies the properties of symmetry, dummy player, and additivity over games.*

Let us now bring together the core and the Shapley value: when does the Shapley value yield allocations that are in the core of the game ? The only pertaining results to our knowledge are that of Shapley [30] and Aubin [5]. The former involves the property of strong subadditivity:

Definition 8 *A coalitional game is* strongly subadditive *if it is based on a strongly subadditive*[3] *cost function:*

$$c(S) + c(T) \geq c(S \cup T) + c(S \cap T)$$

for all coalitions $S \subseteq N$ and $T \subseteq N$.

Theorem 2 ([30]) *If a game (N, c) is strongly subadditive, its core contains the Shapley value.*

The second condition that ensures that the Shapley value is in the core, is:

Theorem 3 ([5]) *If for all coalitions S, $|S| \geq 2$,*

$$\sum_{T \subseteq S} (-1)^{|S|-|T|} c(T) \leq 0$$

then the core contains the Shapley value.

The implications of these two results are discussed in the next section.

Let us end this section with the algebraic definition of the Shapley value, which provides both an interpretation (see [28] or [24]), and an explicit computational approach.

Definition 9 *The Shapley value K^{Sh} for the game (N, c) is defined as:*

$$K_i^{Sh} = \sum_{S \in \mathcal{C}_i} \frac{(s-1)!(n-s)!}{n!} \big(c(S) - c(S \setminus \{i\}) \big), \quad i \in N$$

where $s = |S|$, and $\mathcal{C}_i$ represents all coalitions of N that contain i.

Note that this requires the evaluation of c for each of the 2^n possible coalitions, unless the problem has some specific structure. Depending on what c is, this task may become impossibly long, even for moderate n.

[3] By definition, a strongly subadditive set function is subadditive. We follow Shapley [30] in our terminology; note that he calls convex, a function satisfying the reverse relation of strong subadditivity.

4.3 Risk capital allocations and games

Clearly, we intend to model risk capital allocation problems as coalitional games. We can associate the portfolios of a firm with the players of a game, and the risk measure ρ with the cost function c :

$$c(S) \triangleq \rho\left(\sum_{i\in S} X_i\right) \qquad \text{for } S \subseteq N \tag{1}$$

Allocation principles naturally become values.

Note that given (1), ρ being coherent and thus subadditive in the sense $\rho(X+Y) \leq \rho(X) + \rho(Y)$ of Definition 1, implies that c is subadditive in the sense $c(S \cup T) \leq c(S) + c(T)$ given above.

The core Allocations satisfying the "no undercut" property lie in the core of the game, and if none does, the core is empty. There is only a interpretational distinction between the two concepts: while a "real" player can threaten to leave the full coalition N, a portfolio cannot walk away from a bank. However, if the allocation is to be fair, undercutting should be avoided. Again, this holds also for coalitions of individual players/portfolios.

The non-emptiness of the core is therefore crucial to the existence of coherent allocation principles. From Theorem 1, we have:

Theorem 4 *If a risk capital allocation problem is modelled as a coalitional game whose cost function c is defined with a coherent risk measure ρ through (1), then its core is non-empty.*

Proof: Let $0 \leq \lambda_S \leq 1$ for $S \in \mathcal{C}$, and $\sum_{S\in\mathcal{C}} \lambda_S 1_S = 1_N$. Then

$$\begin{aligned}
\sum_{S\in\mathcal{C}} \lambda_S\, c(S) &= \sum_{S\in\mathcal{C}} \rho\left(\sum_{i\in S} \lambda_S X_i\right) \\
&\geq \rho\left(\sum_{S\in\mathcal{C}}\left(\sum_{i\in S} \lambda_S X_i\right)\right) \\
&= \rho\left(\sum_{i\in N}\left(\sum_{S\in\mathcal{C},\, S\ni i} \lambda_S X_i\right)\right) \\
&= c(N)
\end{aligned}$$

By Theorem 1, the core of the game is non-empty. □

The Shapley value With the allocation problem modelled as a game, the Shapley value yields a risk capital allocation principle. Much more, it is a coherent allocation principle, but for the "no undercut" axiom. Symmetry is satisfied by definition. The riskless allocation axiom of Definition 3 is implied by the dummy player axiom: from our definitions of section 3, the reference, riskless instrument (cash and equivalents) is a dummy player.

Note that additivity over games is a property that the Shapley value possesses but that is not required of coherent allocation principle. As discussed in section 5.3.2, additivity conflicts with the coherence of the risk measures.

The Shapley value as coherent allocation principle From the above, the Shapley value provides us with a coherent allocation principle *if* it maps games to elements of the core. It is the case when the conditions of either Theorems 2 or 3 are satisfied. The case of Theorem 2 is perhaps disappointing, as the strong subadditivity of c implies an overly stringent condition on ρ:

Theorem 5 *Let ρ be a positively homogeneous risk measure, such that $\rho(0) = 0$. Let c be defined over the set of subsets of random variables in L^∞, through $c(S) \triangleq \rho\left(\sum_{i \in S} X_i\right)$. Then if c is strongly subadditive, ρ is linear.*

Proof: Consider any random variables X, Y, Z in L^∞. The strong subadditivity of c implies

$$\rho(X+Z) + \rho(Y+Z) \;\geq\; \rho(X+Y+Z) + \rho(Z)$$

but also

$$\begin{aligned} \rho(X+Z) + \rho(Y+Z) &= \rho(X + (Y+Z) - Y) + \rho(Y+Z) \\ &\leq \rho(X + (Y+Z)) + \rho((Y+Z) - Y) \\ &= \rho(X+Y+Z) + \rho(Z) \end{aligned}$$

so that

$$\rho(X+Z) + \rho(Y+Z) \;=\; \rho(X+Y+Z) + \rho(Z)$$

By taking $Z = 0$, we obtain the additivity of ρ. Then, combining

$$\rho(-X) \;=\; \rho(X-X) - \rho(X) \;=\; -\rho(X)$$

with the positive homogeneity of ρ, we obtain that ρ is homogeneous, and thus linear. $\square$

That risks be plainly additive is difficult to accept, since it eliminates all possibility of diversification effects.

Unfortunately, the condition of Theorem 3 is also a strong one, at least in no way implied by the coherence of the risk measure ρ.

We thus fall short of a convincing proof of the existence of coherent allocations. However, we consider next an other type of coalitional games, where an slightly different definition of coherence yields much stronger existence results.

5 Allocation to fractional players

In the previous section, portfolios were modelled as players of a game, each of them *indivisible*. This indivisibility assumption is not a natural one, as we could consider fractions of portfolios, as well as coalitions involving fractions of portfolios. The purpose of this section is to examine a variant of the allocation game which allows *divisible* players.

This time, we dispense with the initial separation of risk-capital allocation problems and games, and introduce the two simultaneously. As before, players and cost functions are used to model respectively portfolios and risk measures, and values give us allocation principles.

5.1 Games with fractional players

The theory of coalitional games has been extended to continuous players who need neither be in nor out of a coalition, but who have a "scalable" presence. This point of view seems much less incongruous if the players in question represent portfolios: a coalition could consist of sixty percent of portfolio A, and fifty percent of portfolio B. Of course, this means "x percent of *each* instrument in the portfolio".

Aumann and Shapley's book "Values of Non-Atomic Games" [7] was the seminal work on the game concepts discussed in this section. There, the interval

$[0,1]$ represents the set of all players, and coalitions are measurable subintervals (in fact, elements of a σ-algebra). Any subinterval contains one of smaller measure, so that there are no *atoms*, i.e. smallest entities that could be called players; hence the name "non-atomic games".

Some of the non-atomic game theory was later recast in a more intuitive setting: an n-dimensional vector $\lambda \in \mathbb{R}^n_+$ represents the "level of presence" of the each of n players in a coalition. The original papers on the topic are Aubin's [5] and [6], Billera and Raanan's [10], Billera and Heath's [9], and Mirman and Tauman's [18].

Aubin called such games *fuzzy*; we call them (coalitional) games with fractional players:

Definition 10 *A* **coalitional game with fractional players** *(N, Λ, r) consists of*

- *a finite set N of players, with $|N| = n$;*
- *a positive vector $\Lambda \in \mathbb{R}^n_+$, each component representing for one of the n players his full involvement.*
- *a real-valued cost function $r\colon \mathbb{R}^n \to \mathbb{R}$, $r\colon \lambda \mapsto r(\lambda)$ such that $r(0) = 0$.*

Players are portfolios; the vector Λ represents, for each portfolio, the "size" of the portfolio, in a reference unit. (The Λ could also represent the business volumes of the business units). The ratio $\frac{\lambda_i}{\Lambda_i}$ then denotes a presence or activity level, for player/portfolio i, so that a vector $\lambda \in \mathbb{R}^n_+$ can be used to represent a "coalition of parts of players". We still denote by X_i the random variable of the net worth of portfolio i at a future time T), and X_n keeps its riskless instrument definition of section 3, with time T net worth equal to αr_f, with α some constant. Then the cost function r can be identified with a risk measure ρ through

$$r(\lambda) \triangleq \rho\left(\sum_{i\in N} \frac{\lambda_i}{\Lambda_i} X_i\right)$$

so that $r(\Lambda) = \rho(N)$. By extension, we also call $r(\lambda)$ a risk measure. The expression $\frac{X_i}{\Lambda_i}$ is the *per-unit* future net worth of portfolio i.

The definition of coherent risk measure (Definition 1) is adapted as:

Definition 11 *A risk measure r is* **coherent** *if it satisfies the four properties:*

Subadditivity[4] *For all λ^* and λ^{**} in $\mathbb{R}^n$,*

$$r(\lambda^* + \lambda^{**}) \;\leq\; r(\lambda^*) + r(\lambda^{**})$$

Monotonicity *For all λ^* and λ^{**} in $\mathbb{R}^n$,*

$$\sum_{i\in N} \frac{\lambda_i^*}{\Lambda_i} X_i \;\leq\; \sum_{i\in N} \frac{\lambda_i^{**}}{\Lambda_i} X_i \;\;\Rightarrow\;\; r(\lambda^*) \;\geq\; r(\lambda^{**})$$

where the left-hand side inequality is again understood as in footnote 2.

Degree one homogeneity *For all $\lambda \in \mathbb{R}^n$, and for all $\gamma \in \mathbb{R}_+$,*

$$r(\gamma\lambda) \;=\; \gamma\, r(\lambda)$$

Translation invariance *For all $\lambda \in \mathbb{R}^n$,*

$$r(\lambda) \;\;=\;\; r\left(\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \\ 0 \end{bmatrix}\right) - \frac{\lambda_n}{\Lambda_n}\alpha$$

One can check that r is coherent if and only if ρ is.

5.2 Coherent cost allocation to fractional players

The portfolio sizes given by Λ allow us to treat allocations on a *per-unit* basis. We thus introduce a vector $k \in \mathbb{R}^n$, each component of which represents the *per unit* allocation of risk capital to each portfolio. The capital allocated to each portfolio is obtained by a simple Hadamard (i.e. component-wise) product

$$\Lambda .\!* \; k \;=\; K \tag{2}$$

Let us also define, in a manner equivalent to the concepts of section 4:

[4] Note that under degree one homogeneity, subadditivity is equivalent to convexity $r(\alpha\lambda^* + (1-\alpha)\lambda^{**}) \;\leq\; \alpha r(\lambda^*) + (1-\alpha) r(\lambda^{**})$

Definition 12 *A* **fuzzy value** *is a mapping assigning to each coalitional game with fractional players $(N,\ \Lambda,\ r)$ a unique per-unit allocation vector*

$$\phi : (N, \Lambda, r) \longmapsto \begin{bmatrix} \phi_1(N, \Lambda, r) \\ \phi_2(N, \Lambda, r) \\ \vdots \\ \phi_n(N, \Lambda, r) \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

with

$$\Lambda^t k = r(\Lambda) \tag{3}$$

Again, we use the k-notation when the arguments are clear from the context.

Clearly, a fuzzy value provides us with an allocation principle, if we generalize the latter to the context of divisible portfolios.

We can now define the *coherence* of fuzzy values:

Definition 13 *Let r be a coherent risk measure. A fuzzy value*

$$\phi : (N, \Lambda, r) \longmapsto k \in \mathbb{R}^n$$

is **coherent** *if it satisfies the properties defined below, and if k is an element of the fuzzy core:*

- **Aggregation invariance** *Suppose the risk measures r and $\bar{r}$ satisfy $r(\lambda) = \bar{r}(\Gamma\lambda)$ for some $m \times n$ matrix Γ and all λ such that $0 \leq \lambda \leq \Lambda$ then*

$$\phi(N, \Lambda, r) = \Gamma^t \phi(N, \Gamma\Lambda, \bar{r}) \tag{4}$$

- **Continuity** *The mapping ϕ is continuous over the normed vector space M^n of continuously differentiable functions $r : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ that vanish at the origin.*

- **Non-negativity under r non-decreasing**[5] *If r is non-decreasing, in the sense that $r(\lambda) \leq r(\lambda^*)$ whenever $0 \leq \lambda \leq \lambda^* \leq \Lambda$, then*

$$\phi(N, \Lambda, r) \ \geq \ 0 \tag{5}$$

[5] Called monotonicity by some authors.

- **Dummy player allocation** *If i is a dummy player, in the sense that*

$$r(\lambda) - r(\lambda^*) = (\lambda_i - \lambda_i^*)\frac{\rho(X_i)}{\Lambda_i}$$

whenever $0 \leq \lambda \leq \Lambda$ and $\lambda^* = \lambda$ except in the i^{th} component, then

$$k_i = \frac{\rho(X_i)}{\Lambda_i} \tag{6}$$

- **Fuzzy core** *The allocation $\phi(N, \Lambda, r)$ belongs to the* fuzzy core *of the game (N, Λ, r) if for all λ such that $0 \leq \lambda \leq \Lambda$,*

$$\lambda^t \phi(N, \Lambda, r) \ \leq \ r(\lambda) \tag{7}$$

as well as $\Lambda^t \phi(N, \Lambda, r) = r(\Lambda)$.

The properties required of a coherent fuzzy value can be justified essentially in the same manner as was done in Definition 3. Aggregation invariance is akin to the symmetry property: equivalent risks should receive equivalent allocations. Continuity is desirable to ensure that similar risk measures yield similar allocations. Non-negativity under non-decreasing risk measures is a natural requirement to enforce that "more risk" imply "more allocation". The dummy player property is the equivalent of the riskless allocation of Definition 3, and is necessary to give "risk capital" the sense we gave it in section 2: an amount of riskless instrument necessary to make a portfolio acceptable, riskwise. Finally, note that the fuzzy core is a simple extension of the concept of core: allocations obtained from the fuzzy core through (2) allow no undercut from any player, coalition of players, *nor coalition with fractional players.* Such allocations are fair, in the same sense that core element were considered fair in section 4.3.

Much less is known about this allocation problem than is known about the similar problem described in section 4. On the other hand, one solution concept has been well investigated: the Aumann-Shapley pricing principle.

5.3 The Aumann-Shapley Value

Aumann and Shapley extended the concept of Shapley value to non-atomic games, in their original book [7]. The result was called the Aumann-Shapley value, and was later recast in the context of fractional players games, where it is defined as:

$$\phi_i^{AS}(N,\Lambda,r) = k_i^{AS} = \int_0^1 \frac{\partial r}{\partial \lambda_i}(\gamma\Lambda)\, d\gamma \tag{8}$$

for player i of N. The per-unit cost k_i^{AS} is thus an average of the marginal costs of the i^{th} portfolio, as the level of activity or volume increases uniformly for all portfolios from 0 to Λ. The value has a simpler expression, given our assumed coherence of the risk measure r; indeed, consider the result from standard calculus:

Lemma 1 *If f is a k–homogeneous function, i.e. $f(\gamma x) = \gamma^k f(x)$, then $\frac{\partial f(x)}{\partial x_i}$ is $(k-1)$-homogeneous.*

As a result, since r is 1–homogeneous,

$$\phi_i^{AS}(N,\Lambda,r) = k_i^{AS} = \frac{\partial r(\Lambda)}{\partial \lambda_i} \tag{9}$$

and the per-unit allocation vector is the gradient of the mapping r evaluated at the full presence level Λ:

$$\phi(N,\Lambda,r)^{AS} = k^{AS} = \nabla r(\Lambda) \tag{10}$$

We call this gradient "Aumann-Shapley per-unit allocation", or simply "Aumann-Shapley prices". The amount of risk capital allocated to each portfolio is then given by the components of the vector

$$K^{AS} = k^{AS} .* \Lambda \tag{11}$$

5.3.1 Axiomatic characterizations of the Aumann-Shapley value

As in the Shapley value case, a characterization consists of a set of properties, which uniquely define the Aumann-Shapley value. Many characterizations exist (see Tauman [32]); we concentrate here on that of Aubin [5] and [6], and Billera and Heath [9]. Both characterizations are for values of games with fractional players as defined above; only their assumptions on r differ from our assumptions: their cost functions are taken to vanish at zero and to be continuously differentiable, but are not assumed coherent. Aubin also implicitly assumes r to be homogeneous of degree one. Let us define:

A fuzzy value ϕ is linear if for any two games (N, Λ, r_1) and (N, Λ, r_2) and scalars γ_1 and γ_2, it is additive and 1-homogeneous in the risk measure: $\phi(N, \Lambda, \gamma_1 r_1 + \gamma_2 r_2) = \gamma_1\, \phi(N, \Lambda, r_1) + \gamma_2\, \phi(N, \Lambda, r_2)$

Then, the following properties of a fuzzy value are sufficient to *uniquely define the Aumann-Shapley value (8):*

Aubin's	**Billera & Heath's**
• linearity	• linearity
• aggregation invariance	• aggregation invariance
• continuity	• non-negativity under r non decreasing

In fact, both Aubin, and Billera and Heath prove that the Aumann-Shapley value satisfies all four properties in the table above.

So, is the Aumann-Shapley value a coherent fuzzy value when r is a coherent risk measure ? Note first that the coherence of r implies its homogeneity, as well as $r(0) = 0$. Being continuously differentiable is not automatic however; *let us assume for now that r does have continuous derivatives.* The eventual nondifferentiability will be discussed later.

Clearly, two properties are missing from the set above for ϕ to qualify as coherent: the dummy player property and the fuzzy core property. The former causes no problem: given (9), the very meaning of a dummy player in Definition 13 implies:

Lemma 2 *When the allocation process is based on a coherent risk measure r, the Aumann-Shapley prices (9) satisfy the dummy player property.*

Concerning the fuzzy core property, one very interesting result of Aubin is the following:

Theorem 6 ([5]) *The fuzzy core (7) of a fuzzy game (N, r, Λ) with positively homogeneous r is equal to the subdifferential $\partial r(\Lambda)$ of r at Λ.*

As Aubin noted, the theorem has two very important consequences:

Theorem 7 ([5]) *If the cost function r is convex (as well as positively homogeneous), then the fuzzy core is non-empty, convex, and compact.*

If furthermore r is differentiable at Λ, then the core consists of a single vector, the gradient $\nabla r(\Lambda)$.

The direct consequence of this is the Aumann-Shapley value is indeed a coherent fuzzy value, given that it exists:

Corollary 1 *If (N, r, Λ) is a game with fractional players, with r a coherent cost function that is differentiable at Λ, then the Aumann-Shapley value (10) is a* coherent fuzzy value.

Proof: The corollary follows directly from (9), theorem 7, and the fact that under positive homogeneity, r is subadditive if and only if it is convex.

This corollary is our most useful result from a practical point of view. It says that if we use a coherent, but also differentiable risk measure, and if we deem important the properties of allocation given in Definition 13, then the allocation $\nabla r(\Lambda) .* \Lambda$ is a right way to go.

As a final note, let us mention that the condition of non-increasing marginal costs, given in [9] for the the membership of $\phi^{AS}(N, \Lambda, r)$ in the fuzzy core, in fact implies that r be linear, whenever it is homogeneous.

5.3.2 On linear values and uniqueness

¿From the results given above, the Aumann-Shapley value is the *only linear* coherent allocation principle, when the cost function is adequately differentiable. However, linearity over risk measures is not required of a coherent allocation principle: while 1-homogeneity is quite acceptable, the additivity part $\phi(N, \Lambda, r_1 + r_2) = \phi(N, \Lambda, r_1) + \phi(N, \Lambda, r_2)$ causes the following problem. Because of the riskless condition, a coherent risk measure cannot be the sum of two other coherent risk measures, as it leads to the contradiction (written in the less cluttered but equivalent ρ notation)

$$\begin{aligned} \rho(X) - \alpha = \rho(X + \alpha\, r_f) &= \rho_1(X + \alpha\, r_f) + \rho_2(X + \alpha\, r_f) \\ &= \rho_1(X) + \rho_2(X) - 2\alpha \\ &= \rho(X) - 2\alpha \end{aligned}$$

Therefore, the very definition of additivity would imply that we consider non-coherent risk measures. On the other hand, *convex combinations* of coherent risk measures are coherent (see [13]), so we could make the following condition part of the definition of coherent allocations:

Definition 14 *A fuzzy value ϕ satisfies the* **convex combination property** *if for any two games $(N,\, \Lambda,\, r_1)$ and $(N,\, \Lambda,\, r_2)$ and any scalar $\gamma \in [0, 1]$,*

$$\phi\big(N, \Lambda, \gamma\, r_1 + (1-\gamma) r_2\big) = \gamma\, \phi(N, \Lambda, r_1) + (1-\gamma)\, \phi(N, \Lambda, r_2)$$

That condition implies linearity, when combined with the 1-homogeneity with respect to r of $\phi^{AS}(N, \Lambda, r)$ (which holds given the aggregation invariance property). This would make the Aumann-Shapley value the *unique* coherent allocation principle. However, we see no compelling, intuitive reason to include linearity (under a form or another) in the definition of coherent fuzzy allocation, allowing for the existence of *nonlinear* coherent fuzzy allocation principles, a topic left for further investigation.

The same remarks on uniqueness and linearity apply to the Shapley value and allocation in the non-divisible players context. Note that the debate on the pertinence of linearity is far from new: Luce and Raiffa [15], wrote in 1957 that "(additivity) strikes us as a flaw in the concept of (Shapley) value".

5.3.3 On the differentiability requirement

Concerning the differentiability of the risk measures/cost functions, recent results are encouraging. Tasche [31] and Scaillet [26] give conditions under which a coherent risk measure, the expected shortfall, is differentiable. The conditions are relatively mild, especially in comparison with the temerarious assumptions common in the area of risk management. Explicit first derivatives are provided, which have the following interpretation: they are expectations of the risk factors, conditioned on the portfolio value being below a certain quantile of its distribution. This is very interesting: it shows that when Aumann-Shapley value is used with a shortfall risk measure, the resulting (coherent) allocation is again of a shortfall type:

$$K_i = E\left[-X_i \;\mid\; \sum_i X_i \le q_\alpha \right]$$

where q_α is a quantile of the distribution of $\sum_i X_i$.

Even when r is not differentiable, something can often be saved. Indeed, suppose that r is not differentiable at Λ, but is the supremum of a set of parameterized functions that are themselves convex, positively homogeneous and differentiable at Λ:

$$r(\lambda) \ = \ \sup_{p \in P} w(\lambda, p) \tag{12}$$

where P is a compact set of parameters of the functions w, and $w(\lambda, p)$ is upper semicontinuous in p. Then Aubin [5] proved:

The fuzzy core is the closed convex hull of all the values $\phi^{AS}(N, \Lambda, w(\Lambda, p))$ of the functions w that are "active" at Λ, i.e. that are equal to $r(\Lambda)$.

Thus, should (12) arise, —which is not unlikely, think of Lagrangian relaxation when r is defined by an optimization problem—, the above result provides a *set* of coherent values to choose from.

5.3.4 Alternative paths to the Aumann-Shapley value

It is very interesting that the recent report of Tasche [31] comes fundamentally to the same result obtained in this section, namely that given some differentiability conditions on the risk measure ρ, the correct way of allocating risk capital is through the Aumann-Shapley prices (9). Tasche's justification of this contention is however completely different; he defines as "suitable", capital allocations such that if the risk-adjusted return of a portfolio is "above average", then, at least locally, increasing the share of this portfolio improves the overall return of the firm. Note that the work of Schmock and Straumann [27] points again to the same conclusion. In the approach of [31] and [27], *the Aumann-Shapley prices are in fact the unique satisfactory allocation principle.*

Others important results on the topic include that of Artzner and Ostroy [4], who, working in a non-atomic measure setting, provide alternative characterizations of differentiability and subdifferentiability, with the goal of establishing the existence of allocations through, basically, Euler's theorem. See also the forthcoming Delbaen [13].

On Euler's theorem[6], note also that the feasibility (3) of the allocation vector follows directly from it, and that out of consideration for this, some authors have called the allocation principle (9) the *Euler principle.* See for example the attachment to the report of Patrik, Bernegger, and Rüegg [23], which provides some properties of this principle.

We shall end this section by drawing the attention of the reader to the importance of the coherence of the risk *measure* ρ (and the r derived from it) for the allocation.

The subadditivity of the risk measure: is a necessary condition for the existence of an allocation with no undercut, in both the atomic and fractional players contexts.

The homogeneity of the risk measure: ensures the simple form (9) of the Aumann-Shapley prices.

Both subadditivity and homogeneity: are used to prove that the core in non-empty (Theorem 4), in the atomic game setting. In the fuzzy game setting, the two properties are used to show that the Aumann-Shapley value is in the fuzzy core (under differentiability). They are also used in the non-negativity proof of the appendix.

The riskless property: is central to the definition of the riskless allocation (dummy player) property.

6 The non-negativity of the allocation

Given our definition of risk measure, a portfolio may well have a negative risk measure, with the interpretation that the portfolio is then *safer* than deemed necessary.

Similarly, there is no justification *per se* to enforce that the risk capital allocated to a portfolio be non-negative; that is, the allocation of a negative amount does not pose a conceptual problem. Unfortunately, in the *application*

[6] Which states that if F is a real, n–variables, homogeneous function of degree k, then $x_1 \frac{\partial F(x)}{\partial x_1} + x_2 \frac{\partial F(x)}{\partial x_2} + \cdots + x_n \frac{\partial F(x)}{\partial x_n} = kF(x)$

we would like to make of the allocated capital, non-negativity is a problem. If the amount is to be used in a RAPM-type quotient $\frac{\text{return}}{\text{allocated capital}}$, negativity has a rather nasty drawback, as a portfolio with an allocated capital slightly below zero ends up with a negative risk-adjusted measure of large magnitude, whose interpretation is less than obvious. A negative allocation is therefore not so much a concern with the allocation itself, than with the use we would like to make of it.

A crossed-fingers, and perhaps most pragmatic approach, is to assume that the coherent allocation is inherently non-negative. In fact, one could reasonably expect non-negative allocations to be the norm in real-life situations. For example, provided no portfolio of the firm ever decreases the risk measure when added to any subset of portfolios of the firm:

$$c(S \cup \{i\}) \geq c(S) \quad \forall\, S \subseteq N,\ \forall\, i \in N \setminus S$$

then the Shapley value is necessarily non-negative. The equivalent condition for the Aumann-Shapley prices is the property of non-negativity under non-decreasing r (equation (5)): if the antecedent always holds, the per-unit allocations are non-negative.

Another approach would be to enforce non-negativity by requiring more of the risk measure. For example, the core and the non-negativity of the K_i's form a set of linear inequalities (and one linear equality), so that the existence of a non-negative core solution is equivalent to the existence of a solution to a linear system. Specifically, a hyperplane separation argument proves that such a solution will exist if the following condition on ρ holds:

$$\forall\, \lambda \in \mathbb{R}^n_+, \quad \rho\left(\sum_{i \in N} X_i\right) \min_{i \in N} \{\lambda_i\} \;\leq\; \rho\left(\sum_{i \in N} \lambda_i X_i\right) \tag{13}$$

The proof is given in addendum. The condition could be interpreted as follows. First assume that $\rho\left(\sum_{i \in N} X_i\right) > 0$, which is reasonable, if we are indeed to allocate some risk capital. Then (13) says that there is no positive linear combination of (each and every) portfolios, that runs no risk. In other words, a perfectly hedged portfolio cannot be attained by simply re-weighting the portfolios, if all portfolios are to have a positive weight. However, unless one is willing to impose such a condition on the risk measure, the fact is that the issue of the non-negativity remains unsatisfactorily resolved for the moment.

7 Allocation with an SEC-like risk measure

In this section, we provide some examples of applications of the Shapley and Aumann-Shapley concepts to a problem of margin (i.e. risk capital) allocation.

The risk measure we use is derived from the Securities and Exchange Commission (SEC) rules for margin requirements (Regulation T), as described in the National Association of Securities Dealers (NASD) document [19]. These rules are used by stock exchanges to establish the margins required of their members, as guarantee against the risk that the members' portfolios involve (the Chicago Board of Options Exchange is one such exchange). The rules themselves are not constructive, in that they do not specify how the margin should be computed; this computation is left to each member of the exchange, who must find the smallest margin complying with the rules. Rudd and Schroeder [25] proved in 1982 that a linear optimization problem (L.P.) modelled the rules adequately, and was sufficient to establish the minimum margin of a portfolio, that is, to evaluate its *risk measure*. It is worth mentioning that given this L.P.-based risk measure, the corresponding coalitional game has been called *linear production game* by Owen [22], see also [10].

For the purpose of the article, we restrict the risk measure to simplistic portfolios of calls on the same underlying stock, and with the same expiration date. This restriction of the SEC rules is taken from Artzner, Delbaen, Eber and Heath [3] who use it as an example of a *non-coherent* risk measure. In the case of a portfolio of calls, the margin is calculated through a representation of the calls by a set of *spread options*, each of which carrying a fixed margin. To obtain a coherent measure of risk, we prove later that it is sufficient to represent the calls by a set of spreads *and* butterfly options. Note that such a change to the margins rules was proposed by the NASD and very recently accepted by the SEC, see [20].

7.1 Coherent, SEC-like margin calculation

We consider a portfolio consisting of C_P calls at strike price P, where P belongs to a set of strike prices $\mathcal{P} = \{P_{\min}, P_{\min} + 10, \ldots, P_{\max} - 10, P_{\max}\}$. This assumption about the format of the strike prices set $\mathcal{P}$, including the intervals of 10, makes the notation more palatable, without loss of generality. For convenience, we denote the set $\mathcal{P} \setminus \{P_{\min}, P_{\max}\}$ by $\mathcal{P}^-$. We also make the simplifying assumption that there are as many long calls as short calls in the portfolio, i.e. $\sum_{P \in \mathcal{P}} C_P = 0$. Both assumptions remain valid throughout section 7.

We will denote by $\boldsymbol{C_P}$ the vector of the C_P parameters, $P \in \mathcal{P}$. While $\boldsymbol{C_P}$ fully describes the *portfolio*, it certainly does not describe the *future value of the portfolio*, which depends on the price of the underlying stock at a future date. Although risk measures were defined as a mappings on random variables, we nevertheless write $\rho(\boldsymbol{C_P})$ since the ρ considered here can be defined by using *only* $\boldsymbol{C_P}$. On the other hand, there is a simple linear relationship between $\boldsymbol{C_P}$ and the future worths (under an appropriate discretization of the stock price space), so that an expression such as $\rho\left(\boldsymbol{C_P^*} + \boldsymbol{C_P^{**}}\right)$ is also justified. Only in the case of the property "monotonicity" need we treat with more care the distinction between number of calls and future worth.

We can now define our SEC-like margin requirement. To evaluate the margin (or risk measure) ρ of the portfolio $\boldsymbol{C_P}$, we first replicate its calls with *spreads* and *butterflies*, defined as follows:

Variable	Instrument	Calls equivalent
$S_{H,K}$	Spread, long in H, short in K	One long call at price H, one short call at strike K
B_H^{long}	Long butterfly, centered at H	One long call at $H-10$, two short calls at H, one long call at $H+10$
B_H^{short}	Short butterfly, centered at H	One short call at $H-10$, two long calls at H, one short call at $H+10$

The variables shall represent the number of each specific instrument. All H and K are understood to be in $\mathcal{P}$, or $\mathcal{P}^-$ for the butterflies; $H \neq K$ for the spreads.

As in the SEC rules, fixed margins are attributed to the instruments used for the replicating portfolio, i.e. spreads and butterflies in our case. Spreads carry a margin of $(H-K)^+ = \max(0, H-K)$; short butterflies are given a margin of 10, while long butterflies require no margin. In simple language, each instrument requires a margin equal to the worst potential loss, or negative payoff, it could yield.

By definition, the margin of a portfolio of spreads and butterflies is the sum of the margins of its components.

On the basis of [25], the margin $\rho(\boldsymbol{C_P})$ of the portfolio can be evaluated with the linear optimization problem (SEC-LP):

$$\begin{array}{lll} \text{minimize} & f^t\boldsymbol{Y} & \\ & & (\text{SEC}-\text{LP}) \\ \text{subject to} & A\boldsymbol{Y} = \boldsymbol{C_P} & \\ & \boldsymbol{Y} \geq \boldsymbol{0} & \end{array}$$

where: $\boldsymbol{Y}$ stands for $\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{S} \\ \boldsymbol{B}^{\textbf{long}} \\ \boldsymbol{B}^{\textbf{short}} \end{bmatrix}$ where $\boldsymbol{S}$ is a column vector of all spreads variables considered (appropriately ordered: bull spreads, then bear spreads), and $\boldsymbol{B}^{\textbf{long}}$ and $\boldsymbol{B}^{\textbf{short}}$ are appropriately ordered column vectors of butterflies variables; $f^t\boldsymbol{Y}$ is shorthand notation for

$$f^t\boldsymbol{Y} = \sum_{H,K\in\mathcal{P}} (H-K)^+ \, S_{H,K} \; + \sum_{H\in\mathcal{P}^-} 10\, B_H^{\text{short}};$$

and A is

$$A = \left[\begin{array}{rrcr|rrcr|rrcr|rrcr}
1 & 1 & \cdots & 0 & -1 & -1 & \cdots & 0 & 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
-1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2 & 1 & \cdots & 0 & 2 & -1 & \cdots & 0 \\
0 & -1 & \cdots & 0 & 0 & 1 & \cdots & 0 & 1 & -2 & \cdots & 0 & -1 & 2 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & -2 & 0 & 0 & \cdots & 2 \\
0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1
\end{array}\right]$$

The objective function represent the margin; the equality constraints ensure that the portfolio is exactly replicated. The risk measure thus defined is coherent; the proof is given next.

7.2 Proof of the coherence of the measure

We prove here that the risk measure ρ obtained through (SEC-LP) is *coherent*, in the sense of Definition 1. We prove each of the four property in turn, below.

1) Subadditivity: For any two portfolios $\boldsymbol{C_P^*}$ and $\boldsymbol{C_P^{**}}$,

$$\rho\left(\boldsymbol{C_P^*} + \boldsymbol{C_P^{**}}\right) \;\leq\; \rho(\boldsymbol{C_P^*}) + \rho(\boldsymbol{C_P^{**}})$$

Proof: If solving (SEC-LP) with $\boldsymbol{C_P^*}$ as right-hand side of the equality constraints yields a solution $\boldsymbol{S^*}$, and solving with $\boldsymbol{C_P^{**}}$ yields a solution $\boldsymbol{S^{**}}$, then $\boldsymbol{S^*} + \boldsymbol{S^{**}}$ is a feasible solution for the (SEC-LP) with $\boldsymbol{C_P^*} + \boldsymbol{C_P^{**}}$ as right-hand side. Subadditivity follows directly, given the linearity of the objective function.

2) Degree one homogeneity: For any $\gamma \geq 0$ and any portfolio $\boldsymbol{C_P}$,

$$\rho(\gamma \boldsymbol{C_P}) = \gamma \rho(\boldsymbol{C_P})$$

Proof: This is again a direct consequence of the linear optimization nature of ρ, as $\gamma \boldsymbol{S}$ is a solution of the (SEC-LP) with $\gamma \boldsymbol{C_P}$ as right-hand side of the constraints, when $\boldsymbol{S}$ is a solution of the (SEC-LP) with $\boldsymbol{C_P}$ as right-hand side. Of course, the very definition of homogeneity implies that we allow fractions of calls to be sold and bought.

3) Translation invariance:[7] Adding to any portfolio of calls $\boldsymbol{C_P}$ an amount of riskless instrument worth α today, decreases the margin of $\boldsymbol{C_P}$ by α.

Proof: There is little to prove here; we rather need to define the behaviour of ρ in the presence of a riskless instrument, and naturally choose the translation invariance property to do so. This property simply anchors the meaning of "margin".

4) Monotonicity: For any two portfolios $\boldsymbol{C_P^*}$ and $\boldsymbol{C_P^{**}}$ such that the future worth of $\boldsymbol{C_P^*}$ is always less than or equal to that of $\boldsymbol{C_P^{**}}$,

$$\rho(\boldsymbol{C_P^*}) \;\geq\; \rho(\boldsymbol{C_P^{**}})$$

[7] Note that in the interest of a tidier notation, we departed from our previous usage and did not use the last component of $\boldsymbol{C_P}$ to denote the riskless instrument

Before proving monotonicity, let us first introduce the values V_P, for $P \in \{P_{\min}+10, \ldots, P_{\max}, P_{\max}+10\}$, which represent the future payoffs, or worths, of the portfolio for the future prices P of the underlying. (Obviously, the latter set of prices may be too coarse a representation of possible future prices, and is used to keep the notation compact; starting with a finer $\mathcal{P}$ would relieve this problem) Again, we write $\boldsymbol{V_P}$ to denote the vector of all V_P's. The components of $\boldsymbol{V_P}$ are completely determined by the number of calls in the portfolio:

$$V_P \;=\; \sum_{p=P_{\min}}^{P-10} C_p(P-p) \qquad \forall\, P \in \{P_{\min}+10, \ldots, P_{\max}, P_{\max}+10\}$$

which is alternatively written $\boldsymbol{V_P} \triangleq M\boldsymbol{C_P}$, with the square, invertible matrix M:

$$M \;=\; \begin{bmatrix} 10 & 0 & 0 & \cdots \\ 20 & 10 & 0 & \cdots \\ 30 & 20 & 10 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The antecedent of the monotonicity property is, of course, the componentwise $\boldsymbol{V_P^*} \leq \boldsymbol{V_P^{**}}$.

We will also use the following lemma:

Lemma 3 *Under subadditivity, two equivalent formulations of monotonicity are, for any three portfolios* $\boldsymbol{C_P}$, $\boldsymbol{C_P^*}$ *and* $\boldsymbol{C_P^{**}}$*:*

$$\boldsymbol{V_P^*} \;\leq\; \boldsymbol{V_P^{**}} \;\;\Longrightarrow\;\; \rho\,(M^{-1}\boldsymbol{V_P^*}) \;\geq\; \rho\,(M^{-1}\boldsymbol{V_P^{**}})$$

and

$$0 \;\leq\; \boldsymbol{V_P} \;\;\Longrightarrow\;\; \rho\,(M^{-1}\boldsymbol{V_P}) \;\leq\; 0$$

Proof: The upper condition is sufficient, as it implies

$$\rho(\mathbf{0}) \;\geq\; \rho(M^{-1}\boldsymbol{V_P}),$$

and $\rho(\mathbf{0}) = 0$ from the very structure of (SEC-LP). The upper condition is necessary, as

$$\begin{aligned} \rho\,(M^{-1}\boldsymbol{V_P^{**}}) \;&=\; \rho\,\left(M^{-1}(\boldsymbol{V_P^*} + (\boldsymbol{V_P^{**}} - \boldsymbol{V_P^*}))\right) \\ &\leq\; \rho\,(M^{-1}\boldsymbol{V_P^*}) + \rho\,\left(M^{-1}(\boldsymbol{V_P^{**}} - \boldsymbol{V_P^*})\right) \\ &\leq\; \rho\,(M^{-1}\boldsymbol{V_P^*}). \end{aligned}$$

□

Proof of monotonicity: As a consequence of the above lemma, it is sufficient to prove that if a portfolio of calls *always* has non-negative future payoff, then its associated margin is non-positive.

A look at (SEC-LP) shows that the margin assigned to the portfolio will be non-positive (in fact, zero), if and only if a non-negative, feasible solution of (SEC-LP) exists in which all spreads variables $S_{H,K}$ with $H > K$ and all short butterflies variables B^{short} have value zero. This means that there exists a solution to the linear system:

$$\widetilde{A}\boldsymbol{Y} = \boldsymbol{C_P} \qquad (14)$$

$$\boldsymbol{Y} \geq 0 \qquad (15)$$

where $\widetilde{A}$ is made of the first and third parts of the A that was defined for (SEC-LP):

$$A = \begin{bmatrix} 1 & 1 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & -2 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & 1 & -2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -2 \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

and $\boldsymbol{Y}$ is the appropriate vector of spreads and butterflies variables. We obtain a new, equivalent system of equations $M\widetilde{A}\boldsymbol{Y} = M\boldsymbol{C_P} = \boldsymbol{V_P}$ by pre-multiplying by the invertible matrix M introduced above. Recall now that we have made the assumption that the portfolio contains as many short calls as long calls, i.e. $\sum_{P\in\mathcal{P}} C_P = 0$. Thus, we need only prove that there exists a non-negative solution to the system

$$M\widetilde{A}\boldsymbol{Y} = \boldsymbol{V_P} \quad \text{whenever} \quad \boldsymbol{V_P} \geq 0 \quad \text{and} \quad e^t M^{-1}\boldsymbol{V_P} = 0$$

(e^t is a row vector of 1's). A simple observation of $M\widetilde{A}$ shows that its columns span the same subspace as the set of columns

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Observing furthermore that

$$e^t M^{-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 1 \end{bmatrix}^t$$

so that any $\boldsymbol{V_P}$ satisfying $e^t M^{-1} \boldsymbol{V_P} = 0$ has identical last two components, the right-hand side of $M\widetilde{A}\boldsymbol{Y} = \boldsymbol{V_P}$ can always be expressed as a non-negative linear combination of the columns of $M\widetilde{A}$. □

7.3 Computation of the allocations

Given this risk measure as a linear optimization problem, the Shapley value is easy to compute when the "total portfolio" is divided in a small number of subportfolios. First, the margin of every possible coalition of subportfolios is calculated. Then, the margin allocated to each subportfolio is computed, using the formula of the Shapley value given in Definition 9.

The computation of the Aumann-Shapley value is even simpler. Note that by working in the fractional players framework, we implicitly assume that fractions of portfolios are sensible instruments. We choose the vector of full presence of all players Λ to be the vector of ones e. Recall that the Aumann-Shapley *per unit* margin allocated to the i^{th} subportfolio is

$$k_i^{AS} = \frac{\partial r(e)}{\partial \lambda_i} \tag{16}$$

where $r(\lambda)$ is the margin required of the sum of all the subportfolios i, each scaled by a scalar λ_i, so that $r(e) = \rho(\boldsymbol{C_P})$. In vector notation, $k^{AS} = \nabla r(e)$.

Let us first define the linear operator L which maps the level of presence of the subportfolios to numbers of calls in the global portfolio. If there are $|\mathcal{P}|$ different calls (equivalently here, $|\mathcal{P}|$ strike prices), and n subportfolios, then L is an $|\mathcal{P}| \times n$ matrix, such that $Le = \boldsymbol{C_P}$. Examples are the three-by-five matrices at the top of the tables given in section 7.4.

Now, the optimal *dual* solution δ^* of the linear program (SEC-LP), obtained automatically when computing the margin of the total portfolio, provides the rates of change of the margin, when the presence of *each specific call* varies. However, using the complementarity condition satisfied at the optimal solution pair $(\boldsymbol{Y^*}, \delta^*)$, we can write

$$\begin{aligned} f^t \boldsymbol{Y^*} &= -(\delta^*)^t \boldsymbol{C_P} && (17) \\ &= (-(\delta^*)^t L)e && (18) \end{aligned}$$

so that the components of $L^t\delta^*$ give the marginal rates of change of the objective value of (SEC-LP), as a function of subportfolio presence, evaluated at the point of full presence of all subportfolios.

Put in one sentence, the most interesting result of this section is that *the Aumann-Shapley allocation is only a matrix product away from the lone evaluation of the margin for the total portfolio.*

Finally, concerning the uniqueness of the allocation and the differentiability of the risk measure, we can only say that they depend directly on the uniqueness of the optimal solution of the dual problem of (SEC-LP). Although there is not special reason for multiple optimal dual solutions to occur here, it can well happen, in which case we have obtained one of many acceptable allocations, per section 5.3.3.

7.4 Numerical examples of coherent allocation

We can obtain a somewhat more practical feeling of Shapley and Aumann-Shapley allocations by looking at examples.

For all allocation examples below, the reference "total" portfolio is the same; its values of $C_P, P \in \{10, 20, 30, 40, 50\}$ are:

	C_{10}	C_{20}	C_{30}	C_{40}	C_{50}
Total	-1	-2	8	-7	2

meaning one short call at strike 10, two short calls at strike 20, eight long calls at strike 30, etc. It carries a margin of 40: $\rho(\boldsymbol{C_P}) = 40$.

In each of the tables below, we give the division of the total portfolio in three subportfolios such that $\boldsymbol{C_{P_1}} + \boldsymbol{C_{P_2}} + \boldsymbol{C_{P_3}} = \boldsymbol{C_P}$, followed by the Shapley allocation and the Aumann-Shapley allocation. The tables are followed by some observations. Consider first the division:

	C_{10}	C_{20}	C_{30}	C_{40}	C_{50}	Shapley	Aumann – Shapley
$\boldsymbol{C_{P_1}}$	−1	0	6	−6	1	15	20
$\boldsymbol{C_{P_2}}$	0	−2	2	0	0	20	20
$\boldsymbol{C_{P_3}}$	0	0	0	−1	1	5	0
Total	−1	−2	8	−7	2	40	40

In this example, coalitions of subportfolios incur margins as follows:

$$\rho(\boldsymbol{C_{P_1}} + \boldsymbol{C_{P_2}}) = 40 \qquad \rho(\boldsymbol{C_{P_1}} + \boldsymbol{C_{P_3}}) = 20 \qquad \rho(\boldsymbol{C_{P_2}} + \boldsymbol{C_{P_3}}) = 30$$
$$\rho(\boldsymbol{C_{P_1}}) = 20 \qquad \rho(\boldsymbol{C_{P_2}}) = 20 \qquad \rho(\boldsymbol{C_{P_3}}) = 10$$

Consider a second example:

	C_{10}	C_{20}	C_{30}	C_{40}	C_{50}	Shapley	Aumann – Shapley
$\boldsymbol{C_{P_1}}$	−1	0	2	−2	1	20	20
$\boldsymbol{C_{P_2}}$	0	−1	6	−5	0	0	10
$\boldsymbol{C_{P_3}}$	0	−1	0	0	1	20	10
Total	−1	−2	8	−7	2	40	40

Here, coalitions of subportfolios portfolios incur the margins:

$$\rho(\boldsymbol{C_{P_1}} + \boldsymbol{C_{P_2}}) = 30 \qquad \rho(\boldsymbol{C_{P_1}} + \boldsymbol{C_{P_3}}) = 50 \qquad \rho(\boldsymbol{C_{P_2}} + \boldsymbol{C_{P_3}}) = 20$$
$$\rho(\boldsymbol{C_{P_1}}) = 20 \qquad \rho(\boldsymbol{C_{P_2}}) = 10 \qquad \rho(\boldsymbol{C_{P_3}}) = 30$$

Finally, the third example is:

	C_{10}	C_{20}	C_{30}	C_{40}	C_{50}	Shapley	Aumann – Shapley
$\boldsymbol{C_{P_1}}$	−1	−1	4	−2	0	26.66	30
$\boldsymbol{C_{P_2}}$	0	−1	4	−3	0	6.66	10
$\boldsymbol{C_{P_3}}$	0	0	0	−2	2	6.66	0
Total	−1	−2	8	−7	2	40	40

where the coalitions of subportfolios incur:

$$\rho(\boldsymbol{C_{P_1}} + \boldsymbol{C_{P_2}}) = 40 \qquad \rho(\boldsymbol{C_{P_1}} + \boldsymbol{C_{P_3}}) = 30 \qquad \rho(\boldsymbol{C_{P_2}} + \boldsymbol{C_{P_3}}) = 10$$
$$\rho(\boldsymbol{C_{P_1}}) = 30 \qquad \rho(\boldsymbol{C_{P_2}}) = 10 \qquad \rho(\boldsymbol{C_{P_3}}) = 20$$

On these examples, we note that:

- **The Shapley and Aumann-Shapley allocations do not agree.** An equal allocation in this setting would have been fortuitous.
- **Null allocations do occur.** Null (or negative) allocations cannot be ruled out, and do happen here, both for the Shapley and Aumann-Shapley principles. For use in a risk-adjusted return calculation (return divided by allocated capital), these would indeed be problematic.

- **The Aumann-Shapley allocation is a marginal rate of risk capital.** It can be checked that, for example in the first case, the risk capital of the whole portfolio does not change when the presence of the third subportfolio varies slightly around unity.
- **The Shapley allocation may not be in the core.** Again, this was to be expected. The Shapley allocation of the third example is not part of the core, as can be checked manually.

As a final note, let us mention that we could have verified whether the Shapley allocation is part or not of the *fuzzy* core (whenever it is part of the core). One way of doing this is through a bilevel linear program, which, unfortunately, is known to be computationally demanding for large instances; see Marcotte [16] or Marcotte, Savard [17]. Further details on this issue are reserved for later.

8 Conclusion

In this article, we have discussed the allocation of risk capital from an axiomatic perspective, defining in the process what we call *coherent allocation principles.*

Our original goal is to establish a framework within which financial risk allocation principles could be compared as meaningfully as possible. Our stand is that this can be achieved by binding the concept of coherent *risk measures* to the existing game theory results on allocation.

We suggest two sets of axioms, each defining the coherence of risk capital allocation in a specific setting: either the constituents of the firm are considered indivisible entities (in the coalitional game setting), or, to the contrary, they are considered to be divisible (in the context of games with fractional players). In the former case, we find that the Shapley value is a coherent allocation principle, though only under rather restrictive conditions on the risk measure used.

In the fractional players setting, the Aumann-Shapley value is a coherent allocation principle, under a much laxer differentiability condition on the risk measure; under linearity, it is also the unique coherent principle. In fact, given that the allocation process starts with a coherent risk measure, this coherent allocation simply corresponds to the gradient of the risk measure with respect to the presence level of the constituents of the firm. As a consequence, the Aumann-Shapley approach, beyond its theoretical soundness, further has a computational fe, in that it is as easy to evaluate, as the risk itself is.

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Appendix

A Proof of the non-negativity condition

The following result from section 6 is proved here.

Theorem 8 *A sufficient condition for a non-negative, "no undercut" allocation to exist is:*

$$\forall\, \lambda \in \mathbb{R}^n_+, \quad \rho\left(\sum_{i\in N} X_i\right) \min_{i\in N}\{\lambda_i\} \;\leq\; \rho\left(\sum_{i\in N} \lambda_i X_i\right)$$

Proof: Let us recall that we denoted by $1_S \in \mathbb{R}^n$ the characteristic vector of the coalition S:

$$(1_S)_i = \begin{cases} 1 & \text{if } \; i \in S \\ 0 & \text{otherwise} \end{cases}$$

A non-negative, "no undercut" allocation K exists when

$$\begin{aligned} \exists\, K \in \mathbb{R}^n \text{such that} & \\ K^t 1_S \leq \rho\left(\sum_{i\in S} X_i\right) & \quad \forall\, S \subsetneq N \\ K^t 1_N = \rho\left(\sum_{i\in N} X_i\right) & \\ K \geq 0 & \end{aligned} \tag{19}$$

Using Farkas's lemma, this is equivalent to

$$1_N y_N + \sum_{S\subsetneq N} 1_S y_S \geq\; 0, \quad \forall\, y_N \in \mathbb{R}, \quad \forall\, y_S \in \mathbb{R}_+,\; S \subsetneq N$$
$$\Longrightarrow \quad \rho\left(\sum_{i\in N} X_i\right) y_N + \sum_{S\subsetneq N} \rho\left(\sum_{i\in S} X_i\right) y_S + \geq\; 0 \tag{20}$$

which in turn is equivalent to

$$y_N \geq -\sum_{S\ni i} y_S, \quad \forall\, y_S \geq 0,\; S \subsetneq N,\; and\, \forall\, i \in N,$$
$$\Longrightarrow \quad \sum_{S\subsetneq N} \rho(\sum_{i\in S} X_i) y_S \geq -\rho\left(\sum_{i\in N} X_i\right) y_N \tag{21}$$

Now, using the homogeneity and the subadditivity of ρ,

$$\begin{aligned}\sum_{S\subsetneq N}\rho\left(\sum_{i\in S}X_i\right)y_S &= \sum_{S\subsetneq N}\rho\left(y_S\sum_{i\in S}X_i\right)\\ &\geq \rho\left(\sum_{S\subsetneq N}\left(y_S\sum_{i\in S}X_i\right)\right)\\ &= \rho\left(\sum_{i\in N}\left(X_i\sum_{S\ni i}y_S\right)\right)\end{aligned}$$

Therefore, a sufficient condition for (19) (or (20) or (21)) to hold, is

$$y_N \geq -\sum_{S\ni i} y_S, \quad \forall\, y_S \geq 0,\ S\subsetneq N,\ \forall\, i\in N,$$

$$\Longrightarrow \quad \rho\left(\sum_{i\in N}\left(X_i\sum_{S\ni i}y_S\right)\right)\geq \rho\left(\sum_{i\in N}X_i\right)(-y_N)$$

Finally, using the definition $\lambda_i \triangleq \sum_{S\ni i} y_S$, we can write the *sufficient* condition for (19)

$$\forall\, \lambda_i \geq 0,\ i\in N, \quad \rho\left(\sum_{i\in N}\lambda_i X_i\right)\geq \rho\left(\sum_{i\in N}X_i\right)\left(\min_{i\in N}\lambda_i\right)$$

Note that in the last step, we also used $\rho\left(\sum_{i\in N}X_i\right)\geq 0$, a *necessary* condition for the existence of a non-negative, "no undercut" allocation; checking $y_N = 1,\ y_S = 0\ \ \forall\, S\subsetneq N$ in (20) shows this point. □